Summary of Real Analysis — Batch B (2007–08)

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- **Basic LUB Property**

1. Sets bounded above and below, LUB Property of $\mathbb{R}$.
2. Archimedean property of $\mathbb{N}$. Two versions.
3. Density of rational numbers in $\mathbb{R}$. Density of irrational numbers in $\mathbb{R}$.
4. The non-existence of solutions of $X^2 = 2$ **in** $\mathbb{Q}$.
5. Existence and uniqueness of non-negative n-th roots of non-negative real numbers.
6. Nested interval theorem.

- **Sequences and their convergence**

1. Definition of sequences and their convergence. Importance of looking at the convergence definition geometrically.
2. Uniqueness of the limit.
3. If $x_n \to x$ and $x_n \geq 0$, then $x \geq 0$.
4. Sandwich Lemma. Let (x_n) , (y_n) and (z_n) be sequences such that (i) $x_n \to \alpha$ and $y_n \to \alpha$ and (ii) $x_n \leq z_n \leq y_n$. Then $z_n \to \alpha$.
5. Some examples of convergent sequences.
6. Bounded sequences; every convergent sequence is bounded.
7. We showed: The sequence $((-1)^n) = (-1, 1, -1, 1, \ldots)$ is bounded but not convergent.
8. Let (x_n) be such that $x_n \to x$. Assume that $x \neq 0$. Then there exists N such that for all $n \geq N$, we have $|x_n| \geq |x|/2$.

   We proved this in two ways. One geometric which looked at cases when x is positive and negative. The second one used triangle inequality.
9. Algebra of convergent sequences: Let $x_n \to x$, $y_n \to y$ and $\alpha \in \mathbb{R}$. Then
   (a) $x_n + y_n \to x + y$.
   (b) $\alpha x_n \to \alpha x$.
   (c) $x_n \cdot y_n \to xy$.

(d) $\frac{1}{x_n} \to \frac{1}{x}$ provided that $x \neq 0$. By Item 8, the terms $1/x_n$ make sense for all sufficiently large n.

10. Let (a_n) be bounded and (x_n) converge to 0. Then $a_n x_n \to 0$.
11. Let (x_n) be increasing. Then it is convergent iff it is bounded above.
12. Let (x_n) be decreasing. Then it is convergent iff it is bounded below.
13. Some Important Limits.
    (a) Let $0 \leq r < 1$ and $x_n := r^n$. Then $x_n \to 0$.
    (b) Let $x_n \to \ell$. Fix $N \in \mathbb{N}$. Define $y_n := x_n$ if $n > N$. Let y_k be any real number for $1 \leq k \leq N$. Then $y_n \to x$.
    (c) $x_n \to 0$ iff $|x_n| \to 0$.
    (d) Let $-1 < t < 1$. Then $t^n \to 0$.
    (e) Let $|r| < 1$. Then $nr^n \to 0$.
    (f) $n^{1/n} \to 1$.
    (g) Fix $a \in \mathbb{R}$. Then $\frac{a^n}{n!} \to 0$.
    (h) Let $a > 0$. Then $a^{1/n} \to 1$. Hint: f $a > 1$ then $1 \leq a^{1/n} \leq n^{1/n}$ for $n \geq a$.
14. Definition of divergence to ∞ (or to $-\infty$). We showed that $(n!)^{1/n}$ diverges to ∞.
15. Let $x_n \to 0$. Let (s_n) be the sequence of arithmetic means (or averages) defined by $s_n := \frac{x_1 + \cdots + x_n}{n}$. Then $s_n \to 0$.
16. Let $x_n \to x$. Then the sequence (s_n) of arithmetic means converges to x.
17. Let $a \in \mathbb{R}$. Consider $x_1 = a$, $x_2 = \frac{1+a}{2}$, and by induction $x_n := \frac{1+x_{n-1}}{2}$. Then $x_n \to 1$.
18. Definition of a subsequence. (Do you recall it?) Most important observation: $n_k \geq k$ for all k.
19. If $x_n \to x$, and if (x_{n_k}) is a subsequence, then $x_{n_k} \to x$ as $k \to \infty$.
20. We looked at the sequence $a^{1/n}$ again.
21. Given any sequence (x_n) there exists a monotone subsequence.
22. Bolzano-Weierstrass Theorem: If (x_n) is a bounded sequence, it has a convergent subsequence.
23. Definition of a Cauchy sequence of real numbers. Any Cauchy sequence is bounded.
24. Let (x_n) be Cauchy. Let a subsequence (x_{n_k}) converge to x. Then $x_n \to x$.
25. A real sequence (x_n) is Cauchy iff it is convergent.
26. Given any real number x there exist sequences (s_n) of rational numbers and (t_n) of irrational numbers such that $s_n \to x$ and $t_n \to x$.

