An Outline of Real Analysis

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

This is essentially a compilation of the hand-outs given to the students of Real Analysis Course (1st Semester), at the Department of Mathematics and Statistics, during the year 2008-2009. Most of the material was done in about 55 hours. Those which are not covered in the course are marked with an asterisk.

Reference Books:

1. K.A. Ross, *Analysis – Theory of Calculus*, Springer India Edition.
2. Bartle and Sherbert, *Introduction to Real Analysis*, Wiley International Ed.
3. R.R. Goldberg, *Methods of Real Analysis*, Oxford-IBH Publishing Co.
4. Tom Apostol, *Mathematical Analysis*, Narosa Publishing House.
5. W. Rudin, *Principles of Mathematical Analysis*, Wiley International.

The first two are easy books for a beginner with a lot of graded problems. The third is also a good book for a beginner but it does not have enough problems. The fourth and the fifth are classics. The fourth, though no so easy as the first three, is a must for anybody who is seriously interested in Analysis. The last is terse and does not have enough exercises for a beginner in Analysis to practice.

Need to reorder the sections. For example, Riemann integral should follow immediately after infinite series, followed by uniform convergence, limsup liminf, Dedekind cuts and lastly Metric spaces.

My aim is to give very many concrete functions, sequences, series and restore 'Hard-analysis' by replacing the prevalent practice of 'soft analysis' courses.

An Important Note: I did not have a hand-written manuscript when I typed these notes. Also, I rarely proof-read. The best way you can repay me for using this set of notes is to bring typos/mistakes to my notice.

1 LUB property of $\mathbb{R}$ and its consequences

1. Field of real numbers; We reviewed the order relations and the standard results on the manipulation of inequalities.

2. We say that a real number α is an upper bound of A if, for each $x \in A$, we have $x \leq \alpha$. (Geometrically, this means that elements of A are to the left of α on the number line.)

   A real number β is *not* an upper bound of A if there exists at least one $x \in A$ such that $x > \alpha$.

   If α is an upper bound of A and $\alpha' > \alpha$, then α' is an upper bound of A.

3. Lower bounds of a nonempty subset of $\mathbb{R}$ are defined analogously.

   If α is a lower bound of A, where can you find elements of A in the number line with reference to α? When do you say a real number is not a lower bound of A?

4. There exists a lower bound for $\mathbb{N}$ in $\mathbb{R}$. Does there exist an upper bound for $\mathbb{N}$ in $\mathbb{R}$? The answer is 'No' and it requires a proof which involves the single most important property of $\mathbb{R}$. See Item 19

5. $\emptyset \neq A \subset \mathbb{R}$ is said to be *bounded above* in $\mathbb{R}$ if there exists $\alpha \in \mathbb{R}$ which is an upper bound of A, that is, if there exists $\alpha \in \mathbb{R}$ such that for each $x \in A$, we have $x \leq \alpha$.

6. Subsets of $\mathbb{R}$ bounded below are defined analogously.

7. A is *not* bounded above in $\mathbb{R}$ if for each $\alpha \in \mathbb{R}$, there exists $x \in A$ (which depends on α) such that $x > \alpha$. Can you visualize this in number line?

8. When do you say $A \subset \mathbb{R}$ is not bounded below in $\mathbb{R}$?

9. Exercise:

   (a) If $\emptyset \neq A \subset \mathbb{R}$ is finite then an upper bound of A belongs to A.

   (b) Let α be an upper bound of $A \subset \mathbb{R}$. If $\alpha \in A$, then $\alpha = \text{l.u.b. } A$.

   (c) Any lower bound of a nonempty subset A of $\mathbb{R}$ is less than or equal to an upper bound of A.

10. Let $\emptyset \neq A \subset \mathbb{R}$ be bounded above. A real number $\alpha \in \mathbb{R}$ is said to be a *least upper bound* A if (i) α is an upper bound of A and (ii) if β is an upper bound of A, then $\alpha \leq \beta$.

11. If α and β are least upper bounds of A, then $\alpha = \beta$, that is, least upper bound of a (nonempty) subset (bounded above) is unique. We denote it by l.u.b. A.

12. $\alpha \in \mathbb{R}$ is the least upper bound of A iff (i) α is an upper bound of A and (ii) if $\beta < \alpha$, then β is not an upper bound of A, that is, if $\beta < \alpha$, then there exists $x \in A$ such that $x > \beta$.

13. If an upper bound α of A belongs to A, then l.u.b. $A = \alpha$.

14. A greatest lower bound of a subset of $\mathbb{R}$ bounded below in $\mathbb{R}$ is defined analogously.

15. What are the results for glb's analogous to those in Items 11–13?

16. Let $A = (0, 1) := \{x \in \mathbb{R} : 0 < x < 1\}$. Then l.u.b. $A = 1$ and g.l.b. $A = 0$. To prove the first observe that if $0 < \beta < 1$, then $(1 + \beta)/2 \in A$.

17. **The LUB property of $\mathbb{R}$:** Given any nonempty subset of $\mathbb{R}$ which is bounded above, there exists $\alpha \in \mathbb{R}$ such that $\alpha =$ l.u.b. A.

    Thus, any subset of $\mathbb{R}$ which has an upper bound in $\mathbb{R}$ has the lub in $\mathbb{R}$.

    Note that l.u.b. A need not be in A.

    The LUP property of $\mathbb{R}$ is the single most important property of the real number system and all key results in real analysis depend on it. It is also known as the Order-completeness of $\mathbb{R}$.

18. The field $\mathbb{Q}$ though is an ordered field does not enjoy the lub property. We shall see later that the subset $\{x \in \mathbb{Q} : x^2 < 2\}$ is bounded above in $\mathbb{Q}$ and does not have an lub *in* $\mathbb{Q}$.

19. As a first application of the LUB property, we established the Archimedean property (AP1) of $\mathbb{N}$: $\mathbb{N}$ is *not* bounded above in $\mathbb{R}$. That is, given any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x > n$.

    Sketch of proof: Proof by contradiction. Assume that $\mathbb{N}$ is bounded above in $\mathbb{R}$. Using LUB property, let $\alpha \in \mathbb{R}$ be l.u.b. $\mathbb{N}$. Then for each $k \in \mathbb{N}$, we have $k \leq \alpha$. For each $k \in \mathbb{N}$, $k + 1 \in \mathbb{N}$. Hence for each $k \in \mathbb{N}$, we have $k + 1 \leq \alpha$, so that for each $k \in \mathbb{N}$, we have $k \leq \alpha - 1$. That is, $\alpha - 1$ is an upper bound for $\mathbb{N}$. Now complete the proof.

20. 2nd Version of the Archimedean Property (AP2) of $\mathbb{N}$: Given $x, y \in \mathbb{R}$ with $x > 0$, there exists $n \in \mathbb{N}$ such that $nx > y$. (This version is the basis of all units and measurements!)

    Sketch of a proof: Proof by contradiction. If false, then for each $n \in \mathbb{N}$, we must have $nx \leq y$ so that $n \leq y/x$. That is, y/x is an upper bound for $\mathbb{N}$.

21. In fact, both the Archimedean principles are equivalent. We now show that AP2 implies AP1.

    Enough to show that no $\alpha \in \mathbb{R}$ is an upper bound of $\mathbb{N}$. Given α, let $x = 1$ and $y = \alpha$. Then by AP2, there exists $n \in \mathbb{N}$ such that $nx > y$, that is, $n > \alpha$.

22. Given $x > 0$, there exists $n \in \mathbb{N}$ such that $x > 1/n$.

23. Use Item 22 to show: if $x \geq 0$, then $x = 0$ iff $x \leq 1/n$ for each $n \in \mathbb{N}$. Typical use in Analysis: when we want to show two real numbers a, b are equal, we show that $|a - b| \leq 1/n$ for all $n \in \mathbb{N}$.

24. Exercise:

    (a) Show that, for $a, b \in \mathbb{R}$, $a \leq b$ iff $a \leq b + \varepsilon$ for all $\varepsilon > 0$.

    (b) Prove by induction that $2^n > n$ for all $n \in \mathbb{N}$. Hence conclude that for any given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $2^{-n} < \varepsilon$.

    (c) Show that $\mathbb{Z}$ is neither bounded above nor bounded below.

25. We proved the existence of the greatest integer function:
Let $x \in \mathbb{R}$. There exists a unique $m \in \mathbb{Z}$ such that $m \leq x < m+1$. This unique integer m is denoted by $[x]$ and called the greatest integer less than or equal to x.

Sketch of a proof: Consider the set $S := \{k \in \mathbb{Z} : k \leq x\}$. Then S is nonempty (why?) and bounded above by x. If α is the lub of S, then there exists $k \in S$ such that $\alpha - 1 < k$. Then $k+1 > x$, for otherwise, $k+1 \in S$ and hence $k+1 \leq \alpha$, a contradiction. Then $k \leq \alpha \leq x < k+1$. k is as required. Why is it unique?

Proposition 1 (Greatest Integer Function). *Let $x \in \mathbb{R}$. Then there exists a unique $m \in \mathbb{Z}$ such that $m \leq x < m+1$.*

Proof. Let $S := \{k \in \mathbb{Z} : k \leq x\}$. We claim that $S \neq \emptyset$. For, otherwise, for each $k \in \mathbb{Z}$, we must have $k > x$. It follows that for each $k \in \mathbb{Z}$, $-k < -x$. As k varies over $\mathbb{Z}$, $-k$ also varies over $\mathbb{Z}$. Hence $-x$ is an upper bound for $\mathbb{Z}$. In particular, since $\mathbb{N} \subset \mathbb{Z}$, $\mathbb{N}$ is bounded above by $-x$. This contradicts the Archimedean property. We therefore conclude that $S \neq \emptyset$.

S is bounded above by x. Let $\alpha \in \mathbb{R}$ be its least upper bound. Then there exists $k \in S$ such that $k > \alpha - 1$. Since $k \in S$, $k \leq x$. We claim that $k+1 > x$. For, if false, then $k+1 \leq x$. Therefore, $k+1 \in S$. Since α is an upper bound for S, we must have $k+1 < \alpha$ or $k < \alpha - 1$. This contradicts our choice of k. Hence we have $x < k+1$. The proposition follows if we take $m = k$.

m is unique. Let n also satisfy $n \leq x < n+1$. If $m \neq n$, without loss of generality, assume that $m < n$ so that $n \geq m+1$. (Why?) Since $m \leq x < m+1$ holds, we deduce that $m \leq x < m+1 \leq n$. In particular, $n > x$, a contradiction to our assumption that $n \leq x < n+1$. ☐

26. Exercise: Show that any nonempty subset of $\mathbb{Z}$ which is bounded above in $\mathbb{R}$ has a maximum. *Hint:* Go through the last proof carefully. The integer k of that proof is the maximum of S.

Formulate an analogue for the case of subsets of $\mathbb{Z}$ bounded below in $\mathbb{R}$ and prove it.

27. **Density of $\mathbb{Q}$ in $\mathbb{R}$.**

Theorem 2. *Given $a, b \in \mathbb{R}$ with $a < b$, there exists $r \in \mathbb{Q}$ such that $a < x < b$.*

Sketch of a proof: Assuming the existence of such an r, we write it as $r = m/n$ with $n > 0$. So, we have $x < m/n < y$, that is, $nx < y < ny$. Thus we are claiming that the interval $[nx, ny]$ contains an integer. It is geometrically obvious that a sufficient condition for an interval $J = [\alpha, \beta]$ to have an integer in it is that its length $\beta - \alpha$ should be greater than 1. This gives us an idea how to look for an n. The n we are looking for is the one next to $[\alpha]$. We start with the proof.

Use AP2 to choose $n \in \mathbb{N}$ such that $n(b-a) > 1$. Let $k = [na]$ and $m := k+1$. Then $na < m < nb$. Why? If $k+1 = m \geq nb$, then

$$1 = (k+1) - k \geq nb - na = n(b-a) > 1, \text{ a contradiction.}$$

28. As a corollary we prove the density of irrationals: Given $a, b \in \mathbb{R}$ with $a < b$, there exists $t \notin \mathbb{Q}$ such that $a < t < b$.

Sketch of a proof: Consider the real numbers $a - \sqrt{2} < b - \sqrt{2}$ and apply the last result.

29. Exercises:

(a) Let $a \in \mathbb{R}$. Let $C_a := \{r \in \mathbb{Q} : r < a\}$. Show that l.u.b. $C_a = a$. Is the map $a \mapsto C_a$ of $\mathbb{R}$ into the power set $P(\mathbb{R})$ one-one?

(b) Given any open interval (a, b), $a < b$, show that the set $(a, b) \cap \mathbb{Q}$ is infinite.

(c) Let $t > 0$ and $a < b$ be real numbers. Show that there exists $r \in \mathbb{Q}$ such that $a < tr < b$.

30. We recalled the proof of the irrationality of $\sqrt{2}$. The proof carried over to show that $\sqrt{3}$ is irrational. Finally, we saw that the same argument established the irrationality of $\sqrt{p}$ for any prime p.

31. We used the fundamental theorem of arithmetic to prove $\sqrt{2}$ is irrational and extended the argument to show that $\sqrt{n}$ is irrational where n is not a square (that is, $n \neq m^2$, for any integer m).

32. Thus there exists no solution **in** $\mathbb{Q}$ to the equations $X^2 = n$ where n is not a square. We contrast this with the next result which says that for any positive real number and a positive integer n, n-th roots exists in $\mathbb{R}$.

33. **Existence of n-th roots of positive real numbers.**

Theorem 3. *Let $\alpha \in \mathbb{R}$ be nonnegative and $n \in \mathbb{N}$. The there exists a unique nonnegative $x \in \mathbb{R}$ such that $x^n = \alpha$.*

Proof. The crucial part of the theorem is the existence of such an x. Uniqueness holds even in any ordered field. If $\alpha = 0$, the result is obvious, so we assume that $\alpha > 0$ in the following.

Draw the graph of $y = x^n$. Keep looking at it through the proof. We define

$$S := \{t \in \mathbb{R} : t \geq 0 \text{ and } t^n \leq \alpha\}.$$

Since $0 \in S$, we see that S is not empty. It is bounded above. For, by Archimedean property of $\mathbb{R}$, we can find $N \in \mathbb{N}$ such that $N > \alpha$. We claim that α is an upper bound for S. If this is false, then there exists $t \in S$ such that $t > N$. But, then we have

$$t^n > N^n \geq N > \alpha,$$

a contradiction, since for any $t \in S$, we have $t^n \leq \alpha$. Hence w conclude that N is an upper bound for S. Thus, S is a nonempty subset of $\mathbb{R}$ which is bounded above. By the LUB property of $\mathbb{R}$, there exists $x \in \mathbb{R}$ such that x is the LUB of S. We claim that $x^n = \alpha$.

Exactly one of the following is true: (i) $x^n < \alpha$, (ii) $x^n > \alpha$ and (iii) $x^n = \alpha$. We shall show that the first two possibilities do not arise. The idea is as follows. Look at

Figure again. If Case (i) holds, that is, if $x^n < \alpha$, then it is geometrically clear that for y very near to x and greater than x, we must have $y^n < \alpha$. In particular, we can find a positive integer $k \in \mathbb{N}$ such that $(x + 1/k)^n < \alpha$. It follows that $x + 1/k \in S$. This is a contradiction, since x is supposed to be an upper bound for S. In the second case, when $x^n > \alpha$, by similar considerations, we can find $k \in \mathbb{N}$ such that $(x - 1/k)^n > \alpha$. Since $x - 1/k < x$ and x is the least upper bound for S, there exists $t \in S$ such that $t > x - 1/k$. We then see

$$t^n > (x - 1/k)^n > \alpha.$$

This again leads to a contradiction, since $t \in S$.

So, to complete the proof rigorously, we need only prove the existence of a positive integer k in each of the first two cases.

Case (i): Assume that $x^n < \alpha$. For any $k \in \mathbb{N}$, we have

$$\begin{aligned}
(x + 1/k)^n &= x^n + \sum_{j=1}^{n} \binom{n}{j} x^{n-j}(1/k^j) \\
&\leq x^n + \sum_{j=1}^{n} \binom{n}{j} x^{n-j}(1/k) \\
&= x^n + C/k, \quad \text{where } C := \sum_{j=1}^{n} \binom{n}{j} x^{n-j}.
\end{aligned}$$

If we choose k such that $x^n + C/k < \alpha$, that is, for $k > C/(\alpha - x^n)$, it follows that $(x + 1/k)^n < \alpha$.

Case (ii): Assume that $x^n > \alpha$. For any $k \in \mathbb{N}$, we have $(-1)^j(1/k^j) > -1/k$ for $j \geq 1$. We use this below.

$$\begin{aligned}
(x - 1/k)^n &= x^n + \sum_{j=1}^{n} \binom{n}{j} (-1)^j x^{n-j}(1/k^j) \\
&\geq x^n - \sum_{j=1}^{n} \binom{n}{j} x^{n-j}(1/k) \\
&= x^n - C/k, \quad \text{where } C := \sum_{j=1}^{n} \binom{n}{j} x^{n-j}.
\end{aligned}$$

If we choose k such that $x^n - C/k > \alpha$, that is, if we take $k > C/(x^n - \alpha)$, it follows that $(x - 1/k)^n > \alpha$.

We now show that if x and y are non-negative real numbers such that $x^n = y^n = \alpha$, then $x = y$. Look at the following algebraic identity:

$$(x^n - y^n) \equiv (x - y) \cdot [x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}].$$

If x and y are nonnegative with $x^n = y^n$ and if $x \neq y$, say, $x > y$ then the left hand side is zero while both the factors in brackets on the right are strictly positive, a contradiction.

