Rank Theorem

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Let $A\colon \mathbb{R}^m \to \mathbb{R}^n$ be linear. Assume that the rank of A is r. Then we can choose bases for $\mathbb{R}^m$ and $\mathbb{R}^n$ in such a way that the matrix representation of A with respect to the bases looks like $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$. To prove this, let us choose a basis $\{w_1, \ldots, w_r\}$ of the image of A. We choose $v_i \in \mathbb{R}^m$ such that $Av_i = w_i$, $1 \le i \le r$. We now choose a basis $\{v_{r+1}, \ldots, v_m\}$ of the kernel of A. It is easy to see that $\{v_i : 1 \le i \le n\}$ is a basis of $\mathbb{R}^m$. We extend the set $\{w_i : 1 \le i \le r\}$ of linearly independent vectors to a basis of $\mathbb{R}^n$. Then $\{v_i : 1 \le i \le m\}$ and $\{w_j : 1 \le j \le n\}$ are bases of $\mathbb{R}^m$ and $\mathbb{R}^n$ as required. Note that if we use the system of coordinates given rise to by these bases, then the linear map A is the map $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_r, 0, \ldots, x_r)$.

The rank theorem says that if $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is smooth and has constant rank r in a neighbourhood of a point $p \in \mathbb{R}^m$, then there exist local coordinates around p and $f(p)$ with respect which the map is represented as $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0)$. To make this precise, we need the following

Definition 1. Let $U \subset \mathbb{R}^m$ be open. Let $f\colon U \to \mathbb{R}^m$ be smooth and $p \in U$. If $Df(p)$ is nonsingular, then by the inverse mapping theorem, there exists a neighbourhood U' of p in U such that f maps U' bijectively onto an open set $f(U')$ and $f^{-1}\colon U' \to U$ is smooth. Thus f is a local diffeomorphism at p. We say that f is a local coordinate system at p: for, if we define $y_i(p') := \langle f(p'), e_i \rangle$, for $p' \in U'$, then we have a new system of coordinates on U'.

Theorem 2 (Rank Theorem)**.** *Let $U \subset \mathbb{R}^m$ be open. Let $f\colon U \to \mathbb{R}^n$ be smooth. If the derivative $Df(p')$ has a constant rank r in a neighbourhood of $p \in U$, then there is a local coordinate system g around p and a local coordinate system h at $q := f(p)$ such that*

$$h \circ f \circ g^{-1}\colon (x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0).$$

Proof. Without loss of generality, we may assume that $p = 0 \in \mathbb{R}^m$ and $f(p) = 0 \in \mathbb{R}^n$. We write $f = (f_1, \ldots, f_n)$. Reindexing the variables, if necessary, we may assume that the matrix $J_1 := (\frac{\partial f_i}{\partial x_j})_{1 \le i,j \le r}$ is has nonzero determinant at 0.

We define

$$F(x_1, \ldots, x_m) = (f_1(x), \ldots, f_r(x), x_{r+1}, \ldots, x_m).$$

Then the Jacobian of F is $\begin{pmatrix} J_1 & * \\ 0 & I_{m-r} \end{pmatrix}$. Its determinant is the same as that of J_1 and hence is nonzero. The map F therefore gives rise to a local diffeomorphism g at 0 and therefore to a local coordinate system g. We observe that the map $\varphi := f \circ g^{-1}$ is given by

$$(z_1, \ldots, z_m) \mapsto (z_1, \ldots, z_r, \varphi_{r+1}(z), \ldots, \varphi_n(z)).$$

The Jacobian of $f \circ g^{-1}$ is of the form $\begin{pmatrix} I_r & 0 \\ * & A(z) \end{pmatrix}$ where $A(z) = (\frac{\partial \varphi_i}{\partial z_j})_{r+1 \leq i \leq n, r+1 \leq j \leq m}$. We have so far used the fact that the rank of f is at least r and used this information to find a new system of coordinates in the domain space.

Since we have $\text{Rank}(Df) = \text{Rank}(D(f \circ g^{-1})) = r$ in an open set W containing $0 \in \mathbb{R}^m$, we must have

$$\frac{\partial \varphi_i}{\partial z_j} = 0, \text{ on } W, \text{ for } r+1 \leq i \leq n, r+1 \leq j \leq m. \tag{1}$$

We write $y = (y_1, \ldots, y_n) \in \mathbb{R}^n = \mathbb{R}^r \times \mathbb{R}^{n-r}$ as $y = (u, v)$. Consider now the map h defined in a neighbourhood of $0 \in \mathbb{R}^n$ by

$$(u, v) := (y_1, \ldots, y_n) \mapsto (u, y_{r+1} - \varphi_{r+1}(u, 0), \ldots, y_n - \varphi_n(u, 0)).$$

The Jacobian of h is of the form $\begin{pmatrix} I_r & 0 \\ * & I_{n-r} \end{pmatrix}$. Thus h gives rise to a system of local coordinates at $0 \in \mathbb{R}^n$. We also find that the map $h \circ f \circ g^{-1}$ is represented by

$$\begin{aligned} (z_1, \ldots, z_m) &\mapsto (z_1, \ldots, z_r, \varphi_{r+1}(z), \ldots, \varphi_n(z)) \\ &\mapsto (z_1, \ldots, z_r, \varphi_{r+1}(z) - \varphi_{r+1}(z_1, \ldots, z_r, 0, \ldots, 0), \ldots, \\ &\qquad \varphi_n(z) - \varphi_n(z_1, \ldots, z_r, 0, \ldots, 0)) \,. \end{aligned}$$

If we now restrict ourselves to a sufficiently small open ball around $0 \in \mathbb{R}^m$, then

$$\varphi_i(z_1, \ldots, z_n) - \varphi_i(z_1, \ldots, z_r, 0, \ldots, 0) = 0, \text{ for } r+1 \leq i \leq n,$$

in view of (1). Therefore we conclude the map $h \circ f \circ g^{-1}$ is represented on this neighbourhood by

$$(z_1, \ldots, z_m) \mapsto (z_1, \ldots, z_r, 0, \ldots, 0).$$

□

An immediate corollary is when f is of maximal rank at a point p, since in such a case, f is of constant rank in a neighbourhood of p. (Why?) Applying the rank theorem to this situation yields the following

Corollary 3. *Let $f \colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ be smooth. Assume that f is of maximal rank at $p \in U$.*

Case (1): *$Df(p)$ is one-one. Then there exist local coordinate systems around p and $q := f(p)$ such that the map f is represented as $(z_1, \ldots, z_m) \mapsto (z_1, \ldots, z_m, 0, \ldots, 0)$. That is, f looks like the restriction of canonical inclusion of $\mathbb{R}^m$ to $\mathbb{R}^n$ with respect to the new systems of coordinates.*

Case (2): *$Df(p)$ is onto. Then there exist local coordinate systems around p and $q := f(p)$ such that the map f is represented as $(z_1, \ldots, z_m) \mapsto (z_1, \ldots, z_n)$. That is, f looks like the restriction of canonical projection of $\mathbb{R}^m$ to $\mathbb{R}^n$ with respect to the new systems of coordinates. In particular, f is open on a neighbourhood of p.* □