- **Continuity**

1. Sequential definition of continuity.
2. Examples of continuous functions such as $f(x) = x^2$, $f(x) = 1/x$.
3. The characteristic function of $\mathbb{Q}$ defined by $f(x) = 1$ if $x \in \mathbb{Q}$ and 0 if $x \notin \mathbb{Q}$ is not continuous at any point of $\mathbb{R}$.

4. Algebra of continuous functions: Let $f, g\colon (a,b) \to \mathbb{R}$ be continuous at $c \in (a,b)$. Let $\alpha \in \mathbb{R}$. Then
   (a) $f + g$ is continuous at c.
   (b) αf is continuous at c. (In view of these two properties, the set of functions from $(a,b) \to \mathbb{R}$ continuous at c is a real vector space.)
   (c) The product fg is continuous at c.
   (d) If we further assume $f(c) \neq 0$, then $1/f$ is continuous at c.
   (e) $|f|$ is continuous at c.
   (f) Let $h(x) := \max\{f(x), g(x)\}$. Then h is continuous at c. Similarly, the function $k(x) := \min\{f(x), g(x)\}$ is continuous at c. *Hint:* Observe that for any two real numbers $\max\{a,b\} = [(a+b)+|a-b|]/2$ and $\min\{a,b\} = [(a+b)-|a-b|]/2$.
   (g) Let $f\colon (a,b) \to \mathbb{R}$ be continuous at c. Assume that $f(c) \in (\alpha, \beta)$ and that $g\colon (\alpha,\beta) \to \mathbb{R}$ is continuous at $f(c)$. Then the composition $g \circ f$ is continuous at c.
5. Sequential definition of continuity is equivalent to the ε-δ definition of continuity.
6. Some examples to work with ε-δ definition: $f(x) = x^n$, $g(x) = 1/x$ for $x > 0$ and $h(x) = 1/x$ for $x \geq 1$.
7. Let $f(x) := x$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. Then f is continuous only at $x = 0$.
8. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Assume that $f(r) = 0$ for $r \in \mathbb{Q}$. Then $f = 0$.
9. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be continuous. If $f(x) = g(x)$ for $x \in \mathbb{Q}$, then $f = g$.
10. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous which is also an additive homomorphism, that is, $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Then $f(x) = \lambda x$ where $\lambda = f(1)$.
11. Consider $f\colon (0,1) \to \mathbb{R}$ defined by $f(x) = 1/q$ if $x = p/q$ in reduced form and $f(x) = 0$ if $x \notin \mathbb{Q}$. Then f is continuous only at the irrationals.
12. Let $f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Show that f is continuous at 0.
13. Let $f\colon (a,b) \to \mathbb{R}$ be continuous at c with $f(c) \neq 0$. Then there exists $\delta > 0$ such that $f(x) > |f(c)|/2$ for all $x \in (c - \alpha, c + \delta)$.
14. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x - [x]$, where $[x]$ stands for the greatest integer less than or equal to x. At what points f is continuous? *Hint:* Draw a picture.
15. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \min\{x - [x], 1 + [x] - x\}$, that is, the minimum of the distances of x from $[x]$ and $[x] + 1$. At what points f is continuous? *Hint:* Draw a picture.
16. If $A \subset \mathbb{R}$ is a nonempty subset, define $f(x) := \inf\{|x - a| : a \in A\}$. Then f is continuous.

- **Two important Results:**

  1. **Intermediate Value Theorem.** Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function such that $f(a) < 0 < f(b)$. Then there exists $c \in (a,b)$ such that $f(c) = 0$.

     We gave two proofs of this result. One used the nested interval theorem and the other LUB property.

Let $g\colon [a, b] \to \mathbb{R}$ be a continuous function. Let λ be a real number between $g(a)$ and $g(b)$. Then there exists $c \in (a, b)$ such that $g(c) = \lambda$.

2. **Weierstrass Theorem.** Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function. Then f is bounded. In fact, there exists $x_1, x_2 \in [a, b]$ such that $f(x_1) \le f(x) \le f(x_2)$ for all $x \in [a, b]$. (In other words, a continuous function f on a closed and bounded interval is bounded and attains its maximum and minimum.)

   We proved the boundedness of f in three way: (i) Sequences and Bolzano-Weiestrass theorem (ii) LUB Property and (iii) Nested interval theorem.