This completes the proof of the theorem. □

34. How to get the algebraic identity $(x^n - y^n) \equiv (x-y)\cdot[x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}]$? Recall how to sum a geometric series:$s_n := 1 + t + t^2 + \cdots + t^{n-1}$. Multiply both sides by t to get $ts_n = t + \cdots + t^n$. Subtract one from the other and get the formula for s_n. In the formula for s_n substitute y/x for t and simplify.

35. Observe that the argument of the proof in Item 33 can be applied to the set $A := \{x \in \mathbb{Q} : x^2 < 2\} \subset \mathbb{Q}$ (which is bounded above in $\mathbb{Q}$ by 2) to conclude that if $x \in \mathbb{Q}$ is the lub of A, then $x^2 = 2$. (In the quoted proof, the numbers $x \pm \frac{1}{k} \in \mathbb{Q}$!) This contradicts Item 30-32. Hence we conclude that the ordered field $\mathbb{Q}$ does not enjoy the LUB property.

36. Let $[c, d] \subset [a, b]$ be intervals. Then we have $a \le c \le d \le b$.

37. **Nested Interval Theorem.**

Theorem 4. *Let $J_n := [a_n, b_n]$ be intervals in $\mathbb{R}$ such that $J_{n+1} \subseteq J_n$ for all $n \in \mathbb{N}$. Then $\cap J_n \neq \emptyset$.*

Proof. Let A be the set of left endpoints of J_n. Thus, $A := \{a \in \mathbb{R} : a = a_n \text{ for some } n\}$. A is nonempty.

We claim that b_k is an upper bound for A for each $k \in \mathbb{N}$, i.e., $a_n \le b_k$ for all n and k. If $k \le n$ then $[a_n, b_n] \subseteq [a_k, b_k]$ and hence $a_n \le b_n \le b_k$. (Draw pictures!) If $k > n$ then $a_n \le a_k \le b_k$. Thus the claim is proved. By the LUB axiom there exists $c \in \mathbb{R}$ such that $c = \sup A$. We claim that $c \in J_n$ for all n. Since c is an upper bound for A we have $a_n \le c$ for all n. Since each b_n is an upper bound for A and c is the least upper bound for A we see that $c \le b_n$. Thus we conclude that $a_n \le c \le b_n$ or $c \in J_n$ for all n. Hence $c \in \cap J_n$. □

38. Exercises on LUB and GLB properties of $\mathbb{R}$.

    (a) What can you say about A if l.u.b. $A =$ g.l.b. A?

    (b) Prove that $\alpha \in \mathbb{R}$ is the lub of A iff (i) α is an upper bound of A and (ii) for any $\varepsilon > 0$, there exists $x \in A$ such that $x > \alpha - \varepsilon$.

    Formulate an analogue for glb.

    (c) Let A, B be nonempty subsets of $\mathbb{R}$ with $A \subset B$. Prove

$$\text{g.l.b. } B \le \text{g.l.b. } A \le \text{l.u.b. } A \le \text{l.u.b. } B.$$

    (d) Let A, B be nonempty subsets of $\mathbb{R}$. Assume that $a \le b$ for all $a \in A$ and $b \in B$. Show that l.u.b. $A \le$ g.l.b. B.

    (e) Let $A, B \subset \mathbb{R}$ be bounded above. Find a relation between l.u.b. $(A \cup B)$, l.u.b. A and l.u.b. B.

    Analogous question when the sets are bounded below.

    (f) For $A \subset \mathbb{R}$, we define $-A := \{y \in \mathbb{R} : \exists x \in A \text{ such that } y = -x\}$, that is, $-A$ is the set of all negatives of elements of A.

        i. Let $A = \mathbb{Z}$. What is $-A$?

        ii. If $A = [-1, 2]$ what is $-A$?

Assume that A is bounded above and $\alpha :=$ l.u.b. A. Show that $-A$ is bounded below and that g.l.b. $(-A) = -\alpha$.
Can you formulate the analogous result (for $-B$) if $\beta =$ g.l.b. B?

(g) Formulate the GLB property of $\mathbb{R}$ (in a way analogous to the LUB property of $\mathbb{R}$).

(h) Show that LUB property holds iff the GLB property holds true in $\mathbb{R}$.

(i) Let $A, B \subset \mathbb{R}$ be nonempty. Define

$$A + B := \{x \in \mathbb{R} : \exists\, (a \in A, b \in B) \text{ such that } x = a + b\} = \{a + b : a \in A, b \in B\}.$$

  i. Let $A = [-1, 2] = B$. What is $A + B$?
  ii. Let $A = B = \mathbb{N}$. What is $A + B$?
  iii. Let $A = B = \mathbb{Z}$. What is $A + B$?
  iv. Let $A = B = \mathbb{Q}$. What is $A + B$?
  v. Let $A = B$ be the set of all irrational numbers. What is $A + B$?

(j) Let $\alpha =$ l.u.b. A and $\beta =$ l.u.b. B. Show that $A + B$ is bounded above and that l.u.b. $(A + B) = \alpha + \beta$.

(k) Let A, B be nonempty subsets of positive real numbers. Let $\alpha :=$ l.u.b. A and $\beta :=$ l.u.b. B. Define $A \cdot B := \{ab : a \in b \in B\}$. Show that l.u.b. $(A \cdot B) = \alpha \cdot \beta$.

(l) Let $\alpha :=$ l.u.b. A. Let $b \in \mathbb{R}$. Let $b + A := \{b + a : a \in A\}$. Find l.u.b. $(b + A)$.

(m) Let $\alpha :=$ l.u.b. A. Let $b \in \mathbb{R}$ be positive. Let $bA := \{ba : a \in A\}$. Find l.u.b. (bA). Investigate what result is possible result when $b < 0$.

(n) Let $A \subset \mathbb{R}$ with g.l.b. $A > 0$. Let $B := \{x^{-1} : x \in A\}$. Show that B is bounded above and relate its lub with the glb of A.

(o) Let $\emptyset \neq A \subset \mathbb{R}$ be bounded above in $\mathbb{R}$. Let B be the set of upper bounds of A. Show that B is bounded below and that l.u.b. $A =$ g.l.b. B.

(p) Show that g.l.b. $\{1/n : n \in \mathbb{N}\} = 0$.

(q) Show that l.u.b. $\{1 - \frac{1}{n^2} : n \in \mathbb{N}\} = 1$.

(r) Let $A := \{x \in \mathbb{R} : x^2 - 5x + 6 < 0\}$. Find the lub and glb of A.

(s) Find the glb of $\{x + x^{-1} : x > 0\}$. Is the set bounded above?

(t) Let $A := \{\frac{1}{3} \pm \frac{n}{3n+1} | n \in \mathbb{N}\}$. Show that l.u.b. $A = 2/3$ and g.l.b. $A = 0$.

(u) Find the glb and lub of the set of real numbers in $(0, 1)$ in whose decimal expansions only 0's and 1's appear.

(v) Let $x, y \in \mathbb{R}$ be such that $x \leq y + \frac{1}{n}$ for all $n \in \mathbb{N}$. Show that $x \leq y$.

(w) A subset $A \subset \mathbb{R}$ is said to be bounded in $\mathbb{R}$ iff it is both bounded above and bounded below. Show that A is bounded iff there exists $M > 0$ such that $-M \leq x \leq M$ for all $x \in A$, that is, A is bounded iff there exists $M > 0$ such that $|x| \leq M$ for all $x \in A$.

39. **Absolute value of a real number.** For $x \in \mathbb{R}$, we define

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \leq 0. \end{cases}$$

40. The following are easy to see.

    (a) $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$.
    (b) $|a|^2 = a^2$ for any $a \in \mathbb{R}$. In particular, $|x| = \sqrt{x^2}$, the unique nonnegative square root of x^2.
    (c) $\pm a \leq |a|$ for all $a \in \mathbb{R}$.
    (d) $-|a| \leq a \leq |a|$ for all $a \in \mathbb{R}$.
    (e) $|x| < \varepsilon$ iff $x \in (-\varepsilon, \varepsilon)$.
    (f) $|x - a| < \varepsilon$ iff $x \in (a - \varepsilon, a + \varepsilon)$.
    (g) **Triangle Inequality.** $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$. Equality holds iff both a and b are of the same side of 0.
    (h) $||a| - |b|| \leq |a - b|$.

41. $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$ and $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$ for any $a, b \in \mathbb{R}$.

    We also understood this expression in a geometric way.

42. To do analysis, one should learn to be comfortable in dealing with inequalities. We do the following as samples.

    (a) $\{x \in \mathbb{R} : |x - a| = |x - b|\}$ (where $a \neq b$) $= \{\frac{a+b}{2}\}$.
    (b) $\{x \in \mathbb{R} : \frac{x+2}{x-1} < 4\} = (-\infty, 1) \cup (2, \infty)$.
    (c) $\{x \in \mathbb{R} : |\frac{2x-3}{3x-2}| = 2\} = \{1/4, 7/8\}$.
    (d) $\{x \in \mathbb{R} : |\frac{3-2x}{2+x}| < 2\} = (-1/4, \infty)$.
    (e) $\{x \in \mathbb{R} : x^4 - 5x^2 + 4 < 0\} = (-2, -1) \cup (1, 2)$.

43. The main purpose of this exercise is to make you acquire confidence in dealing with inequalities.

    Exercise: Identify the following subsets of $\mathbb{R}$:

    (a) $\{x \in \mathbb{R} : |3x + 2| > 4|x - 1|\}$. Ans: $(2/7, 6)$
    (b) $\{x \in \mathbb{R} : |\frac{x}{x+1}| > \frac{x}{x+1}$ where $x \neq -1\}$. Ans: (-1,0)
    (c) $\{x \in \mathbb{R} : |\frac{x+1}{x+5}| < 1$ where $x \neq -5\}$. Ans: $(-3, \infty)$.
    (d) $\{x \in \mathbb{R} : x^2 > 3x + 4\}$. Ans: $(-\infty, -1) \cup (4, \infty)$.
    (e) $\{x \in \mathbb{R} : 1 < x^2 < 4\}$. Ans: $(-2, -1) \cup (1, 2)$
    (f) $\{x \in \mathbb{R} : 1/x < x\}$. Ans: $(-1, 0) \cup (1, \infty)$
    (g) $\{x \in \mathbb{R} : 1/x < x^2\}$. Ans: $(-\infty, 0) \cup (1, \infty)$
    (h) $\{x \in \mathbb{R} : |4x - 5| < 13\}$. Ans: $(-2, 9/2)$
    (i) $\{x \in \mathbb{R} : |x^2 - 1| < 3\}$. Ans: $|x| < 2$
    (j) $\{x \in \mathbb{R} : |x + 1| + |x - 2| = 7\}$. Ans: $x = -3$ or $x = 4$
    (k) $\{x \in \mathbb{R} : |x| + |x + 1| < 2\}$. Ans: $(-3/2, 1/2)$.
    (l) $\{x \in \mathbb{R} : 2x^2 + 5x + 2 > 0\}$. Ans: $\mathbb{R}$
    (m) $\{x \in \mathbb{R} : \frac{2x}{3} - \frac{x^2-3}{2x} + \frac{1}{2} < \frac{x}{6}\}$. Ans: (-3,0)

2 Sequences and their convergence

1. Let X be any set. A sequence in X is a function $f\colon \mathbb{N} \to X$. We let $x_n := f(n)$ and call x_n the n-th term of the sequence. One usually denotes f by (x_n).

2. Let (x_n) be a real sequence, that is, as sequence in $\mathbb{R}$. We say that (x_n) converges to $x \in \mathbb{R}$ if for any given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $x_n \in (x - \varepsilon, x + \varepsilon)$, that is, for $n \geq n_0$, we have $|x - x_n| < \varepsilon$. The number x is called a *limit* of the sequence (x_n). We then write $x_n \to x$. We also say that (x_n) is convergent to x. We write this as $\lim_n x_n = x$.

   One can similarly define that a sequence (z_n) of complex numbers converges to $z \in \mathbb{C}$ if for any given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have $|x - x_n| < \varepsilon$.

   Convention. If a sequence is not convergent, we also say that it is *divergent.*

3. Uniqueness of the limit: If a sequence (x_n) of real numbers converges to $x \in \mathbb{R}$ as well as to $y \in \mathbb{R}$, then $x = y$. We give 2 proofs.

   Proof 1: For any $k \in \mathbb{N}$, choose n_k and m_k such that

   $$n \geq n_k \implies |x - x_n| < 1/k \text{ and } n \geq m_k \implies |y - x_n| < 1/k.$$

   Let $n_0 := \max\{n_k, m_k\}$. If $n \geq n_0$, we have $|x - y| \leq |x - x_n| + |x_n - y| < 2/k$. $\square$

   Proof 2: If $x < y$, let $2\varepsilon = y - x$. Then $(x - \varepsilon, x + \varepsilon) \cap (y - \varepsilon, y + \varepsilon) = \emptyset$. Let n_1, n_2 be such that $n \geq n_1 \implies |x - x_n| < \varepsilon$ and $n \geq n_2 \implies |y - x_n| < \varepsilon$. Then for any $n \geq \max\{n_1, n_2\}$, we have $x_n \in (x - \varepsilon, x + \varepsilon) \cap (y - \varepsilon, y + \varepsilon)$. $\square$

   Both these proofs can be adapted to prove the uniqueness result for complex sequences.

4. Exercise: Prove that each of the following sequences (a_n) converges to a limit a by finding, for each given $\varepsilon > 0$, an $n_0 \in \mathbb{N}$ depending on ε such that $|a_n - a| < \varepsilon$ for $n > n_0$.

   (i) $a_n = 1/(n+1)$ (ii) $a_n = (n+1)/(2n+3)$
   (iii) $a_n = n/(n^2 - n + 1)$ (iv) $a_n = 1/2^n$
   (v) $a_n = 2/\sqrt{n}$ (vi) $a_n = \sqrt{n+1} - \sqrt{n}$

5. Let $x_n \to x$, $x_n, x \in \mathbb{C}$. Fix $N \in \mathbb{N}$. Define a sequence (y_n) such that $y_n := x_n$ if $n \geq N$ while y_k could be any real/complex number for $1 \leq k < N$. The $y_n \to x$. Thus, if we alter a finite number of terms of a convergent sequence, the new sequence still converges to the limit of the original sequence.

6. Let (x_n) be a sequence of real/complex numbers. Then $x_n \to 0$ iff $|x_n| \to 0$.

7. Let (x_n) be a sequence of real/complex numbers. $x_n \to x$ iff $x_n - x \to 0$ iff $|x_n - x| \to 0$.

8. Exercise: Let $0 \leq b_n \to 0$. Assume that $|a_n - a| \leq b_n$ for all large values of n. Prove that $a_n \to a$.

9. Let (x_n) be a sequence of real/complex numbers. If $x_n \to x$, then $|x_n| \to |x|$. However the converse is not true. The sequence $(x_n) := ((-1)^n)$ is not convergent while the sequence $(|x_n|)$, being a constant sequence, is convergent. To see the first result, if $x_n \to \ell$, observe that for any $\varepsilon > 0$, $\pm 1 \in (\ell - \varepsilon, \ell + \varepsilon)$. What if $\varepsilon \leq 1$?

10. n_0 of the definition of convergence depends on ε and is not unique. (If n_0 'works' and $n_1 \geq n_0$ also 'works!')

    If we consider $x_n := 1/(2n-7)$, we showed that we can take $n_0 > \frac{1}{2}\left(\frac{1}{\varepsilon}+7\right)$ as well as $n_0 > \frac{1}{2\varepsilon}+7$. Thus, each person may arrive at different n_0 depending upon the way he estimates the terms!

11. Let (x_n) be a sequence of real numbers. Let $x_n \to x > 0$. Then there exists $n_0 \in \mathbb{N}$ such that $x_n > x/2$ for $n \geq n_0$. What is the analogous result if $x < 0$? Can you think of a (single) formulation which encompasses both these results?

12. Let (x_n) be a complex sequence such that $x_n \to x$. Assume that $x \neq 0$. Then there exists N such that for all $n \geq N$, we have $|x_n| \geq |x|/2$.

    This is done in the last item in a geometric way for the case of real sequences. To do the general case, let n_0 correspond to $\varepsilon := |x|/2$. Then for $n \geq n_0$, we have

    $$|x| \leq |x - x_n| + |x_n| < \varepsilon + |x_n| \text{ so that } |x_n| > |x| - \varepsilon = \frac{|x|}{2}.$$

13. A sequence (x_n) of real/complex numbers is said to be bounded if there exists $C > 0$ such that $|x_n| < C$ for all $n \in \mathbb{N}$.

14. Every convergent sequence of real/complex numbers is bounded. Proof: Let n_0 correspond to $\varepsilon = 1$. Then

    $$|z_n| < |z - z_n| + |z| = 1 + |z| \text{ for } n \geq n_0.$$

    Then $C := \max\{|z_1|, \ldots, |z_{n_0-1}|, 1+|z|\}$ is as required. □

    But the converse is not true. Look at $((-1)^n) = (-1, 1, -1, 1, \ldots)$. See Item 9.

15. Algebra of convergent sequences in $\mathbb{C}$: Let $x_n \to x$, $y_n \to y$ and $\alpha \in \mathbb{C}$. Then

    (a) $x_n + y_n \to x + y$.

    (b) $\alpha x_n \to \alpha x$.

    (c) $x_n \cdot y_n \to xy$.

    (d) $\frac{1}{x_n} \to \frac{1}{x}$ provided that $x \neq 0$. By Item 12 the terms $1/x_n$ make sense for all sufficiently large n.

    We proved all these in the class. To give a taste of the proofs, let us look the product and the quotient.

    $$\begin{aligned} |x_n y_n - xy| &\leq |x_n y_n - x y_n| + |x y_n - xy| \\ &\leq |y_n||x_n - x| + |x||y_n - y| \\ &< C|x_n - x| + (1+|x|)|y_n - y| \text{ using Item 14.} \end{aligned}$$

    For the quotient, we need to estimate $|\frac{1}{x_n} - \frac{1}{x}|$. We have

    $$\begin{aligned} |\frac{1}{x_n} - \frac{1}{x}| &= |\frac{x - x_n}{x x_n}| \\ &= \frac{1}{|x_n|}\frac{1}{|x|}|x - x_n| \\ &\leq \frac{2}{|x|}\frac{1}{|x|}|x - x_n|, \text{ say, for } n \geq n_1, \end{aligned}$$

where we have used Item 14. Let n_2 be such that $|x - x_n| < \frac{\varepsilon|x|^2}{2}$ and take $n_0 = \max\{n_1, n_2\}$. $\square$

16. The set $\mathcal{C}$ of convergent sequences of real/complex numbers form a real/complex vector space under the operations: $(x_n) + (y_n) := (x_n + y_n)$ and $\alpha \cdot (x_n) := (\alpha x_n)$.

    Moreover, the map $(x_n) \mapsto \lim x_n$ from $\mathcal{C}$ to $\mathbb{R}$ (or to $\mathbb{C}$) is a linear transformation.

17. Exercise:

    (a) Given that $x_n \to 1$, identify the limits of the sequences whose n-th terms are (a) $1 - x_n$, (b) $2x_n + 5$, (c) $(4 + x_n^2)/x_n$.

    (b) Let (x_n) and (y_n) be convergent. Let $s_n := \min\{x_n, y_n\}$ and $t_n := \max\{x_n, y_n\}$. Are the sequences (s_n) and (t_n) convergent?

    (c) Show that the set of bounded (real) sequences form a real vector space.

    (d) True or False: If (x_n) and $(x_n y_n)$ are bounded, then (y_n) is bounded.

    (e) Let $x_n \geq 0$, $x_n \to x$. Prove that $\sqrt{x_n} \to \sqrt{x}$.

    (f) True or False: If (x_n) and (y_n) are sequences such that $x_n y_n \to 0$, then one of the sequences converges to 0.

    (g) Let (x_n) be a sequence. Prove that $x_n \to 0$ iff $x_n^2 \to 0$.

    (h) Let (x_n) and (y_n) be two real/complex sequences. Let (z_n) be a new sequence defined $(x_1, y_1, x_2, y_2, \ldots)$. (Can you write down explicit expression for z_n?) Show that (z_n) is convergent iff both the sequences converge to the same limit.

    (i) Let (x_n) be a sequence. Assume that $x_n \to 0$. Let $\sigma \colon \mathbb{N} \to \mathbb{N}$ be a bijection. Define a new sequence $y_n := x_{\sigma(n)}$. Show that $y_n \to 0$. *Hint:* Given ε, let n_0 be such that $|x_n| < \varepsilon$ for $n \geq n_0$. Let $N := \max\{\sigma^{-1}(1), \ldots, \sigma^{-1}(n_0)\}$. Estimate y_n for $n \geq N$.

    (j) Let $x_n := \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{n+1}\right)$. Show that (x_n) is convergent. *Hint:* What are x_1, x_2 etc?

18. Definition of a Cauchy sequence in $\mathbb{R}$ or $\mathbb{C}$. A sequence (x_n) (in $\mathbb{C}$) is said to be Cauchy if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$ we have $|x_n - x_m| < \varepsilon$.

    Example: Any convergent (real) sequence is Cauchy. (In fact, these are the only examples in $\mathbb{R}$! See the next item.)

19. **Cauchy Completeness of $\mathbb{R}$.** A real sequence (x_n) is Cauchy iff it is convergent.

    Let $E := \{x \in \mathbb{R} : \exists\, N \text{ such that } n \geq N \implies x < x_n\}$. Let $\delta > 0$ and let $n_0 = n_0(\delta)$ be such that
    $$n \geq n_0 \implies x_n \in (x_{n_0} - \delta, x_{n_0} + \delta). \tag{1}$$

    Claim 1. $x_{n_0} - \delta \in E$. For, if we take $N = n_0(\delta)$, then $n \geq n_0 \implies x_n > x_{n_0} - \delta$.

    Claim 2. $x_{n_0} + \delta$ is an upper bound of E. If not, let $x \in E$ be such that $x > x_{n_0} + \delta$. This means that there exists some N such that for all $n \geq N$ $x_n \geq x > x_{n_0} + \delta$. In particular, for all $n \geq \max\{n_0, N\}$, we have $x_n > x_{n_0} + \delta$. This contradicts (1).

    Let $\ell :=$ l.u.b. E. Claim 3: $x_n \to \ell$. Let $\varepsilon > 0$ be given. Let $n_0 = n_0(\varepsilon)$ correspond to ε. Then for all n
    $$|x_n - \ell| \leq |x_n - x_{n_0}| + |x_{n_0} - \ell|. \tag{2}$$

If $n \geq n_0$, then $x_{n_0} - \varepsilon \in E \implies x_{n_0} - \varepsilon \leq \ell$ and $\ell \leq x_{n_0} + \varepsilon$ by Claims 1 and 2, that is, $|x_{n_0} - \ell| \leq \varepsilon$. The first term of the LHS of (2) is less than ε thanks to Cauchy condition. Hence, for $n \geq n_0$, the RHS of (2) is at most 2ε. □

20. Any Cauchy sequence is bounded.

    This is obvious since any Cauchy sequence is convergent and convergent sequences are bounded. We give a direct proof. If (x_n) is Cauchy, let N correspond to $\varepsilon = 1$. Then for $n \geq N$, $|x_n| \leq |x_n - x_N| + |x_N| < 1 + |x_N|$. Let $C := \max\{|x_1|, \ldots, |x_{N-1}|, 1 + |x_N|\}$. Then $|x_n| \leq C$ for all n.

21. Exercise:

    (a) Prove that the sum of two Cauchy sequences and the product of two Cauchy sequences are Cauchy.

    (b) Let $|x_n| \leq \frac{1+n}{1+n+2n^2}$. Prove that (x_n) is Cauchy.

    (c) If (x_n) is a Cauchy sequences of integers, what can you say about the sequence?

    (d) Let (x_n) be a sequence and let $a > 1$. Assume that

    $$|x_{k+1} - x_k| < a^{-k} \text{ for all } k \in \mathbb{N}.$$

    Show that (x_n) is Cauchy.

22. We say a sequence (x_n) of real numbers is increasing if for each n, we have $x_n \leq x_{n+1}$. Clearly, any increasing sequence is bounded below. Hence such a sequence is bounded iff it is bounded above.

    Define decreasing sequences. When is it bounded?

23. Let (x_n) be increasing. Then it is convergent iff it is bounded above.

    Let $x(\mathbb{N}) := \{x_n : n \in \mathbb{N}\}$ be the image of the sequence x. Let ℓ be the lub of this set. Given $\varepsilon > 0$, since $\ell - \varepsilon$ is not an upper bound of $x(\mathbb{N})$. Let $x_N > \ell - \varepsilon$. Since the sequence is increasing, for all $n \geq N$, we have $x_N \leq x_n$ and hence $\ell - \varepsilon < x_N \leq x_n \leq \ell < \ell + \varepsilon$, that is, $x_n \to \ell$.

    What is the analogous result in the case of decreasing sequences?

24. A typical use: Let $0 \leq r < 1$. Let $x_n := r^n$. Then (r^n) is decreasing, bounded below; hence convergent, say, to ℓ. Then $rx_n \to r\ell$, but $rx_n \equiv x_{n+1}$ so that $rx_n \to \ell$. By uniqueness of the limits, $r\ell = \ell$. Conclude $\ell = 0$.

25. **The Number e.** We were lucky in the last example to find the limit explicitly. In general it may not be possible. In fact, some real numbers are defined as the limit of such sequences. For instance, consider $x_n := (1 + \frac{1}{n})^n$. Then the sequence (x_n) is increasing and bounded above. The real number which is the limit of this sequence is denoted by e.

    (a) By binomial theorem

    $$\begin{aligned} x_n &= 1 + \sum_{k=1}^{n} \frac{n!}{k!\,(n-k)!} n^{-k} \\ &= 1 + \sum_{k=1}^{n} \frac{1}{k!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{k-1}{n}). \end{aligned} \tag{3}$$

(b) Conclude from (3) that $x_n < x_{n+1}$.

$$\begin{aligned} x_n &= 1 + \sum_{k=1}^{n} \frac{1}{k!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{k-1}{n}) \\ &< 1 + \sum_{k=1}^{n} \frac{1}{k!}(1 - \frac{1}{n+1})(1 - \frac{2}{n+1}) \cdots (1 - \frac{k-1}{n+1}) \\ &< x_{n+1}. \end{aligned}$$

(c) From (3), conclude that $x_n \leq 1 + \sum_{k=1}^{n} \frac{1}{k!}$.

(d) $1 + \sum_{k=1}^{n} \frac{1}{k!} < 1 + 1 + \sum_{k=1}^{n+1} \frac{1}{2^k} = 1 + \frac{1-2^{-n}}{1-1/2} < 1 + \frac{1}{1/2} = 3$.

(e) Conclude that (x_n) is increasing and bounded above and hence $\lim x_n$ exists. Let $e = \lim x_n$.

(f) Let $y_n := \sum_{k=0}^{n} \frac{1}{k!}$. Here $0! = 1$. From 25d conclude that $\lim y_n$ exists. From 25c, we know that $x_n \leq y_n$ and hence $e = \lim x_n \leq \lim y_n$.

(g) For $n > m$ omitting terms for $k \geq m+1$ from (3) we get:

$$x_n \geq 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \cdots + \frac{1}{m!}(1 - \frac{1}{n}) \cdots (1 - \frac{m-1}{n}).$$

Fix m and let $n \to \infty$ to see that $e \geq y_m$. Now let $m \to \infty$ to see that $e \geq \lim y_m$.

(h) Hence $e := \lim_{n\to\infty}(1 + \frac{1}{n})^n = \lim_{n\to\infty}\left(\sum_{k=0}^{n} \frac{1}{k!}\right)$.

(i) e is irrational. If not, let $e = m/n$. We have

$$\begin{aligned} 0 \leq e - s_n &= \sum_{k=n+1}^{\infty} \frac{1}{k!} \\ &\leq \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right) s = \frac{1}{n!n}. \end{aligned}$$

Then $0 < n!(e - s_m) < \frac{1}{n}$. Observe that $n!e$ (because fo hypothesis) and $n!s_n$ lie in $\mathbb{N}$.

26. Exercises:

(a) Let $x_n := \frac{1}{n+1} + \frac{1}{n+2} \cdots + \frac{1}{2n}$. Show that (x_n) is convergent to a limit at most 1. *Hint:* Show that the sequence is increasing.

(b) Let (x_n) be a sequence of positive real numbers. Assume that $x_{n+1}/x_n \to \ell$ with $\ell < 1$. Show that $x_n \to 0$.

(c) Let $a_n := \frac{n!}{n^n}$. Show that $a_n \to 0$. *Hint:* (a_n) is decreasing and a_{n+1}/a_n has a limit less than 1.

27. **Sandwich Lemma.** Let (x_n) , (y_n) and (z_n) be sequences such that (i) $x_n \to \alpha$ and $y_n \to \alpha$ and (ii) $x_n \leq z_n \leq y_n$. Then $z_n \to \alpha$.

Sketch of a proof: Given $\varepsilon > 0$, choose n_1, n_2 such that $n \geq n_1 \implies x_n \in (x - \varepsilon, x + \varepsilon)$ and $n \geq n_2 \implies y_n \in (x - \varepsilon, x + \varepsilon)$. Let $n_0 = \max\{n_1, n_2\}$. Then for $n \geq n_0$, we observe

$$x - \varepsilon < x_n \leq z_n \text{ and } z_n \leq y_n < x + \varepsilon, \text{ that is, } x_n \in (x - \varepsilon, x + \varepsilon).$$

28. **Typical uses of sandwich lemma.**

    (a) We have $\frac{\sin n}{n} \to 0$, as $-1/n \le (\sin n)/n \le 1/n$.

    (b) Given any real number x there exist sequences (s_n) of rational numbers and (t_n) of irrational numbers such that $s_n \to x$ and $t_n \to x$.

    Hint: By density of rationals there exists r such that $x - 1/n < r < x$. Call this r as r_n.

    (c) Let $\alpha :=$ l.u.b. $A \subset \mathbb{R}$. Then there exists a sequence (a_n) in A such that $a_n \to \alpha$.

    Hint: $\alpha - 1/n$ is not an upper bound of A. Let $a_n \in A$ be such that $\alpha - 1/n < a_n \le \alpha$.

    Formulate the analogous result for glb.

    (d) Let (a_n) be a bounded (real/complex) sequence and (x_n) converge to 0. Then $a_n x_n \to 0$.

29. Exercise: Use sandwich lemma to solve the following.

    (a) The sequence $\sqrt{n+1} - \sqrt{n} \to 0$.

    (b) $x_n := \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \to 1$. *Hint:* Use obvious lower and upper bounds for x_n.

    (c) Let $0 < a < b$. The sequence $((a^n + b^n)^{1/n}) \to b$. *Hint:* Use obvious lower and upper bounds and use Item 31f

30. If $x_n \to x$ and $x_n \ge 0$, then $x \ge 0$. (If $x < 0$, use Item 11 to arrive at $x_n < 0$ for $n > n_0$.) However, if each $x_n > 0$ and if $x_n \to x$, then x need not be positive. Example?

    Similar results:
    (i) If $a \le x_n \le b$ and if $x_n \to x$, then $a \le x \le b$.
    (ii) Let $x_n \le y_n$ for all $n > N$ for some N. If $x_n \to x$ and $y_n \to y$, then $x \le y$.

31. **Some Important Limits.**

    (a) Let $0 \le r < 1$ and $x_n := r^n$. Then $x_n \to 0$. If $0 < r < 1$, write $r = 1/(1+h)$ for some $h > 0$. Using binomial theorem, we see that $(1+h)^n > nh$. Hence $r^n \le \frac{1}{nh}$.

    (b) Let $-1 < t < 1$. Then $t^n \to 0$.

    Follows in view of the last result and Item 6.

    (c) Let $|r| < 1$. Then $nr^n \to 0$. More generally, for any $k \in \mathbb{N}$, we have $n^k r^n \to 0$.
    Hint: Observe that $(1+h)^n \ge \frac{n(n-1)}{2} h^2$.

    (d) $n^{1/n} \to 1$. Similar to the previous one. Write $n^{1/n} = 1 + h_n$ with $h_n > 0$. Then, as in the last item,
    $$n = (1+h_n)^n \ge \frac{n(n-1)}{2} h_n^2.$$

    (e) Fix $a \in \mathbb{R}$. Then $\frac{a^n}{n!} \to 0$. Assume $a > 0$. Fix $N > a$. Then, for $n \ge N$, we have
    $$\begin{aligned} \frac{a^n}{n!} &= \left(\frac{a}{1}\frac{a}{2}\cdots\frac{a}{N}\right)\frac{a}{N+1}\cdots\frac{a}{n} \\ &\le Cr^{-N} r^n, \text{ where } C := \left(\frac{a}{1}\frac{a}{2}\cdots\frac{a}{N}\right) \text{ and } r := \frac{a}{N}. \end{aligned}$$

(f) Let $a > 0$. Then $a^{1/n} \to 1$. Hint: If $a > 1$ then $1 \le a^{1/n} \le n^{1/n}$ for $n \ge a$.

We gave also a direct proof in the class. If $a > 1$, then write $a^{1/n} = 1 + h_n$, with $h_n > 0$. Then $a = (1 + h_n)^n \ge nh_n$ so that $h_n \to 0$. If $0 < a < 1$, apply the result to $b^{1/n}$ where $b = 1/a$ and observe that $a^{1/n} = 1/(b^{1/n})$.

32. Let $x_n \to 0$. Let (s_n) be the sequence of arithmetic means (or averages) defined by $s_n := \frac{x_1+\cdots+x_n}{n}$. Then $s_n \to 0$.

Sketch of a proof: Given $\varepsilon > 0$, choose N such that $|x_n| < \varepsilon/2$. Let M be such that $|x_n| \le M$ for all n. Choose n_1 such that $n \ge n_1 \implies (MN)/n < \varepsilon/2$. Observe that for $n \ge \max\{n_1, N\}$

$$\frac{|(x_1 + \cdots + x_N) + (x_{N+1} + \cdots + x_n)|}{n} \le \frac{MN}{n} + \frac{(n-N)}{n}\frac{\varepsilon}{2}.$$

This proof uses an important technique of estimation, namely, that of "Divide and Conquer". The students are advised to go through the proof at least a few times to learn this trick well. Divide and conquer

33. Let $x_n \to x$. Then applying the last result to the sequence $y_n := x_n - x$, we conclude that the sequence (s_n) of arithmetic means converges to x.

34. Exercise: Let $x_n \to x$ and $y_n \to y$. Then

$$\frac{x_1y_n + x_2y_{n-1} + \cdots + x_ny_1}{n} \to xy.$$

35. Let (x_n) be a real sequence. We say that (x_n) diverges to $+\infty$ (or simply diverges to ∞) if for any $R \in \mathbb{R}$ there exists $n_0 \in \mathbb{N}$ such that

$$n \ge n_0 \implies x_n > R.$$

Formulate an analogous notion of diverging to $-\infty$.

36. A sequence of real numbers diverging to ∞ (or to $-\infty$) is divergent, that is, it is not convergent.

37. Examples of divergent sequences:

(a) The sequences $x_n = n$, and $y_n := 2^n$ diverge to infinity. *Hint:* Prove by induction that $2^n > n$.

(b) Let $a > 1$. Then $a^n \to \infty$. *Hint:* $a^n = (1+h)^n > nh$.