- **Applications of the two important results.**

  1. Let $f\colon [a, b] \to [a, b]$ be continuous. Then there exists $x \in [a, b]$ such that $f(x) = x$.
  2. Prove that $x = \cos x$ for some $x \in (0, \pi/2)$.
  3. Prove that $xe^x = 1$ for some $x \in (0, 1)$.
  4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous taking values in $\mathbb{Q}$. Then f is a constant.
  5. Let $f\colon [a, b] \to \mathbb{R}$ be a nonconstant continuous function. Show that $f([a, b])$ is uncountable.
  6. Let $f\colon [0, 1] \to \mathbb{R}$ be continuous. Assume that the image of f lies in $[1, 2] \cup (5, 10)$ and that $f(1/2) \in [0, 1]$. What can you conclude about the image of f?
  7. Existence of n-th roots: Let $\alpha \ge 0$ and $n \in \mathbb{N}$ be given. Then there exists $x \ge 0$ such that $x^n = \alpha$.
  8. Let $f\colon [0, 2\pi] \to [0, 2\pi]$ be continuous such that $f(0) = f(2\pi)$. Show that there exists $x \in [0, 2\pi]$ such that $f(x) = f(x + \pi)$.
  9. Let $p(X)$ be an odd degree polynomial with real coefficients. Then p has a real root.
  10. Let p be a real polynomial function of odd degree. Show that $p\colon \mathbb{R} \to \mathbb{R}$ is onto.
  11. Show that $x^4 + 5x^3 - 7$ has two real roots.
  12. Let $p(X) := a_0 + a_1X + \cdots + a_nX^n$. If $a_0a_N < 0$, show that p has at least two real roots.
  13. Let J be an interval and $f\colon J \to \mathbb{R}$ be continuous and 1-1. Then f is strictly monotone.
  14. Let I be an interval and $f\colon I \to \mathbb{R}$ be strictly monotone. If $f(I)$ is an interval, show that f is continuous.
  15. Use the last item to conclude that the function $x \mapsto x^{1/n}$ from $[0, \infty) \to [0, \infty)$ is continuous.
  16. Let $f\colon [a, b] \to \mathbb{R}$ be continuous. Show that $f([a, b]) = [c, d]$. Can you "identify" c, d?
  17. Does there exists a continuous function $f\colon [0, 1] \to (0, \infty)$ which is onto?
  18. Let $f\colon [a, b] \to \mathbb{R}$ be continuous such that $f(x) > 0$ for all $x \in [a, b]$. Show that there exists δ such that $f(x) > \delta$ for all $x \in [a, b]$.
  19. Does there exists a continuous function $f\colon [a, b] \to (0, 1)$ which is onto?

20. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Assume that $f(x) \to 0$ as $|x| \to \infty$. (Do you understand this?) Show that there exists $c \in \mathbb{R}$ such that either $f(x) \leq f(c)$ or $f(x) \geq f(c)$ for all $x \in \mathbb{R}$. Give an example of a function in which only one of these happens.
21. Are there continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x) \notin \mathbb{Q} \text{ for } x \in \mathbb{Q} \text{ and } f(x) \in \mathbb{Q} \text{ for } x \notin \mathbb{Q}?$$

22. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that (i) $f(\mathbb{R}) \subset (-2,-1) \cup [1,5)$ and (ii) $f(0) = e$. Can you give 'realistic bounds' for f?

- **Sequences in $\mathbb{R}^n$**

1. Review of Euclidean metric on $\mathbb{R}^n$.
2. Let $x_k := (x_{k1}, \ldots, x_{kn}) \in \mathbb{R}^n$ be a sequence in $\mathbb{R}^n$. Then x_k converges to $x = (x_1, \ldots, x_n)$ iff $x_{kj} \to x_j$ for $1 \leq j \leq n$,
3. The limit of a convergenct sequence is unique.
4. Any converegence sequnce is bounded.
5. A sequnece (x_k) in $\mathbb{R}^n$ is convergent iff it is a Cauchy sequence.
6. Bolzano-Weierstrass theroem for bounded sequences in $\mathbb{R}^n$.
7. Examples: $(n^{1/n}, (-1)^n) \in \mathbb{R}^2$, $(a^{1/n}, \sin(1/n)) \in \mathbb{R}^2$ for some $a > 0$ etc.
8. Let $x_k \to x$ and $y_k \to y$ in $\mathbb{R}^n$. Then
   (a) $x_k + y_k \to x + y$ in $\mathbb{R}^n$.
   (b) $x_k \cdot y_k \to x \cdot y$. (Here $u \cdot v$ stands for the standard dot product of vectors u and v in $\mathbb{R}^n$.