(c) $(n!)^{1/n}$ diverges to ∞. *Hint:* Given $a \in \mathbb{R}$, use Item 31e to conclude that $n! > a^n$ for all large values of n.

38. Consider the sequence (x_n) where $x_n = (-1)^n n$. This sequence is divergent, but divergent neither to ∞ nor to $-\infty$.

39. Let $x_n > 0$. Then $x_n \to 0$ iff $1/x_n \to +\infty$. What happens if $x_n < 0$ and $\lim x_n = 0$?

The sequence $x_n := \frac{(-1)^n}{n} \to 0$ but the sequence of reciprocals is $((-1)^n n)$. Refer to Item 38

40. Exercises:

   (a) Let $x_n := \sum_{k=1}^n \frac{1}{k}$. Show that the sequence (x_n) diverges to ∞. *Hint:* Observe that $x_{2n} - x_n \geq 1/2$ for any n.

   (b) Let (x_n) be a sequence in $(0, \infty)$. Let $y_n := \sum_{k=1}^n (x_k + \frac{1}{x_k})$. Show that (y_n) diverges to ∞.

   (c) Let (x_n) and (y_n) be sequences of positive reals. Assume that $\lim x_n/y_n = A > 0$. Show $\lim x_n = +\infty$ iff $\lim y_n = +\infty$.

   (d) Show that $\lim \frac{an^2+b}{cn+d} = \infty$ if $ac > 0$.

   (e) Let (a_n) be a sequence of positive reals. Assume that $\lim \frac{a_{n+1}}{a_n} = \alpha$. Then show that $\lim (a_n)^{\frac{1}{n}} = \alpha$. *Hint:* Choose any $a > \alpha$ and N such that $\frac{a_{n+1}}{a_n} < L$ for any $n > N$. Then $a_n = \frac{a_n}{a_{n-1}} \cdots \frac{a_{N+1}}{a_N} a_N$ so that $a_n \leq a^{n-N} a_N$.

   (f) Use the last item to find the 'limit' of $(n!)^{\frac{1}{n}}$.

41. Definition of a subsequence: Let $x \colon \mathbb{N} \to \mathbb{R}$ be a sequence. Then a subsequence is the restriction of x to an infinite subset S of $\mathbb{N}$.

42. Using the well-ordering principle (thrice!) of $\mathbb{N}$, we observe that an infinite subset $S \subset \mathbb{N}$ can be listed as $\{n_1 < n_2 < \cdots < n_k < n_{k+1} \cdots \}$.

    Let n_1 be the least element of S. Assume that we have chosen $n_1, \ldots, n_k \in S$ such that $n_1 < n_2 < \cdots < n_k$. Since $S_k := S \setminus \{n_1, \ldots, n_k\} \neq \emptyset$, let n_{k+1} be the least element of S_k. Thus we have a recursively defined sequence of integers. Observe that by our choice $n_k \geq k$ for each $k \in \mathbb{N}$. Why this process exhausts S? If $T := S \setminus \{n_k : k \in \mathbb{N}\} \neq \emptyset$, let m be the least element of T. (Note that m is an element of S!) Consider $A := \{k \in \mathbb{N} : n_k \geq m\}$. Since $n_m \geq m$, we deduce that $A \neq \emptyset$. Let k be the least element of A. Then we must have $n_{k-1} < m$. Since $m \notin S_{k-1}$, since $m \leq n_k$ and since n_k is the least element of S_{k-1} we conclude that $m = n_k$. But this is a contradiction to the fact that $m \in T$.

    With this observation, the standard practice is to denote the subsequence as (x_{n_k}).

43. Most useful/handy observation: $n_k \geq k$ for all k.

44. Let (x_n) be a sequence and (x_{n_k}) be a subsequence. What does it mean to say that the subsequence converges to x?

    Let us define a new sequence (y_k) where $y_k := x_{n_k}$. Then we say $x_{n_k} \to x$ iff $y_k \to x$. That is, for a given $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$, we must have $|y_k - x| < \varepsilon$, which is same as saying that

$$\text{for } k \geq k_0, \text{ we have } |x_{n_k} - x| < \varepsilon.$$

45. If $x_n \to x$, and if (x_{n_k}) is a subsequence, then $x_{n_k} \to x$ as $k \to \infty$.

46. **Existence of a monotone subsequence of a real sequence.** Given any real sequence (x_n) there exists a monotone subsequence.

    Sketch of a proof: Consider the set S defined by

$$S := \{n \in \mathbb{N} : x_m < x_n \text{ for } m > n\}.$$

There are two cases: S is finite or infinite.

Case 1. S is finite. Let N be any natural number such that $k \leq N$ for all $k \in S$. Let $n_1 > N$. Then $n_1 \notin S$. Hence there exists $n_2 > n_1$ such that $x_{n_2} \geq x_{n_1}$. Since $n_2 > n_1 > N$, $n_2 \notin S$. Hence we can find an $n_3 > n_2$ such that $x_{n_3} \geq x_{n_2}$. This we way, we can find a monotone nondecreasing (increasing) subsequence, (x_{n_k}).

Case 2. S is infinite. Let n_1 be the least element of S. Let n_2 be the least element of $S \setminus \{n_1\}$ and so on. We thus have a listing of S:

$$n_1 < n_2 < n_3 < \cdots$$

Since n_{k-1} is an element of S and since $n_{k-1} < n_k$, we see that $x_{n_k} < x_{n_{k-1}}$, for all k. We now have a monotone decreasing sequence.

47. **Bolzano-Weierstrass Theorem.** If (x_n) is a bounded real sequence, it has a convergent subsequence.

    Follows from the last result on the existence of monotone subsequences and the convergence of bounded monotone sequences.

48. Let (x_n) be Cauchy. Let a subsequence (x_{n_k}) converge to x. Then $x_n \to x$.

    Sketch: Given $\varepsilon > 0$, let k_0 correspond to ε for the convergent sequence (x_{n_k}). Let n_0 correspond to ε for the Cauchy sequence (x_n). Let $N := \max\{n_0, k_0\}$. Then for $n \geq N$, we have

    $$|x - x_n| \leq |x - x_{n_N}| + |x_{n_N} - x_n| < 2\varepsilon.$$

49. We can now give 2nd proof of the Cauchy completeness of $\mathbb{R}$. Use Item 20, Bolzano-Weierstrass theorem and the last item to arrive at a proof.

50. Do subsequences arise naturally (in Mathematics)? Let (a_n) be a sequence. Prove that (a_n) is divergent iff for each $a \in \mathbb{R}$, there exists an $\varepsilon > 0$ and a subsequence (x_{n_k}) such that $|a - a_{n_k}| \geq \varepsilon$ for all k.

    For another such occurrence, see Item 53b.

51. Some typical uses of subsequences:

    (a) Consider the sequence $a^{1/n}$ again where $a > 1$. We showed that it is decreasing and bounded below, hence convergent to some $\ell \in \mathbb{R}$. The subsequence $(a^{1/2n})$ is also convergent to ℓ. Conclude $\ell^2 = \ell$.

    (b) Assume that the sequence $(n^{1/n})$ is convergent. Show that the limit is 1 by considering the subsequence $((2n)^{1/2n})$.

    (c) Show that the sequence $((-1)^n)$ is divergent.

52. Exercises:

    (a) Prove that the sequence (x_n) where $x_n := \frac{(n^2+13n-41)\cos(2^n)}{n^2+2n+1}$ has a convergent subsequence.

    (b) True or false: For any sequence (x_n), the sequence $y_n := \frac{x_n}{1+|x_n|}$ has a convergent subsequence.

(c) True or false: A sequence (x_n) is bounded iff every subsequence of (x_n) has a convergent subsequence.

(d) Prove that a sequence (x_n) is unbounded iff there exists a subsequence (x_{n_k}) such that $|x_{n_k}| \geq k$ for each $k \in \mathbb{N}$.

(e) Let (a_n) be a sequence. Prove that (a_n) is divergent iff for each $a \in \mathbb{R}$, there exists an $\varepsilon > 0$ and a subsequence (x_{n_k}) such that $|a - a_{n_k}| \geq \varepsilon$ for all k.

(f) Show that if a monotone sequence has a convergent subsequence, then it is convergent.

(g) Let $\{r_n\}$ be an enumeration of all rationals in $[0, 1]$. Show that $\{r_n\}$ is not convergent.

53. Typical uses of Bolzano-Weierstrass theorem.

(a) The sequence $(\sin(n))$ has a convergent subsequence.

(b) True or false: A sequence (x_n) is bounded iff every subsequence of (x_n) has a convergent subsequence.

54. **Compactness:** We say that a subset $K \subset \mathbb{R}$ is compact if every sequence **in** K has a subsequence which converges to an element **of** K. Thus Bolzano-Weierstrass theorem says that any closed and bounded interval is compact.

(a) $A := \mathbb{R}$ is not compact, for consider $x_n := n$.

(b) The interval $(0, 1)$ is not compact, for the sequence $(1/n)$ converges to $0 \notin (0, 1)$. More generally, any open interval $B := (a, b)$ is not compact. For instance, consider $(b - \frac{1}{n})$ for $n \geq N$ where N is chosen so that $b - \frac{1}{N} > a$.

(c) Let (r_n) be a sequence in $C := [-2, 2] \cap \mathbb{Q}$ which converges to $\sqrt{2}$. Then C is not compact.

(d) The set D of irrational numbers is not compact, for, consider $x_n := \frac{\sqrt{2}}{n}$.

55. **(Some sequences defined recursively).** Find the limits (if they exist) of the following recursively defined sequences.

(a) $x_1 = \sqrt{2}$, $x_n = \sqrt{2 + \sqrt{x_{n-1}}}$ for $n \geq 2$.

(b) $x_1 = 1$, $x_n = \sqrt{2x_{n-1}}$ for $n \geq 2$.

(c) For $a > 0$, let x_1 be any positive real number and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$. *Hint:* Show that $x_n^2 \geq a$ by induction. (Use AM $\geq$ GM inequality.) Show that (x_n) is eventually decreasing.

(d) Let $0 < a \leq x_1 \leq x_2 \leq b$. Define $x_n = \sqrt{(x_{n-1}x_{n-2})}$ for $n \geq 3$. Show that $a \leq x_n \leq b$ and $|x_{n+1} - x_n| \leq \frac{b}{a+b} |x_n - x_{n-1}|$ for $n \geq 2$. Prove (x_n) is convergent.

(e) Let $0 < y_1 < x_1$. Define

$$x_{n+1} = \frac{x_n + y_n}{2} \text{ and } y_{n+1} = \sqrt{x_n y_n}, \text{ for } n \in \mathbb{N}.$$

i. Prove that (y_n) is increasing and bounded above while (x_n) is decreasing and bounded below.

ii. Prove that $0 < x_{n+1} - y_{n+1} < 2^{-n}(x_1 - y_1)$ for $n \in \mathbb{N}$.

iii. Prove that x_n and y_n converge to the same limit.

(f) Let $x_1 = a$ and $x_2 = b$. Define $x_{n+2} = (x_n + x_{n+1})/2$. Show that (x_n) is convergent by showing that it is Cauchy.

(g) **Square Roots.** Let $x_1 = 2$, define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

Show that each $x_n^2 \geq 2$ and use it to prove that $x_{n+1} - x_n \geq 0$. Conclude that $x_n \to \sqrt{2}$.

Given $\alpha > 0$, can you modify the sequence to produce one which converges to $\sqrt{\alpha}$?

(h) Let $x_1 \geq 0$, and define recursively $x_{n+1} = \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that if the sequence is convergent then the limit is 2.

(i) Let $0 \leq x_1 \leq x$. Define $x_{n+1} := 1 - \sqrt{1 - x_n}$ for $n \in \mathbb{N}$. Show that if the sequence is convergent, then the limit is either 0 or 1.

56. The following are good topics for students seminars.

(a) **The Number e.** See Item 25.

(b) **Euler's Constant.** Let $\gamma_n = \sum_{k=1}^n \frac{1}{k} - \log n = \sum_{k=1}^n \frac{1}{k} - \int_1^n t^{-1} dt$.

i. Show that γ_n is a decreasing sequence. *Hint:* $\gamma_n - \gamma_{n+1} = \int_n^{n+1} t^{-1} - \frac{1}{n+1} > 0$.

ii. Show that $0 < \gamma_n \leq 1$ for all n: *Hint:* $\gamma_n \leq \gamma_1$ for all n. Also,

$$\gamma_n > \sum_{k=1}^{n} \left[\frac{1}{k} - \int_k^{k+1} t^{-1} dt\right] > 0.$$

iii. $\lim \gamma_n$ exists and is denoted by γ. This γ is called the *Euler's constant*. Till today it is not known whether γ is rational or not!

(c) **Fibonacci's Sequence.** Let $x_0 = 1$, $x_1 = 1$. Define (x_n) recursively by $x_n = x_{n-1} + x_{n-2}$, $n \geq 2$. This (x_n) is called the *Fibonacci sequence*. Let $\gamma_n := \frac{x_n}{x_{n-1}}$, $n \geq 1$.

i. Prove that (x_n) is divergent.

ii. (i) $1 \leq \gamma_n \leq 2$, (ii) $\gamma_{n+1} = 1 + \frac{1}{n}$, (iii) $\gamma_{n+2} - \gamma_n = \frac{\gamma_n - \gamma_{n-2}}{(1+\gamma_n)(1+\gamma_{n-2})}$.

iii. (γ_{2n}) is decreasing.

iv. (γ_{2n+1}) is increasing.

v. (γ_{2n}) and (γ_{2n+1}) are convergent. The limits of both these sequences satisfy the equation $\ell^2 - \ell - 1 = 0$.

vi. $\lim \gamma_n = \frac{1+\sqrt{5}}{2}$.

(d) Let (a_n) be a sequence such that $|a_n - a_m| < \varepsilon$ for all $m, n \geq N$. If $a_n \to a$, show that $|a_n - a| \leq \varepsilon$ for all $n \geq N$. (An easy but often used result.)

57. Miscellaneous Exercises:

(a) Decide for what values of x, the sequences whose n-th term is $x_n := \frac{x+x^n}{1+x^n}$ is convergent.

(b) Find the limit of the sequence whose n-th term is $\frac{1+a+a^2+\cdots+a^{n-1}}{n!}$.

(c) Let $a_n := \frac{n}{2^n}$. Show that $\lim a_n = 0$.

(d) Let $a \in \mathbb{R}$. Consider $x_1 = a$, $x_2 = \frac{1+a}{2}$, and by induction $x_n := \frac{1+x_{n-1}}{2}$. Then $x_n \to ?$ Draw pictures and guess the limit and prove your guess.

(e) Consider the sequence

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}, \ldots,$$

Show that $x_n \to 2$. *Hint:* There is a recursive/inductive definition involved.

(f) Let (x_n) be given. Let $y_n := x_{2n-1}$ and $z_n := x_{2n}$. Show that (x_n) is convergent iff both (y_n) and (z_n) converge to the same limit.

(g) Prove that the sequence $(\sin n)$ is divergent.

(h) Show that a sequence $(z_n = x_n + iy_n)$ in $\mathbb{C}$ is convergent iff the real sequences (x_n) and (y_n) are convergent.

(i) Show that any Cauchy sequence in $\mathbb{C}$ converges to an element of $\mathbb{C}$.

(j) Formulate and prove an analogue of Bolzano-Weierstrass theorem for complex sequences.

3 Continuity

1. Let $J \subset \mathbb{R}$. (An important class of subsets J are intervals of any kind.) Let $f\colon J \to \mathbb{R}$ be a function and $a \in J$. We say that f is continuous at a if for *every* sequence (x_n) **in** J with $x_n \to a$, we have $f(x_n) \to f(a)$.

   We say that f is continuous on J if it is continuous at every point $a \in J$.

2. Examples:

   (a) Let f be a constant function on J. Then f is continuous on J.

   (b) Let $f(x) := x$ for all $x \in J$. Then f is continuous on J.
   More generally, $f(x) := x^n$ is continuous on $\mathbb{R}$.

   (c) Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 1$ if $x \in \mathbb{Q}$ and 0 otherwise. Then f is not continuous at any point of $\mathbb{R}$.

   (d) Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$ Then f is continuous at all nonzero elements of $\mathbb{R}$ and is not continuous at 0.

   (e) Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$ Then f is continuous on $\mathbb{R}$.

   (f) Let $f\colon \mathbb{R}^* \to \mathbb{R}^*$ be given by $f(x) = 1/x$. Then f is continuous on $\mathbb{R}^*$.

   (g) Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} \alpha & \text{if } x < 0 \\ ax^2 - bx + c & \text{if } x \geq 0. \end{cases}$ What value of α ensures the continuity of f at 0?

3. Exercises:

   (a) Let $f\colon J \to \mathbb{R}$ be continuous. Let $J_1 \subset J$. Let g be the restriction of f to J_1. Show that g is continuous on J_1.

   (b) Let $f(x) = 3x$ for $x \in \mathbb{Q}$ and $f(x) = x + 8$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. Find the points at which f is continuous.

   (c) Let $f(x) := x$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. Then f is continuous only at $x = 0$.

   (d) Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Assume that $f(r) = 0$ for $r \in \mathbb{Q}$. Then $f = 0$.

   (e) Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be continuous. If $f(x) = g(x)$ for $x \in \mathbb{Q}$, then $f = g$.

   (f) Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous which is also an additive homomorphism, that is, $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Then $f(x) = \lambda x$ where $\lambda = f(1)$.

   (g) Consider $f\colon (0, 1) \to \mathbb{R}$ defined by $f(x) = 1/q$ if $x = p/q$ in reduced form and $f(x) = 0$ if $x \notin \mathbb{Q}$. Then f is continuous only at the irrationals.

   (h) Let $f(x) = \begin{cases} x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Show that f is continuous at 0.

   (i) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x - [x]$, where $[x]$ stands for the greatest integer less than or equal to x. At what points f is continuous? *Hint:* Draw a picture.

(j) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \min\{x - [x], 1 + [x] - x\}$, that is, the minimum of the distances of x from $[x]$ and $[x] + 1$. At what points f is continuous? *Hint:* Draw a picture.

(k) If $A \subset \mathbb{R}$ is a nonempty subset, define $f(x) := \text{g.l.b. } \{|x - a| : a \in A\}$. Then f is continuous.

(l) Let $f\colon J \to \mathbb{R}$ be continuous. Let $\alpha \in \operatorname{Im}(f)$. Let $S := f^{-1}(\alpha)$. Show that if (x_n) is a sequence in S converging to an element $a \in J$, then $a \in S$.

4. **Algebra of continuous functions:** Let $f, g\colon J \to \mathbb{R}$ be continuous at $a \in J$. Let $\alpha \in \mathbb{R}$. Then

   (a) $f + g$ is continuous at a.

   (b) αf is continuous at a.

   (c) The set of functions from $J \to \mathbb{R}$ continuous at a is a real vector space.)

   (d) The product fg is continuous at a.

   (e) Assume further that $f(a) \neq 0$. Then there exists $\delta > 0$ such that for each $x \in (a - \delta, a + \delta) \cap J \to \mathbb{R}$, we have $f(x) \neq 0$. The function $1/f\colon (a - \delta, a + \delta) \cap J \to \mathbb{R}$ is continuous at a. (Recall that $(1/f)(x) := \frac{1}{f(x)}$.)

   Sketch of a proof of the first part. If false, then for each $\delta = 1/k$, there exists $x_k \in (a - \frac{1}{k}, a + \frac{1}{k}) \cap J$ such that $f(x_k) = 0$. Clearly, $x_k \to a$ but $f(x_k) \to 0 \neq f(a)$.

   (f) $|f|$ is continuous at c.

   (g) Let $h(x) := \max\{f(x), g(x)\}$. Then h is continuous at c. Similarly, the function $k(x) := \min\{f(x), g(x)\}$ is continuous at c.

   Exercise:

   i. Let $f(x) := x$ and $g(x) := x^2$ for $x \in \mathbb{R}$. Find $\max\{f, g\}$ and draw its graph.

   ii. Let $f, g\colon [-\pi, \pi] \to \mathbb{R}$ be given by $f(x) := \cos x$ and $g(x) := \sin x$. Draw the graph of $\min\{f, g\}$.

5. Any polynomial function $f\colon J \to \mathbb{R}$ of the form $f(x) := a_0 + a_1 x + \cdots + a_n x^n$ is continuous on J.

6. A rational function is a function of the form $f(x) = \frac{p(x)}{q(x)}$ where p, q are polynomial functions. The domain of a rational function is the complement (in $\mathbb{R}$) of the set of points at which q takes the value 0. The rational functions are continuous on their domains of definition.

7. Let $f_i\colon J_i \to \mathbb{R}$ be continuous at $a_i \in J_i$, $i = 1, 2$. Assume that $f_1(J_1) \subset J_2$ and $a_2 = f_1(a_1)$. Then the composition $f_2 \circ f_1$ is continuous at a_1.

8. The standard ε-δ definition of continuity. Let $f\colon J \to \mathbb{R}$ be given and $a \in J$. We say that f is continuous at a if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x \in J \text{ and } |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon. \tag{4}$$

9. Our definitions of continuity in Item 1 and Item 8 are equivalent.

Sketch of a proof. Let f be continuous at a according to Item 1. We now show that f is continuous according to ε-δ definition. Assume the contrary that there exists $\varepsilon > 0$ for which no δ as required exists. In particular, for each $\delta = 1/k$, we have $x_k \in J \cap (a - 1/k, a + 1/k)$ with $|f(x_k) - f(a)| \geq \varepsilon$. Since $x_k \to a$, we must have $f(x_k) \to f(a)$, that is, $|f(x_k) - f(a)| \to 0$, a contradiction.

Assume that f is continuous according to ε-δ definition. Let $x_n \in J$ be such that $x_n \to a$. To prove $f(x_n) \to f(a)$, let $\varepsilon > 0$ be given. Then by ε-δ definition, for this $\varepsilon > 0$, there exists $\delta > 0$ such that (4) holds. Since $x_n \to a$, for the $\delta > 0$, there exists N such that

$$n \geq N \implies |x_n - a| < \delta.$$

It follows that if $n \geq N$, $x_n \in (a - \delta, a + \delta)$ and hence $f(x_n) \in (f(a) - \varepsilon, f(a) + \varepsilon)$. That is, $f(x_n) \to f(a)$. □

10. Some examples to work with ε-δ definition. The basic idea to show the continuity of f at a is to obtain an estimate of the form

$$|f(x) - f(a)| \leq C_a |x - a|,$$

where $C_a > 0$ may depend on a. There are situations when this may not work. See 10i. In 10c and 10f, one can choose C_a independent of a.

   (a) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$ for $x \in \mathbb{R}$.
   (b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ for $x \in \mathbb{R}$.
   (c) $f : [-R, R] \to \mathbb{R}$, $f(x) = x^2$ for $|x| < R$.
   (d) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$ for $x \in \mathbb{R}$.
   (e) $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^{-1}$ for $x \in (0, \infty)$.
   (f) $f : (\alpha, \infty) \to \mathbb{R}$, $f(x) = x^{-1}$ for $x \in (\alpha, \infty)$ where $\alpha > 0$ is fixed.
   (g) $f : (0, \infty) \to \mathbb{R}$ given by $f(x) := x^{1/n}$ for a fixed $n \in \mathbb{N}$.
   (h) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$
   (i) Thomae's function. Let $f : (0, 1) \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/q & \text{if } x = p/q \text{ with } p, q \in \mathbb{N} \text{ and } p \text{ and } q \text{ have no common factor.} \end{cases}$$

   Then f is continuous at all irrational points and not continuous at any rational point of $(0, 1)$.

11. We say that a function $f : J \to \mathbb{R}$ is Lipschitz if there exists $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in J$. Any Lipschitz function is continuous.

12. Let $f : J \to \mathbb{R}$ be continuous at c with $f(c) \neq 0$. Then there exists $\delta > 0$ such that $|f(x)| > |f(c)|/2$ for all $x \in (c - \delta, c + \delta) \cap J$.

    In particular, if $f(c) > 0$, then there exists $\delta > 0$ such that $f(x) > f(c)/2$ for all $x \in (c - \delta, c + \delta) \cap J$.

    What is the analogue of this when $f(c) < 0$?

13. Let $f\colon J \to \mathbb{R}$ be continuous at $a \in J$. Then f is *locally bounded*, that is, there exist $M > 0$ and $\delta > 0$ such that

$$\forall\, x \in (a-\delta, a+\delta) \cap J \implies |f(x)| \le M.$$

Apply the ε-δ definition of continuity, say, with $\varepsilon = 1$.

14. The results of the last two items are about the *local* properties of continuous functions. We shall apply them to get some *global* results. See Item 16 and Item 26.

15. To verify whether or not a function is continuous at a point $a \in J$, we do not have to know the values of f at each and every point of J, we need only to know the values $f(x)$ for $x \in J$ 'near' to a, that is, for all $x \in (a-\delta, a+\delta) \cap J$ for some $\delta > 0$.

The following are two of the most important *global* results on continuity. See Item 27.

16. **Intermediate Value Theorem.** Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function such that $f(a) < 0 < f(b)$. Then there exists $c \in (a,b)$ such that $f(c) = 0$.

Proof. Draw some pictures. We wish to locate the "first" c from a such that $f(c) = 0$. Towards this end, we define $E := \{x \in [a,b] : f(y) \le 0 \text{ for } y \in [a,x]\}$.

Using the continuity of f at a for $\varepsilon = -f(a)/2$, we can find a $\delta > 0$ such that $f(x) \in (3f(a)/2, f(a)/2)$ for all $x \in [a, a+\delta)$. This shows that $a+\delta/2 \in E$. Since E is bounded by b there is $c \in \mathbb{R}$ such that $c = \sup E$. Clearly we have $a + \delta/2 \le c \le b$ and hence $c \in (a,b]$. We claim that $c \in E$ and that $f(c) = 0$.

Case 1. $f(c) > 0$. Then by Item 12, there exists $\delta > 0$ such that for $x \in (c-\delta, c+\delta) \cap [a,b]$, we have $f(x) > 0$. Since $c - \delta < c$, there exists $x \in E$ such that $c - \delta < x$. Since $x \in E$, we have $f(t) \le 0$, for $t \in [a,x] = [a, c-\delta] \cup (c-\delta, x]$, a contradiction.

Case 2. $f(c) < 0$. Then by Item 12, there exists $\delta > 0$ such that for $x \in (c-\delta, c+\delta) \cap [a,b]$, we have $f(x) < 0$. Since $c - \delta < c$, there exists $x \in E$ such that $f(t) \le 0$, for $t \in [a,x]$. Hence $f(t) \le 0$ for all $t \in [a,x] \cup (c-\delta, c+\delta/2] = [a, c+\delta/2]$. That is, $c + \delta/2 \in E$, contradicting $c = \text{l.u.b. } E$.

Hence we are forced to conclude that $f(c) = 0$. □

2nd Proof. Let $J_0 := [a,b]$. Let c_1 be the mid point of $[a,b]$. Now there are three possibilities for $f(c_1)$. It is zero, negative or positive. If $f(c_1) = 0$, then the proof is over. If not, we choose one of the intervals $[a, c_1]$ or $[c_1, b]$ so that f assumes values with opposite signs at the end points. To spell it out, if $f(c_1) < 0$, then we take the subinterval $[c_1, b]$. If $f(c_1) > 0$, then we take the subinterval $[a, c_1]$. The chosen subinterval will be called J_1 and we write it as $[a_1, b_1]$.

We now bisect the interval J_1 and choose one of the two subintervals as $J_2 := [a_2, b_2]$ so that f takes values with opposite signs at the end points. We continue this process recursively. We thus obtain a sequence (J_n) of intervals with the following properties:
(i) If $J_n = [a_n, b_n]$, then $f(a_n) \le 0$ and $f(b_n) \ge 0$.
(ii) $J_{n+1} \subset J_n$.
(iii) $\ell(J_n) = 2^{-n}\ell(J_0) = 2^{-n}(b-a)$.

By nested interval theorem there exists a unique $c \in \cap J_n$. Since $a_n, b_n, c \in J_n$, we have

$$|c - a_n| \le \ell(J_n) = 2^{-n}(b-a) \text{ and } |c - b_n| \le \ell(J_n) = 2^{-n}(b-a).$$

Hence it follows that $\lim a_n = c = \lim b_n$. Since $c \in J$ and f is continuous on J, we have

$$\lim_{n\to\infty} f(a_n) = f(c) \text{ and } \lim_{n\to\infty} f(b_n) = f(c).$$

Since $f(a_n) \leq 0$ for all n, it follows that $\lim_n f(a_n) \leq 0$, that is, $f(c) \leq 0$. In an analogous way, $f(c) = \lim f(b_n) \geq 0$. We are forced to conclude that $f(c) = 0$. The proof is complete. □

17. **Intermediate Value Theorem (Standard Version).** Let $g\colon [a,b] \to \mathbb{R}$ be a continuous function. Let λ be a real number between $g(a)$ and $g(b)$. Then there exists $c \in (a,b)$ such that $g(c) = \lambda$.

    Apply the previous version to the function $f(x) = g(x) - \lambda$.

18. We made a crucial use of the LUB property of $\mathbb{R}$ in the proofs of the theorems above. They are not true for example in $\mathbb{Q}$. Let us be brief. Consider the interval $[0,2] \cap \mathbb{Q}$ and th continuous function $f(x) = x^2 - 2$. Then $f(0) < 0$ while $f(2) > 0$. We know that there exists no rational number α whose square is 2. Recall also that we have shown that $\mathbb{Q}$ does not enjoy the LUB property (Item 35).

19. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous taking values in $\mathbb{Z}$ or in $\mathbb{Q}$. Then f is a constant.

20. Let $f\colon [a,b] \to \mathbb{R}$ be a nonconstant continuous function. Show that $f([a,b])$ is uncountable.

21. Let $\alpha \geq 0$ and $n \in \mathbb{N}$. Then there exists $\beta \geq 0$ such that $\beta^n = \alpha$.

    Proof. Choose $N \in \mathbb{N}$ such that $N > \alpha$. Consider $f\colon [0,\infty) \to [0,\infty)$ defined by $f(x) = x^n - \alpha$. Then $f(x) \leq 0$ and $f(N) > 0$. Intermediate value theorem yields applied to the pair $(f, [0,N])$ yields the result. □

22. Any polynomial of odd degree with real coefficients has a real zero.

    Proof. It is enough to prove that a monic polynomial

    $$P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0, (a_j \in \mathbb{R}, 0 \leq j \leq n-1),$$

    of odd degree has a real zero.

    Write $P(X) = X^n(1 + \frac{a_{n-1}}{X} + \cdots + \frac{a_0}{X^n})$. If $N \in \mathbb{N}$, then $|\frac{a_j}{N^{n-j}}| \leq |\frac{a_j}{N}|$ for any $1 \leq j \leq n$. Let $C := \sum_{j=0}^{n-1} |a_j|$. We then have

    $$|\frac{a_{n-1}}{N} + \cdots + \frac{a_0}{N^n}| \leq \frac{C}{N}.$$

    We can choose $N \in \mathbb{N}$ such that $\frac{C}{N} < 1/2$. If $|X| > N$, we have the estimate $|\frac{a_{n-1}}{N} + \cdots + \frac{a_0}{N^n}| < 1/2$. That is, we have

    $$1/2 \leq 1 + \frac{a_{n-1}}{X} + \cdots + \frac{a_0}{X^n} \leq 3/2. \tag{5}$$

    Consequently, $P(X) \leq X^n/2 < 0$ if $X < -N$ and $P(X) \geq X^n/2 > 0$ if $X > N$. Now the intermediate value theorem asserts the existence of a zero of P in $(-N, N)$. □

23. **Fixed point theorem.** Let $f\colon [a,b] \to [a,b]$ be continuous. Then there exists $c \in [a,b]$ such that $f(c) = c$. (Such a c is called a fixed point of f.)

Proof. Consider $g(x) := f(x) - x$. Then $g(a) \geq 0$ and $g(b) \leq 0$. Apply intermediate value theorem. □

24. Exercises:

(a) Let $f\colon [a,b] \to [a,b]$ be continuous. Then there exists $x \in [a,b]$ such that $f(x) = x$.

(b) Prove that $x = \cos x$ for some $x \in (0, \pi/2)$.

(c) Prove that $xe^x = 1$ for some $x \in (0,1)$.

(d) Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous taking values in $\mathbb{Z}$ or in $\mathbb{Q}$. Then f is a constant.

(e) Let $f\colon [a,b] \to \mathbb{R}$ be a nonconstant continuous function. Show that $f([a,b])$ is uncountable.

(f) Are there continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) \notin \mathbb{Q}$ for $x \in \mathbb{Q}$ and $f(x) \in \mathbb{Q}$ for $x \notin \mathbb{Q}$?

(g) Let $f\colon [0,1] \to \mathbb{R}$ be continuous. Assume that the image of f lies in $[1,2] \cup (5,10)$ and that $f(1/2) \in [0,1]$. What can you conclude about the image of f?

(h) Existence of n-th roots: Let $\alpha \geq 0$ and $n \in \mathbb{N}$ be given. Then there exists $x \geq 0$ such that $x^n = \alpha$.

(i) Let $f\colon [0, 2\pi] \to [0, 2\pi]$ be continuous such that $f(0) = f(2\pi)$. Show that there exists $x \in [0, 2\pi]$ such that $f(x) = f(x + \pi)$.

(j) Let $p(X)$ be an odd degree polynomial with real coefficients. Then p has a real root.

(k) Let p be a real polynomial function of odd degree. Show that $p\colon \mathbb{R} \to \mathbb{R}$ is onto.

(l) Show that $x^4 + 5x^3 - 7$ has two real roots.

(m) Let $p(X) := a_0 + a_1 X + \cdots + a_n X^n$. If $a_0 a_N < 0$, show that p has at least two real roots.

(n) Let J be an interval and $f\colon J \to \mathbb{R}$ be continuous and 1-1. Then f is strictly monotone.

(o) Let I be an interval and $f\colon I \to \mathbb{R}$ be strictly monotone. If $f(I)$ is an interval, show that f is continuous.

(p) Use the last item to conclude that the function $x \mapsto x^{1/n}$ from $[0, \infty) \to [0, \infty)$ is continuous.

25. **Weierstrass Theorem.** Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Then f is bounded.

First Proof. Let $E := \{x \in J := [a,b] : f \text{ is bounded on } [a,x]\}$. The conclusion of the theorem is that $b \in E$.

Since f is continuous at a, using Item 13, we see that f is bounded on $[a, a+\delta)$ for some $\delta > 0$. Hence $a + \delta/2 \in E$. Obviously E is bounded by b. Let $c = \sup E$. Since $a + \delta/2 \in E$ we have $a \leq c$. Since b is an upper bound for E, $c \leq b$. Thus $a \leq c \leq b$. We intend to show that $c \in E$ and $c = b$. This will complete the proof.