- **Continuity**

1. Definition of open sets in $\mathbb{R}^n$.
2. Equivalent conditions of continuity of a function $f\colon X \to Y$ at a point, where X, Y are metric spaces.
3. Examples of continuous functions:
   (a) The projections $x \mapsto x_i$ are continuous.
   (b) Algebra of real valued continuous functions.
   (c) Any polynomial function from $\mathbb{R}^n$ to $\mathbb{R}$ is continuous.
4. The set $U := \{(x,y) \in \mathbb{R}^2 : xy > 0\}$ is open in $\mathbb{R}^2$.
5. Let $F\colon \mathbb{R}^m \to \mathbb{R}^n$ be continuous. if $F(x) := (f_1(x), \ldots, f_n(x))$, then F is continuous iff each f_i is continuous.
6. **Theorem.** Let $K \subset \mathbb{R}^m$ be closed and bounded. Let $f\colon K \to \mathbb{R}^n$ be continuous. Then f is bounded. If $n = 1$, then there exist points $x_1, x_2 \in K$ such that $f_1(x) \leq f(x) \leq f_2(x)$ for all $x \in K$.

- **Uniform Continuity**

1. Any linear map $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is continuous. A key step was: There exists $C > 0$ such that $\| f(x) \| \leq C \| x \|$ for all $x \in \mathbb{R}^n$. We concluded that f is in fact uniformly continuous.

2. Lipschitz maps between metric spaces.
   (a) Any Lipschitz map is uniformly continuous.
   (b) Any linear map $f\colon \mathbb{R}^m \to \mathbb{R}^n$ is Lipchitz.
   (c) Let $J \subset \mathbb{R}$ be an interval. Let $f\colon J \to \mathbb{R}$ be differentiable with bounded derivative, that is, there exists $M > 0$ such that $|f'(x)| \leq M$ for all $x \in J$. Then f is Lipschitz.
      - The sine function $\sin\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz.
      - The inverse of tan $\tan^{-1}\colon (-\pi/2, \pi/2) \to \mathbb{R}$ is Lipschitz.

3. Examples of uniformly continuous functions:
   (a) The identity function on any metric space is uniformly continuous.
   (b) Let $a > 0$. The function $f\colon (a, \infty) \to \mathbb{R}$ defined by $f(x) = 1/x$ is uniformly continuous on (a, ∞). In fact, it is Lipschitz.
   (c) The function $f\colon (0, 1) \to (1, \infty)$ given by $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$.
   (d) The function $f\colon (a, b) \to \mathbb{R}$ given by $f(x) = x^2$ is Lipschitz and uniformly contunuous on (a, b).
   (e) But, the function $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.
   (f) Let $p(x) := \sum_{k=0}^{n} a_x x^k$ be a poynomial with real coefficients. Let $J \subset \mathbb{R}$ be any *bounded* interval. Consider p as a function on J. Then p is Lipschitz and hence uniformly continuous on J. *Hint:* p' is bounded on the closure of J.
   (g) Let $\emptyset \neq A \subset X$ be a metric space. Let $f(x) := d_A(x) \equiv d(x, A) := \inf\{d(x, a) : a \in A\}$. Then d_A is uniformly continuous on X.

4. The first serious application of the notion of uniform continuity in an elementary course in real analysis was in the proof of the Riemann integrability of a continuous function defined o a closed and bounded interval.

5. Let X and Y be metric spaces. Let $f\colon X \to Y$ be uniformly continuous. Then f carries Cauchy sequences to Cauchy sequences.

6. The function in Item 3e carries Cauchy sequences to Cauchy sequences, but is not uniformly continuous.

7. **Theorem.** Let $K \subset \mathbb{R}^m$ be a closed and bounded set. Let $f\colon K \to \mathbb{R}^n$ be continuous. Then f is uniformly continuous.

8. When we analyzed the proof of the theorem in Item 7, we found that the codmain could be any metric space. But the domain should be a metric space in which Bolzano-Weiesrtrass theorem must hold true.

9. We say that a metric space X is compact if evey sequence (x_n) in X has a subsequence (x_{n_k}) which converges to an $x \in X$.
   (a) Any closed and bounded subset of $\mathbb{R}^n$ is a metric space.
   (b) The metric spaces $\mathbb{R}^n$ are not compact. *Hint:* Consider the sequence (x_k) where $x_k = (k, \ldots, k)$.

(c) $\mathbb{R}$ with discrete metric is closed and bounded. But it is not compact.

10. Our understanding of the proof of the theorem in Item 7 allowed us to arrive at the following more general result:

    Let X be a compact metric space and Y be any metric space. Let $f\colon X \to Y$ be continuous. Then f is uniformly continuous.

11. We defined extensions of functions. The function $f\colon (0,1) \to (1,\infty)$ given by $f(x) = 1/x$ does not have an extension to $[0,1)$.

12. **Theorem.** Let X and Y be metric spaces. Let D be a dense subset of X. Assume that Y is a complete metric space. Let $f\colon D \to Y$ be uniformly continuus. Then f extends uniquely to a (uniformly) continuous function g on X.

Items 1–11 of Uniform Continuity were done on a marathon session on September 28, 2007.