Since f is continuous at c, it is locally bounded, say, on $(c-\delta, c+\delta) \cap J$. Let $x \in E$ be such that $c-\delta < x \leq c$. Then clearly, f is bounded on $[a, c] \subset [a, x] \cup ((c-\delta, c+\delta) \cap J)$. In particular, $c \in E$. If $c < b$ choose $\delta_1 < \delta$ so that $c + \delta_1 < b$. The above argument shows that $c + \delta_1 \in E$ if $c \neq b$. This contradicts the fact that $c = \sup E$. Hence $c = b$. This proves the result. □

Second Proof. If false, there exists a sequence (x_n) in $[a, b]$ such that $|f(x_n)| > n$ for each $n \in \mathbb{N}$. Since $[a, b]$ is compact, there exists a subsequence, say, (x_{n_k}) which converges to x in the compact set $[a, b]$. Since $|f|$ is continuous, we must have $|f(x_{n_k})| \to f(x)$, in particular, the sequence $(|f(x_{n_k})|)$ is bounded, a contradiction. (This proof obviously works as long as the domain of the continuous function is a compact subset of $\mathbb{R}$.) □

Third Proof. If false, then f is not bounded on one of the subintervals, $[a, (a+b)/2]$ and $[(a+b)/2, b]$. (Why?) Choose such an interval and call it J_1. Note that the length of J_1 is half that of $J = [a, b]$. Repeat this argument to get a sequence of nested intervals J_n such that $\ell(J_n) = 2^{-n}\ell(J)$ and f is not bounded on each J_n. Let c be the unique common point of this nested sequence of intervals. Using Item 13, we see that f is bounded on $(c-\delta, c+\delta)$ for some $\delta > 0$. Note that there exists N such that if $n \geq N$, we have $J_n \subset (c-\delta, c+\delta)$. This leads to a contradiction, since f is not bounded on J_n's. □

26. Note that the first proofs of the intermediate value theorem and Weierstrass theorem are quite similar. We defined appropriate subsets of $[a, b]$ and applied the corresponding *local* result (Item 12 and Item 13 respectively) to get the global result. See also Item 14.

27. The last two theorems (Items 16 and 25) are global results in the following sense.

    In the first case, we imposed a restriction on the domain, namely, that it is an interval. If the domain is not an interval the conclusion does not remain valid. In the second, we required that the domain is compact, otherwise the result is not true.

28. **Extreme Values Theorem.** Let the hypothesis be as in Weierstrass theorem. Then there exists $x_1, x_2 \in [a, b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$. (In other words, a continuous function f on a closed and bounded interval is bounded and attains its maximum and minimum.)

    Let $M :=$ l.u.b. $\{f(x) : a \leq x \leq b\}$. If there exists no $x \in [a, b]$ such that $f(x) = M$ then $M - f(x)$ is continuous at each $x \in [a, b]$ and $M - f(x) > 0$ for all $x \in [a, b]$. If we let $g(x) := 1/(M - f(x)$ for $x \in [a, b]$, then g is continuous on $[a, b]$. By 1), there exists $A > 0$ such that $g(x) \leq A$ for all $x \in [a, b]$. But then we have, for all $x \in [a, b]$, $g(x) := \frac{1}{M - f(x)} \leq A$ or $M - f(x) \geq \frac{1}{A}$. Thus we conclude that $f(x) \leq M - (1/A)$ for $x \in [a, b]$. This contradicts our hypothesis that $M =$ l.u.b. $\{f(x) : x \in [a, b]\}$. We therefore conclude that there exists $x_{\in}[a, b]$ such that $f(x_2) = M$.

    Let $m :=$ g.l.b. $\{f(x) : a \leq x \leq b\}$. Arguing similarly we can find an $x_1 \in [a, b]$ such that $f(x_1) = m$. □

    Second proof. Since $M - \frac{1}{n} < M$, there exists $x_n \in J$ such that $M - \frac{1}{n} < f(x_n) \leq M$. Hence $f(x_n) \to M$. Since J is compact, there exists a subsequence (x_{n_k}) such that x_{n_k} converges to some $x \in J$. By continuity of f at x, $f(x_{n_k}) \to f(x)$. Conclude that $f(x) = M$. □

29. Look at the examples: (i) $f\colon (0,1] \to \mathbb{R}$ given by $f(x) = 1/x$. This is a continuous unbounded function. The interval here though bounded is not closed at the end points. (ii) $f\colon (-1,1) \to \mathbb{R}$ defined by $f(x) := \frac{1}{1-|x|}$. (iii) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$.

    None of these examples 'contradict' Item 25.

30. Exercises:

    (a) Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Show that $f([a,b]) = [c,d]$ for some $c, d \in \mathbb{R}$ with $c \le d$. Can you "identify" c, d?

    (b) Does there exist a continuous function $f\colon [0,1] \to (0,\infty)$ which is onto?

    (c) Does there exist a continuous function $f\colon [a,b] \to (0,1)$ which is onto?

    (d) Let $f\colon [a,b] \to \mathbb{R}$ be continuous such that $f(x) > 0$ for all $x \in [a,b]$. Show that there exists δ such that $f(x) > \delta$ for all $x \in [a,b]$.

    (e) Construct a continuous function from $(0,1)$ *onto* $[0,1]$. Can such a function be one-one?

    (f) Let $f\colon [a,b] \to \mathbb{R}$ be continuous such that $f(x) > 0$ for all $x \in [a,b]$. Show that there exists δ such that $f(x) > \delta$ for all $x \in [a,b]$.

    (g) Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Assume that $f(x) \to 0$ as $|x| \to \infty$. (Do you understand this?) Show that there exists $c \in \mathbb{R}$ such that either $f(x) \le f(c)$ or $f(x) \ge f(c)$ for all $x \in \mathbb{R}$. Give an example of a function in which only one of these happens.

    (h) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that (i) $f(\mathbb{R}) \subset (-2,-1) \cup [1,5)$ and (ii) $f(0) = e$. Can you give 'realistic bounds' for f?

Monotone Functions

31. We say that a function $f\colon J \subset \mathbb{R} \to \mathbb{R}$ is strictly increasing if for all $x, y \in J$ with $x < y$, we have $f(x) < f(y)$.

    One defines strictly decreasing in a similar way. A monotone function is either strictly increasing or strictly decreasing.

    We shall formulate and prove the results for strictly increasing functions. Analogous results for decreasing functions f can be arrived at in a similar way or by applying the result for the increasing functions to $-f$.

32. Let $J \subset \mathbb{R}$ be an interval. Let $f\colon J \to \mathbb{R}$ be continuous and 1-1. Let $a, c, b \in J$ be such that $a < c < b$. Then $f(c)$ lies between $f(a)$ and $f(b)$, that is either $f(a) < f(c) < f(b)$ or $f(a) > f(c) > f(b)$ holds.

    Proof. Since f is one-one, we assume without loss of generality that $f(a) < f(b)$. If the result is false, either $f(c) < f(a)$ or $f(c) > f(b)$.

    Let us look at the first case. Then the value $y = f(a)$ lies between the values $f(a)$ and $f(c)$ at the end points of $[a,c]$. Since $f(c) < f(a) < f(b)$, $y = f(a)$ also lies between the values of f at the end points of $[c,b]$. Hence there exists $x \in (c,b)$ such that $f(x) = y = f(a)$. Since $x > a$, this contradicts the fact that f is one-one.

In case, you did not like the way we used y, you may proceed as follows. Fix any y such that $f(c) < y < f(a)$. By intermediate value theorem applied to the pair $(f, [a, c])$, there exists $x_1 \in (a, c)$ such that $f(x_1) = y$. Since $f(a) < f(b)$, we also have $f(c) < y < f(b)$. Hence there exists $x_2 \in (c, b)$ such that $f(x_2) = y$. Clearly $x_1 \neq x_2$.

The second case when $f(c) > f(b)$ is similarly dealt with. □

33. **Theorem.** *Let $J \subset \mathbb{R}$ be an interval. Let $f: J \to \mathbb{R}$ be continuous and 1-1. Then f is monotone.*

    Proof. Fix $a, b \in J$, say with $a < b$. We assume without loss of generality that $f(a) < f(b)$. We need to show that for all $x, y \in J$ with $x < y$ we have $f(x) < f(y)$.

    (i) If $x < a$, then $x < a < b$ and hence $f(x) < f(a) < f(b)$.

    (ii) If $a < x < b$, then $f(a) < f(x) < f(b)$.

    (iii) If $b < x$, then $f(a) < f(b) < f(x)$.

    $$\text{In particular, } f(x) < f(a) \text{ if } x < a \text{ and } f(x) > f(a) \text{ if } x > a. \tag{6}$$

    If $x < a < y$, then $f(x) < f(a) < f(y)$ by (6).

    If $x < y < a$, then $f(x) < f(a)$ by (6) and $f(x) < f(y) < f(a)$ by the last item.

    If $a < x < y$, then $f(a) < f(y)$ by (6) and $f(a) < f(x) < f(y)$ by the last item.

    Hence f is strictly increasing. □

34. We observed that the intermediate value theorem says that the image of an interval under a continuous function is an interval.

    What is the converse of this statement? The converse is in general not true. A partial converse is found in the next item.

35. **Proposition.** *Let J be an interval and $f: J \to \mathbb{R}$ be monotone. Assume that $f(J) = I$ is an interval. Then f is continuous.*

    Proof. We deal with the case when f is strictly increasing. Let $a \in J$. Assume that a is not an endpoint of J. We prove the continuity of f at a using the ε-δ definition.

    Since a is not an endpoint of J, there exists $x_1, x_2 \in J$ such that $x_1 < a < x_2$ and hence $f(x_1) < f(a) < f(x_2)$. It follows that there exists $\eta > 0$ such that $(f(a) - \eta, f(a) + \eta) \subset (f(x_1), f(x_2)) \subset I$.

    Let $\varepsilon > 0$ be given. We may assume $\varepsilon < \eta$. Let $s_1, s_2 \in J$ be such that $f(s_1) = f(a) - \varepsilon$ and $f(s_2) = f(a) + \varepsilon$. Let $\delta := \min\{a - s_1, s_2 - a\}$. If $x \in (a - \delta, a + \delta) \subset (s_1, s_2)$, then, $f(a) - \varepsilon = f(s_1) \leq f(x) < f(s_2) = f(a) + \varepsilon$, that is, if $x \in (a - \delta, a + \delta)$, then $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$.

    If a is an endpoint of J, an obvious modification of the proof works. □

36. Let $f: J \to \mathbb{R}$ be an increasing continuous function on an interval J. Then $f(J)$ is an interval, $f: J \to f(J)$ is a bijection and the inverse $f^{-1}: f(J) \to J$ is continuous.

37. Consider the n-th root function $f: [0, \infty) \to [0, \infty)$ given by $f(x) := x^{1/n}$. We can use the last item to conclude that f is continuous, a fact seen by us in Item 10g.

38. Exercise: Let $f\colon \mathbb{R} \to \mathbb{R}$ be an additive homomorphism. If f is monotone, then $f(x) = f(1)x$ for all $x \in \mathbb{R}$.

Limits

39. Let $J \subset \mathbb{R}$ be an interval. Let $a \in J$. Assume that $f\colon J \setminus \{a\} \to R$ be *any* function. We say that $\lim_{x\to a} f(x)$ exists if there exists $\ell \in \mathbb{R}$ such that for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |f(x) - \ell| < \varepsilon.$$

Note that a need not be in the domain of f. Even if a lies in the domain of f, ℓ need not be $f(a)$. We let $\lim_{x\to a} f(x)$ stand for ℓ and call ℓ as the limit of f as $x \to a$.

40. With the notation of the last item the 'limit' ℓ is unique.

41. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{x^2-4}{x-2}$ for $x \neq 2$ and $f(2) = e$. Then $\lim_{x\to 2} f(x) = 4$. We proved this using ε-δ definition.

42. **Theorem.** *Let $J \subset \mathbb{R}$ be an interval. Let $a \in J$. Assume that $f\colon J \setminus \{a\} \to R$ be* any *function. Then $\lim_{x\to a} f(x) = \ell$ iff for* **every** *sequence (x_n) with $x_n \in J \setminus \{a\}$ with the property that $x_n \to a$, we have $f(x_n) \to \ell$.*

The proof is quite similar to that in Item 9. We still went through the proof. □

43. Let $f\colon J \to \mathbb{R}$ be given $c \in J$. Is there any relation between $\lim_{x\to c} f(x)$ and the continuity of f at c?

Theorem. *Let $J \subset \mathbb{R}$ be an interval and $c \in J$. Then f is continuous at c iff $\lim_{x\to c} f(x)$ exists and the limit is $f(c)$.* □

44. Formulate results analogous to algebra of convergent sequences and algebra of continuous functions. Do you 'see' proofs of them in your mind?

45. Can you think of a result on the existence of a limit for a composition of functions?

If $\lim_{x\to a} f(x) = \alpha$ and if g is defined in an interval containing α and is continuous at α, then $\lim_{x\to a}(g \circ f)(x)$ exists and it is $g(\alpha)$. (Compare Item 7.)

46. How to define one sided limits such as $\lim_{x\to a+} f(x)$?

47. What is the relation between the one sided limits $\lim_{x\to a+} f(x)$, $\lim_{x\to a-} f(x)$ and the limit $\lim_{x\to a} f(x)$?

48. How to assign a meaning to the *symbol* $\lim_{x\to\infty} f(x) = \ell$ for a function $f\colon (\delta, \infty) \to \mathbb{R}$?

How to assign a meaning to $\lim_{x\to-\infty} f(x) = \ell$ for a function $f\colon (-\infty, \delta) \to \mathbb{R}$?

49. How to assign a meaning to the *symbol* $\lim_{x\to a} f(x) = \infty$? *Hint:* Recall how we defined a sequence diverging to infinity (in Item 35).

50. How to assign a meaning to the *symbol* $\lim_{x\to\infty} f(x) = \infty$? And so on!

51. Some of the examples we looked at:

(a) $f\colon \mathbb{R}^* \to \mathbb{R}$ given by $f(x) := x/|x|$. $\lim_{x\to 0+} f(x) = 1$ and $\lim_{x\to 0-} f(x) = -1$.

(b) $\lim_{x\to 0} f(x) = 0$ where $f(x) = |x|$ if $x \neq 0$ and $f(0) = 23$.

(c) $\lim_{x\to 0} \frac{1}{x^2} = 0$.

(d) $\lim_{x\to 0+} \frac{1}{x} = \infty$ and $\lim_{x\to 0-} \frac{1}{x} = -\infty$.

(e) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{(-1)^n}{n} \sin(\pi x)$ for $x \in [n, n+1)$. Then $\lim_{x\to \pm\infty} f(x) = 0$.

52. Using the equation (5) in Item 22, we see that for an odd degree polynomial $P(X)$ with leading coefficient 1

$$\lim_{x\to\infty} P(X) = \infty \quad \text{and} \quad \lim_{x\to -\infty} P(X) = -\infty.$$

53. Exercises:

(a) Find the limits using the ε-δ definition.

i. $\lim_{x\to a} \frac{x^3 - a^3}{x-a}$.

ii. $\lim_{x\to 0} x \sin(1/x)$.

iii. $\lim_{x\to 2} \frac{x^2-4}{x^2-2x}$.

iv. $\lim_{x\to 6} \sqrt{x+3}$.

(b) Let $J \subset \mathbb{R}$ be an interval. Assume that $a \in J$ and that $f\colon J \setminus \{a\} \to \mathbb{R}$ is such that $\lim_{x\to a} f(x) = \ell$. If we define $f(a) = \ell$, then f is continuous at a.

A typical and standard example is $f\colon \mathbb{R}^* \to \mathbb{R}$ given by $f(x) := \frac{\sin x}{x}$. It is 'well-known' that, $\lim_{x\to 0} f(x) = 1$. Hence if we define $g(x) := f(x)$ for $x \neq 0$ and $g(0) = 1$, then $g\colon \mathbb{R} \to \mathbb{R}$ is continuous.

54. We shall have a closer look at the relation between the existence of one sided limits and the continuity in the case of an increasing function in the next few Items.

Look at the graphs of the following increasing functions. Do you see what happens at the points of discontinuity?

(a) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) := [x]$.

(b) $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) := [x]$ for $x \notin \mathbb{Z}$ and $f(x) = x + 91/2)$ if $x \in \mathbb{Z}$.

55. Let $J \subset \mathbb{R}$ be an interval and $f\colon J \to \mathbb{R}$ be increasing. Assume that $c \in J$ is not an endpoint of J. Then
(i) $\lim_{x\to c_-} f = \text{l.u.b. } \{f(x) : x \in J; x < c\}$.
(ii) $\lim_{x\to c_+} f = \text{g.l.b. } \{f(x) : x \in J; x > c\}$.

56. Let the hypothesis be as in the last item. Then the following are equivalent:
(i) f is continuous at c.
(ii) $\lim_{x\to c_-} f = f(c) = \lim_{x\to c_+} f$.
(iii) $\text{l.u.b. } \{f(x) : x \in J; x < c\} = f(c) = \text{g.l.b. } \{f(x) : x \in J; x > c\}$.

What is the formulation if c is an endpoint of J?

57. Let $J \subset \mathbb{R}$ be an interval and $f\colon J \to \mathbb{R}$ be increasing. Assume that $c \in J$ is not an endpoint of J. The jump at c is defined as

$$j_f(c) := \lim_{x \to c_+} f - \lim_{x \to c_-} f \equiv \text{g.l.b. } \{f(x) : x \in J; x > c\} - \text{l.u.b. } \{f(x) : x \in J; x < c\}.$$

How is the jump $j_f(c)$ defined if c is an endpoint?

58. Let $J \subset \mathbb{R}$ be an interval and $f\colon J \to \mathbb{R}$ be increasing. Then f is continuous at $c \in J$ iff $j_f(c) = 0$. □

59. **Theorem.** *Let $J \subset \mathbb{R}$ be an interval and $f\colon J \to \mathbb{R}$ be increasing. Then the set D of points of J at which f is discontinuous is countable.*

Proof. Assume that f is increasing. Then $c \in J$ belongs to D iff the interval $J_c := (f(c_-), f(c_+))$ is nonempty. For, $c, d \in D$, with, say, $c < d$, the intervals J_c and J_d are disjoint. (Why?) Students have problem here.

> Let $c < t < d$. Since $f(c+) \equiv \text{g.l.b. } \{f(x) : x > c\}$, and since $t > c$, we see that $f(c+) \le f(t)$. Similarly, $f(d-) \equiv \text{l.u.b. } \{f(y) : y < d\}$. Since $t < d$, we see that $f(t) \le f(d-)$. Thus, $f(c+) \le f(t) \le f(d-)$.

Thus the collection $\{J_c : c \in D\}$ is a pairwise disjoint family of open intervals. Such a collection is countable. For, choose $r_c \in J_c \cap \mathbb{Q}$. Then the map $c \mapsto r_c$ from D to $\mathbb{Q}$ is one-one. □

Uniform Continuity

60. Let $J \subset \mathbb{R}$ be any subset. A function $f\colon J \to \mathbb{R}$ is *uniformly continuous* **on** J if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$x_1, x_2 \in J \text{ with } |x_1, x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon.$$

61. Unlike continuity, uniform continuity is a *global* concept.

The notion of uniform continuity of f gives us control on the variation of the images of a pair of points which are close to each other independent of where they lie in the space X. Go through Example 62 and Example 63. In these two examples, it is instructive to draw the graphs of the functions under discussion and try to understand the remark above.

62. Let $\alpha > 0$. Let $f\colon (0, \infty) \to \mathbb{R}$ be given by $f(x) = 1/x$. Then f is uniformly continuous on (α, ∞) but not on $(0, \infty)$.

$$|f(x) - f(y)| = \frac{|x-y|}{xy} \le \frac{|x-y|}{\alpha^2}.$$

The function $g\colon (0, \infty) \to \mathbb{R}$ given by $g(x) = 1/x$ is not uniformly continuous. Assume the contrary. Look at the graph of f near $x = 0$. You will notice that if x and y are very close to each other and are also very near to 0 (which is not in the domain of f, though), their values vary very much. This suggests us a method of attack. If f is uniformly continuous on $(0, \infty)$, then for $\varepsilon = 1$, there exists $\delta > 0$. Choose $N > 1/\delta$. Let $x = 1/N$ and $y = 1/2N$. Then $|f(x) - f(y)| \ge 1$.

63. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Let A be any bounded subset of $\mathbb{R}$, say, $A = [-R, R]$. Then f is uniformly continuous on A but not on $\mathbb{R}$! If you look at the graph of f, you will notice that if x is very large (that is, near to ∞), and if y is very near to x, the variations i $f(x)$ and $f(y)$ become large. If f were uniformly continuous on $\mathbb{R}$, then for $\varepsilon = 1$ we can find a δ as in the definition. Choose N so that $N > 1/\delta$. Take $x = N$ and $y = N + 1/N$. Then $|f(x) - f(y)| \geq 2$. If $x, y \in [-R, R]$, then

$$|f(x) - f(y)| = |x + y|\,|x - y| \leq 2R\,|x - y|\,,$$

which establishes the uniform continuity of f on $[-R, R]$.

64. Any Lipschitz function is uniformly continuous. (See Item 11.)

65. Let $J \subset$ be an interval. Let $f\colon J \to R$ be differentiable with bounded derivative, that is, $|f'(x)| \leq L$ for some $L > 0$. Then f is Lipschitz with Lipschitz constant L. It particular, f is uniformly continuous on J. *Hint:* Recall the mean value theorem.

    Specific examples: $f(x) = \sin x$, $g(x) = \cos x$ are Lipschitz on $\mathbb{R}$. The inverse of tan, $\tan^{-1}\colon (-\pi/2, \pi/2) \to \mathbb{R}$ is Lipschitz.

66. Let $f\colon J \to \mathbb{R}$ be uniformly continuous. Then f maps Cauchy sequences in J to Cauchy sequences in $\mathbb{R}$.

    Given $\varepsilon > 0$, choose δ by uniform continuity of f. Since (x_n) is Cauchy, we have n_0 for this δ. This n_0 will do to establish that $(f(x_n))$ is Cauchy in Y.

67. The converse of the proposition is not true. *Hint:* Consider $f(x) = x^2$ on $\mathbb{R}$. Recall that any Cauchy sequence is bounded.

68. Let J be compact. Then any continuous function $f\colon J \to \mathbb{R}$ is uniformly continuous.

    Proof. If f is not uniformly continuous, then there exists $\varepsilon > 0$ such that for all $1/n$ we can find $a_n, b_n \in J$ such that $|a_n - b_n| < 1/n$ but $|f(a_n) - f(b_n)| \geq \varepsilon$. Since J is compact, there exists a subsequence (a_{n_k}) such that $a_{n_k} \to a \in J$. It is easily seen that $b_{n_k} \to a$. By continuity, $f(a_{n_k}) \to f(a)$ and also $f(b_{n_k}) \to f(a)$. In particular, for all sufficiently large k, we must have $|f(a_{n_k}) - f(b_{n_k})| < \varepsilon$, a contradiction. □

69. Exercises:

    (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and periodic with period p: $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. Show that f is uniformly continuous. (Examples are sin and cos with period $p = 2\pi$.)

    (b) Let $f(x) = \frac{1}{x+1}\cos x^2$ on $[0, \infty)$. Show f is uniformly continuous.

    (c) Let $f(x) = x^{1/2}$ on $[0, \infty)$. Is f uniformly continuous?

    (d) Let f be uniformly continuous on $[a, c]$ and also on $[c, b]$. Show that it is uniformly continuous on $[a, b]$.

    (e) Show that $f(x) = \frac{|\sin x|}{x}$ is uniformly continuous on $(-1, 0)$ and $(0, 1)$ but not on $(-1, 0) \cup (0, 1)$.

    (f) If $\emptyset \neq A \subseteq \mathbb{R}$, show that $f = d_A$ is uniformly continuous where $d_A(x) :=$ g.l.b. $\{d(x, a) : a \in A\}$. See Item 3k.

(g) A function $f\colon \mathbb{R} \to \mathbb{R}$ is uniformly continuous iff whenever (x_n) and (y_n) are sequences of $\mathbb{R}$ such that $|x_n - y_n| \to 0$ we have $|f(x_n) - f(y_n)| \to 0$.

(h) Let $f\colon B \subset \mathbb{R} \to \mathbb{R}$ be uniformly continuous on a bounded set B. Show that $f(B)$ is bounded.

(i) Let $f\colon J \subset \mathbb{R} \to \mathbb{R}$ be uniformly continuous with $|f(x)| \geq \eta > 0$ for all $x \in X$. Then $1/f$ is uniformly continuous on J. *Hint:* Adapt the argument in Example 62.

(j) Let $f(x) := \sqrt{x}$ for $x \in [0,1]$. Then f is uniformly continuous but not Lipschitz on $[0,1]$. *Hint:* Can there exist an $L > 0$ such that $|f(x)| \leq L\,|x|$ for all $x \in [0,1]$?

(k) Check for uniform continuity of the functions on their domains:
   (a) $f(x) := \sin(1/x)$, $x \in (0,1]$.
   (b) $g(x) := x\sin(1/x)$, $x \in (0,1]$.

4 Differentiation

1. The basic idea of differential calculus (as perceived by modern mathematics) is to 'approximate' at a point a given function by an affine (linear) function (or a first degree polynomial).

2. Let J be an interval and $c \in J$. Let $f: J \to\to \mathbb{R}$ be given. We wish to approximate $f(x)$ for x near 0 by a polynomial of the form $a+b(x-c)$. To keep the notation simple, let us assume $c = 0$. What is meant by 'approximation'? If $E(x) := f(x) - a - bx$ is the error by taking the value of $f(x)$ as $a + bx$ near 0, what we want is that the error goes to zero much faster than x going to zero. As we have seen earlier this means that $\lim_{x\to 0} \frac{f(x)-a-bx}{x} = 0$.

   If this happens, then it is easy to see that $a = f(0)$. Hence the requirement is that there exists a real number b such that $\lim_{x\to 0} \frac{f(x)-f(0)}{x} = b$.

   If such is the case, we say that f is *differentiable* at $c = 0$ and denote the (unique) real number b by $f'(0)$. It is called the derivative of f at 0.

3. We say that f is differentiable at $c \in J$ if we can approximate the increment $f(x)-f(c) \equiv f(c+h) - f(c)$ in the dependent variable by a linear polynomial $\alpha(x-c) = \alpha h$ in the increment. Approximation here means that the 'error' should go to zero much faster than the increment going to zero.

4. In terms of ε-δ, we say that f is differentiable at c if there exists $\alpha \in \mathbb{R}$ such that for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$x \in J \text{ and } 0 < |x-c| < \delta \implies |f(x) - f(c) - \alpha(x-c)| < \varepsilon|x-c|. \qquad (7)$$

5. Examples.

   (a) Let $f: J \to \mathbb{R}$ be a constant, say, C. Then f is differentiable at $c \in J$ with $f'(c) = 0$.

   (b) $f: J \to \mathbb{R}$ be given by $f(x) = x$. Then $f'(c) = 1$ for $c \in J$. More generally, if $f(x) = ax+b$, then $f'(c) = a$ for $c \in J$.

   (c) If $f: J \to \mathbb{R}$ is given by $f(x) = x^n$, $n \in \mathbb{N}$, then $f'(c) = nc^{n-1}$. For, note that

$$f(c+h) - f(c) = (c+h)^n - c^n = nc^{n-1}h + \text{ terms involving higher powers of } h.$$

6. **Theorem.** *Let $f: J \to \mathbb{R}$ be given. Then f is differentiable at $c \in J$ iff there exists a function $f_1: J \to \mathbb{R}$ continuous at c such that*

$$f(x) = f(c) + f_1(x)(x-c) \text{ for } x \in J. \qquad (8)$$

   In such a case, $f'(c) = f_1(c)$.

   Proof. Assume that f is differentiable at c. Define

$$f_1(x) := \begin{cases} \frac{f(x)-f(c)}{x-c} & \text{for } x \in J \text{ and } x \neq c \\ f'(c) & \text{if } x = c. \end{cases}$$

Complete the proof. □

In spite of its simplicity, this is a very powerful characterization of differentiability at a point. We illustrate its use in the next few items.

7. Let $f(x) = 1/x$ for $x > 0$. Then $f'(c) = -1/c^2$. For,

$$f(x) - f(x) = \frac{-(x-c)}{f(x)f(c)} = f_1(x)(x-c),$$

where $f_1(x) = \frac{-1}{f(x)f(c)}$.

8. Let $f(x) = e^x$, $x \in \mathbb{R}$. Using the standard facts about the exponential function, we show that $f'(c) = e^c$.

$$f(c+h) - f(c) = e^c(e^h - 1) = e^c \left(h \sum_{k=1}^{\infty} \frac{h^{n-1}}{n!} \right).$$

9. Let $f(x) = x^n$ for $x \in \mathbb{R}$. Then

$$f(x) - f(c) = (x-c)f_1(x), \text{ where } f_1(x) = x^{n-1} + \cdots + c^{n-1}.$$

It is clear that f_1 is continuous at $x = c$ and that $f_1(c) = nc^{n-1}$.

10. If f is differentiable at c, then f is continuous at c.

Observe that the RHS of $f(x) = f(c) + f_1(x)(x-c)$ is continuous at c.

11. **Algebra of differentiable functions.** Let $f, g \colon J \to \mathbb{R}$ be differentiable at $c \in J$. Then
(a) $f + g$ is differentiable at c with $(f+g)'(c) = f'(c) + g'(c)$.
(b) αf is differentiable at c with $(\alpha f)'(c) = \alpha f'(c)$.
(c) fg is differentiable at c with $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$.
(d) If f is differentiable at c with $f(c) \neq 0$, then $\varphi := 1/f$ is differentiable at c with $\varphi'(c) = -\frac{f'(c)}{(f(c))^2}$.

12. Exercises:

(a) Show that $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ is not differentiable at $x = 0$.

(b) Let $f \colon \mathbb{R} \to \mathbb{R}$ be such that $|f(x) - f(y)| \leq (x-y)^2$ for all x, y. Show that f is differentiable, the derivative is zero. Hence conclude that f is a constant.

(c) Let $f \colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. Show that f is differentiable at $x = 0$. Find $f'(0)$.

(d) Show that $f(x) = x^{1/3}$ is not differentiable at $x = 0$.

(e) Let $n \in \mathbb{N}$. Define $f \colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x^n$ for $x \geq 0$ and $f(x) = 0$ if $x < 0$. For which values of n,

i. is f continuous at 0?
ii. is f differentiable at 0?
iii. is f' continuous at 0?

iv. is f' differentiable at 0?

(f) Let $f\colon J \to \mathbb{R}$ be differentiable. Let $x_n \le c \le y_n$ be such that $y_n - x_n \to 0$. Show that

$$\lim_n \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(c).$$

13. **Chain Rule.** If $f(J) \subset J_1$, an interval and if $g\colon J_1 \to \mathbb{R}$ is differentiable at $f(c)$, then $g \circ f$ is differentiable at c with $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

    We discussed the proofs of the last two items and said that you would write the proofs on your own. So, I am not sketching the proofs here!

14. Let $D_a(J)$ (respectively $C_a(J)$) denote the set of functions on J differentiable (respectively continuous) at a. Then $D_a(J)$ is a vector subspace of $C_a(J)$.

15. We shall assume that the properties of the following functions and their derivatives are known.

    (a) $f(x) = e^x$, $f'(x) = e^x$.
    (b) $f(x) = \log x$, $x > 0$, and $f'(x) = 1/x$.
    (c) $f(x) = x^\alpha$ where $x > 0$ and $\alpha \in \mathbb{R}$. Then $f'(x) = \alpha x^{\alpha-1}$.
    (d) $f(x) = \sin x$, $f'(x) = \cos x$.
    (e) $f(x) = \cos x$ and $f'(x) = -\sin x$.

16. Exercise: Let $r \in \mathbb{Q}$. Define $f(x) = x^r \sin(1/x)$ for $x \ne 0$ and $f(0) = 0$. For what values of r, is f differentiable at 0?

17. Let $J \subset \mathbb{R}$ be an interval and $f\colon J \to \mathbb{R}$ be a function. We say that a point $c \in J$ is a point of *local maximum* if there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subset J$ and $f(x) \le f(c)$ for all $x \in (c - \delta, c + \delta)$.

    A local minimum is defined similarly.

    A point $x_0 \in J$ is said to be a point of (global) maximum if $f(x) \le f(x_0)$ for all $x \in J$. Global minimum is defined similarly.

18. Look at $f\colon [a, b] \to \mathbb{R}$ where $f(x) = x$. Then b is a point of global maximum but NOT a local maximum. What can you say about a?

    On the other hand, look at $g\colon [-2\pi, 2\pi] \to \mathbb{R}$ defined by $g(x) = \cos x$. The point $x = 0$ is a local maximum as well as a global maximum. What can you say about the points $x = \pm 2\pi$?

19. **Theorem.** *Let $J \subset \mathbb{R}$ be an interval. Let $c \in J$ be a local maximum for a differentiable function $f\colon J \to \mathbb{R}$. Then $f'(c) = 0$.*

    Similar result holds for local minima as well.

    Proof. Compare the two one-sided limits of the difference quotients:

    $$f'(c) = \lim_{h\to 0+} \frac{f(c+h) - f(c)}{h} \le 0 \quad\text{and}\quad f'(c) = \lim_{h\to 0-} \frac{f(c+h) - f(c)}{h} \ge 0.$$

    Question. Where did we use the fact that c is a local maximum in the proof? Compare the result with the function of f of Item 18. What is $f'(b)$? □

20. **Rolle's Theorem.** *Let* $f\colon [a,b] \to \mathbb{R}$ *be such that (i)* f *is continuous on* $[a,b]$*, (ii)* f *is differentiable on* (a,b) *and (iii)* $f(a) = f(b)$*. Then there exists* $c \in (a,b)$ *such that* $f'(c) = 0$.

    The geometric interpretation is that there exists $c \in (a,b)$ such that the slope of the tangent to the graph of f at c equals zero.

    Proof. By Weierstrass theorem, there exists $x_1, x_2 \in [a,b]$ such that $f(x_1) \le f(x) \le f(x_2)$ for all $x \in [a,b]$. If $f(x_1) = f(x_2)$, then the result is trivial. (Why?) If not at least one of x_1, x_2 is different from a and b. (Why?) If $x \neq a$ and $x_2 \neq b$, then take $c = x_2$ and apply the last result. □

21. **Mean Value Theorem.** *Let* $f\colon [a,b] \to \mathbb{R}$ *be such that (i)* f *is continuous on* $[a,b]$ *and (ii)* f *is differentiable on* (a,b) *Then there exists* $c \in (a,b)$ *such that*

    $$f(b) - f(a) = f'(c)(b-a). \tag{9}$$

    Geometric interpretation: There exists c such that the slope of the tangent to the graph of f at c equals that of the chord joining the two points $(a, f(a)$ and $(b, f(b))$.

    Proof. Consider $g = f(x) - \ell(x)$, where $\ell(x) := f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$, the function defining the chord. Apply the MVT. □

22. The mean value theorem is the single most important result in the theory of differentiation. Below are two typical applications.

    (a) Let J be an interval and $f\colon J \to \mathbb{R}$ be differentiable with $f' = 0$. Then f is a constant on J.
    For any $x, y \in J$, by MVT, we have $f(y) - f(x) = f'(z)(y-x)$ for some z between x and y. Hence, $f(y) = f(x)$ for all $x, y \in J$.
    While discussing this, we saw that we an define the concept of differentiability of a function f defined on a set U which is the union of open intervals also. However, it may happen that $f\colon U \to \mathbb{R}$ is differentiable with $f' = 0$ on U but f is not a constant. For instance, consider $U = (-1,0) \cup (0,1)$ and $f(x) = -1$ on $(-1,0)$ and $f(x) = 1$ on $(0,1)$.

    (b) Let J be an interval and $f\colon J \to \mathbb{R}$ be differentiable with $f'(x) > 0$ for $x \in J$. Then f is increasing on J.
    For any $x, y \in J$ with $x < y$ by MVT, we have $f(y) - f(x) = f'(z)(y-x)$ for some z between x and y. The right side is positive.
    What is the corresponding result when $f'(x) < 0$ for $x \in J$?

23. Another application was already seen while discussing uniform continuity, see Item 65.

24. Mean value theorem is quite useful in proving inequalities. Here are some samples.

    (a) $e^x > 1 + x$ for all $x \in \mathbb{R}$. (Consider $f\colon [0,x] \to \mathbb{R}$ where $f(x) = e^x$. If $x < 0$, then consider the interval $[x, 0]$.)

    (b) $e^x > ex$. (Consider $f(x) = e^x$ on $[1, x]$.)

(c) $\frac{y-x}{y} < \log\frac{y}{x} < \frac{y-x}{x}$, $0 < x < y$. (Consider $f(x) = \log x$ on $[x, y]$.)

(d) $\frac{x}{1+x} < \log(1+x) < x$, $x > 0$. (Consider $f(t) := \log(1+t)$ on $[0, x]$.)

(e) $n(b-a)a^{n-1} < b^n - a^n < n(b-a)b^{n-1}$, $0 < a < b$. (Consider $f(t) = t^n$ on $[a, b]$.)

25. Which is greater e^π or π^e? We prove a more general inequality which answers this question: if $e \le a < b$, then $a^b > b^a$.

Using Item 24c, we see that

$$\frac{b-a}{b} < \log(b/a) < \frac{b-a}{a}.$$

We have $a\log(b/a) < b - a$. Hence

$$e^{a\log(b/a)} < e^{b-a} < a^{b-a} \implies (b/a)^a < a^b/a^a \implies b^a < a^b.$$

26. **Cauchy's Form of MVT.** Let $f, g\colon [a, b] \to \mathbb{R}$ be differentiable. Assume that $g'(x) \neq 0$ for any $x \in (a, b)$. Then there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \tag{10}$$

Proof. Note that $g(a) \neq g(b)$. (Why?) Let $h() := f(x) - \lambda g(x)$ where $\lambda \in \mathbb{R}$ is chosen so that $h(b) = h(a)$. Then $\lambda = \frac{f(b)-f(a)}{g(b)-g(a)}$. Apply Rolle's theorem and complete the proof. □

27. Exercises:

(a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable such that $|f'(x)| \le M$ for some $M > 0$ for all $x \in \mathbb{R}$.

i. Show that f is uniformly continuous on $\mathbb{R}$.

ii. If $\varepsilon > 0$ is sufficiently small, then show that the function $g_\varepsilon(x) := x + \varepsilon f(x)$ is one-one.

(b) Let $f\colon (a, b) \to \mathbb{R}$ be differentiable at $x \in (a, b)$. Prove that

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x).$$

Give an example of a function where the limit exists but the function is not differentiable.

(c) Let $f\colon [0, 2] \to \mathbb{R}$ be given by $f(x) := \sqrt{2x - x^2}$. Show that f satisfies the conditions of Rolle's theorem. Find a c such that $f'(c) = 0$.

(d) Use MVT to establish the following inequalities:

i. Let $b > a > 0$. Show that $b^{1/n} - a^{1/n} < (b-a)^{1/n}$. *Hint:* Consider $f(x) = x^{1/n} - (x-1)^{1/n}$ for suitable value of x.

ii. Show that

$$|\sin x - \sin y| \le |x - y|.$$

iii. Show that

$$nx^{n-1}(y-x) \le y^n - x^n \le ny^{n-1}(y-x) \text{ for } 0 \le x \le y.$$

iv. Show that $ex \le e^x$ for all $x \in \mathbb{R}$.

v. Show that $e^x > 1 + x$ for all $x > 0$.

vi. **Bernoulli's Inequality.** Let $\alpha > 0$ and $h \ge -1$. Then

$$(1+h)^\alpha \le 1 + \alpha h, \text{ for } 0 < \alpha \le 1, \quad (11)$$

$$(1+h)^\alpha \ge 1 + \alpha h, \text{ for } \alpha \ge 1. \quad (12)$$

(e) Assume that : $(a,b) \to \mathbb{R}$ is differentiable on except possibly at $c \in (a,b)$. Assume that $\lim_{x\to c} f'(x)$ exists. Prove that $f'(c)$ exists and f' is continuous at c.

(f) Show that the function $f(x) = x^3 - 3x^2 + 17$ is not 1-1 on the interval $[-1,1]$.

(g) Prove that the equation $x^3 - 3x + b = 0$ has at most one root in the interval $[-1,1]$.

(h) Show that $\cos x = x^3 + x^2 + 4x$ has exactly one root in $[0, \pi/2]$.

(i) Let $f(x) = x + 2x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that $f'(0) = 1$ but f is not monotonic in any interval around 0.

(j) Let J be an open interval and $f, g\colon J \to \mathbb{R}$ be differentiable. Assume that $f(a) = 0 = f(b)$ for $a, b \in J$ with $a < b$. Show that $f'(c) + f(c)g'(c) = 0$ for some $c \in (a,b)$.

(k) Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be differentiable. Assume that $f(0) = g(0)$ and $f'(x) \le g'(x)$ for all $x \in \mathbb{R}$. Show that $f(x) \le g(x)$ for $x \ge 0$.

(l) Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable. Assume that $1 \le f'(x) \le 2$ for $x \in \mathbb{R}$ and $f(0) = 0$. Prove that $x \le f(x) \le 2x$ for $x \ge 0$.

(m) Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be differentiable. Let $a \in \mathbb{R}$. Define $h(x) = f(x)$ for $x < a$ and $h(x) = g(x)$ for $x \ge a$. Find necessary and sufficient conditions which will ensure that h is differentiable at a. (This is a gluing lemma for differentiable functions.)

(n) Let $f\colon [2,5] \to \mathbb{R}$ be continuous and be differentiable on (2,5). Assume that $f'(x) = (f(x))^2 + \pi$ for all $x \in (a,b)$. True or false: $f(5) - f(2) = 3$.

(o) If $f'(x) \to \ell$ as $x \to \infty$, then show that $f(x)/x \to \ell$ as $x \to \infty$.

(p) If the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, $a_n \neq 0$, ($a_j \in \mathbb{R}$ for $0 \le j \le n$) has only real roots, then its derivative $P'(x)$ also has only real roots.

(q) Let $f\colon (a,b) \to \mathbb{R}$ be differentiable. Assume that $\lim_{x\to a+} f(x) = \lim_{x\to b-} f(x)$. Show that there exists $c \in (a,b)$ such that $f'(c) = 0$.

(r) Let $f\colon (0,1] \to \mathbb{R}$ be differentiable with $|f'(x)| < 1$. Define $a_n := f(1/n)$. Show that (a_n) converges.

(s) Let f be continuous on $[0,1]$, differentiable on $(0,1)$ with $f(0) = 0$. Prove that if f' is increasing on $(0,1)$, then $g(x) := f(x)/x$ is increasing on $(0,1)$.

(t) Let $f\colon [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b). Assume further that $f(a) = f(b) = 0$. Prove that for any given $\lambda \in \mathbb{R}$, there exists $c \in (a,b)$ such that $f'(c) = \lambda f(c)$.

(u) Show that $f(x) := x|x|$ is differentiable for all $x \in \mathbb{R}$. What is $f'(x)$? Is f' continuous? Does f'' exist?

(v) Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable with $f(0) = -3$. Assume that $f'(x) \le 5$ for $x \in \mathbb{R}$. How large can $f(2)$ possibly be?

(w) Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable with $f(1) = 10$ and $f'(x) \ge 2$ for $1 \le x \le 4$. How small can $f(4)$ possibly be?

(x) Let $f(x) = 1/x$ for $x \neq 0$ and $g(x) = \begin{cases} 1/x & \text{if } x > 0 \\ 1 + (1/x) & \text{if } x < 0. \end{cases}$. Let $h = f - g$. Then $h' = 0$ but h is not a constant. Explain.

(y) Let $f\colon [0,1] \to \mathbb{R}$ be such that $f'(x) \neq 0$ for $x \in (0,1)$. Show that $f(0) \neq f(1)$.

(z) Show that on the graph of any quadratic polynomial f the chord joining the points $(a, f(a))$ and $(b, f(b))$ is parallel to the tangent line at the midpoint of a and b.

28. The best reference (in my opinion) for L'Hospital's Rules is R.R. Goldberg [Reference 3] Section 8.7, pages 224-230. Clean and precise arguments are to be found in this reference.

29. **L'Hospital's Rule.** Let J be an open interval. Let either $a \in J$ or a is an endpoint of J. (Note that is may happen that $a = \pm\infty$!) Assume that
(i) $f, g\colon J \setminus \{a\} \to \mathbb{R}$ is differentiable,
(ii) $g(x) \neq 0 \neq g'(x)$ for $x \in J \setminus \{a\}$ and
(iii) $A := \lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ where A is either 0 or ∞.

Assume that $B := \lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists either in $\mathbb{R}$ or $B = \pm\infty$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \equiv B.$$

Sketch of a proof.
Case 1. We attend to a simple case where $A = 0$, $a \in \mathbb{R}$ and $B \in \mathbb{R}$.

Set $f(a) = 0 = g(a)$. Then f and g are continuous on J. Let (x_n) be a sequence in J such that either $x_n > a$ or $x_n < a$ for all $n \in N$ and $x_n \to a$. By Cauchy's MVT, there exists c_n between a and x_n such that

$$\frac{f(x_n) - f(a)}{g(x_n) - g(a)} = \frac{f'(c_n)}{g'(c_n)}.$$

Since $f(a) = 0 = g(a)$, it follows that

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(a)}{g(x_n) - g(a)} = \frac{f'(c_n)}{g'(c_n)}.$$

Clearly, $c_n \to a$. By hypothesis, the sequence $f'(c_n)/g'(c_n) \to B$ and hence the result.

Case 2. Let us now look at the case when $A = \infty$. Write $h(x) = f(x) - Bg(x)$, $x \in J \setminus \{a\}$. Then $h'(x) = f'(x) - Bg'(x)$ so that

$$\lim_{x\to a} \frac{h'(x)}{g'(x)} = 0.$$

We want to show that $\lim_{x\to a} \frac{h(x)}{g(x)} = 0$. Let $\varepsilon > 0$ be given. Then there exists $\delta_1 > 0$ such that

$$g(x) > 0 \quad \text{and} \quad |\frac{h'(x)}{g'(x)}| < \frac{\varepsilon}{2} \text{ for } x \in (a, a+\delta_1]. \tag{13}$$

If $x \in (a, a+\delta_1)$, then

$$\frac{h(x) - h(\delta_1)}{g(x) - g(\delta_1)} = \frac{h'(c_x)}{g'(c_x)} \text{ for some } c_x \in (x, a+\delta_1). \tag{14}$$

From (13)-(14), we get

$$|\frac{h(x) - h(\delta_1)}{g(x) - g(\delta_1)}| < \frac{\varepsilon}{2} \text{ for } x \in (a, a+\delta_1). \tag{15}$$

Since $\lim_{x\to a} g(x) = \infty$, there exists $\delta_2 < \delta_1$ such that

$$g(x) > g(\delta_1) \text{ for } x \in (a, a+\delta_2). \tag{16}$$

From (13) and (16), we deduce

$$0 < g(x) - g(\delta_1) < g(x), \text{ for } x \in (a, a+\delta_2). \tag{17}$$

From (15) and (17), we get

$$\frac{|h(x) - h(\delta_1)|}{g(x)} < \frac{|h(x) - h(\delta_1)|}{g(x) - g(\delta_1)} < \frac{\varepsilon}{2}, \text{ for } x \in (a, a+\delta_2). \tag{18}$$

Now choose $\delta_3 < \delta_2$ so that

$$\frac{|h(\delta_1)|}{g(x)} < \frac{\varepsilon}{2} \text{ for } x \in (a, \delta_3). \tag{19}$$

Algebra gives us

$$\frac{h(x)}{g(x)} = \frac{h(x) - h(\delta_1)}{g(x)} + \frac{h(\delta_1)}{g(x)}.$$

Using this, if $x \in (a, a+\delta_3)$, we have

$$|\frac{h(x)}{g(x)}| \leq \frac{|h(x) - h(\delta_1)|}{g(x)} + \frac{|h(\delta_1)|}{g(x)}. \tag{20}$$

Hence by (18) and (19)

$$|\frac{h(x)}{g(x)}| < \varepsilon, \text{ for } x \in (a, a+\delta_3). \tag{21}$$

(21) says that $\lim_{x\to a} \frac{h(x)}{g(x)} = 0$. Since

$$\frac{f(x)}{g(x)} = \frac{h(x)}{g(x)} + B,$$

the result follows.

The other cases are left to the reader as instructive exercises. $\square$

30. **Typical (and important) applications.**

    (a) $f(x) = \log x$ and $g(x) = x$ for $x > 0$. We know that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. Also,

    $$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = 0 \text{ so that } \lim_{x\to\infty} \frac{\log x}{x} = 0.$$

    (b) Repeated application of L'Hospital's rule yields

    $$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \cdots = \lim_{x\to\infty} \frac{n!}{e^x} = 0.$$

    (c) Recall that $\lim_{x\to 0+} f(1/x) = \lim_{x\to\infty} f(x)$. (Why?) From the last item we conclude

    $$\lim_{x\to 0+} x^{-n} e^{-1/x} = 0.$$

31. Item 30c gives rise to an interesting example of a function. Consider

    $$f(x) := \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$

    Then we have from Item 30c,

    $$f'(0) = \lim_{x\to 0+} \frac{g(x) - g(0)}{x} = \lim_{x\to 0+} \frac{e^{-1/x}}{x} = 0.$$

    A similar example is

    $$f(x) := \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

    We shall return to these examples later (if time permits).

32. **Darboux theorem.** Let $f\colon [a,b] \to \mathbb{R}$ be differentiable. Assume that $f'(a) < \lambda < f'(b)$. Then there exists $c \in (a,b)$ such that $f'(c) = \lambda$. (Thus, though f' need not be continuous, it enjoys the intermediate value property.)

    Hint: Consider $g(x) = f(x) - \lambda x$. It attains a global minimum at some $c \in [a,b]$. Show that it is a local minimum. (Why can't we work with a maximum?)

33. What are all the differentiable functions $f\colon [0,1] \to \mathbb{R}$ the slopes of the tangents to their graphs are always rational?

34. Let $f\colon (a,b) \to \mathbb{R}$ be differentiable. Assume that $f'(x) \neq 0$ for $x \in (a,b)$. Show that f is monotone on (a,b).

35. To make sure that Darboux theorem can be applied to a larger class of functions, we looked at some functions which are differentiable whose derivatives are not continuous.

36. **Higher derivatives.** You all knew what is meant by higher derivatives. A function all of whose derivatives exist is called an infinitely differentiable function and also called a C^∞-function.

We looked at a couple of examples.

(a) $f(x) := \begin{cases} x^n \sin(1/x) & x \neq 0 \\ 0 & x = 0. \end{cases}$

(b) $f(x) := \begin{cases} x^n & x > 0 \\ 0 & x \leq 0. \end{cases}$

(c) Let $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x^3 & \text{if } x \geq 0. \end{cases}$. Then $f'(0)$ exists but $f''(0)$ does not.

(d) Let $f(x) := |x|$. Define $g_1(x) := \int_0^x f(t)\,dt$. Then, by the fundamental theorem of calculus, g_1 is differentiable with derivative $g_1'(x) = f(x)$. Define recursively, $g_n(x) := \int_0^x g_{n-1}(t)\,dt$. Then g_n is n-times continuously differentiable but not $(n+1)$-times differentiable.

(e) The function in Item 31 is infinitely differentiable with $g^{(n)}(0) = 0$.

37. Exercise: Prove Leibniz formula: If $h = fg$ is a product of two functions with derivatives up to order n, then

$$h^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x). \tag{22}$$

38. **Taylor's Theorem.** Assume that $f\colon [a,b] \to \mathbb{R}$ is such that $f^{(n)}$ is continuous on $[a,b]$ and $f^{(n+1)}(x)$ exists on (a,b). Fix $x_0 \in [a,b]$. Then for each $x \in [a,b]$ with $x \neq x_0$, there exists c between x and x_0 such that

$$f(x) = f(x_0) + \sum_{k=1}^{n} \frac{(x-x_0)^k}{(k)!} f^{(k)}(x_0) + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c). \tag{23}$$

Proof. Define

$$F(t) = f(t) + \sum_{k=1}^{n} \frac{(x-t)^k}{(k)!} f^{(k)}(t) + M(x-t)^{n+1},$$

where M is chosen so that $F(x_0) = f(x)$. This is possible since $x \neq x_0$. Clearly, F is continuous on $[a,b]$, differentiable on (a,b) and $F(x) = f(x) = F(x_0)$. Hence by Rolle's theorem, there exists $c \in (a,b)$ such that

$$0 = F'(c) = \frac{(x-c)^n}{n!} f^{(n+1)}(c) - (n+1)M(x-c)^n.$$

Thus, $M = \frac{f^{(n+1)}(c)}{(n+1)!}$. Hence

$$f(x) = F(x_0) = f(x_0) + \sum_{k=1}^{n} \frac{(x-x_0)^k}{(k)!} f^{(k)}(x_0) + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

This is what we wanted. □

39. The equation (23) is called the n-th order Taylor expansion of the function f at x_0. The term $\frac{f^{(n)}(c)}{n!}(x-x_0)^{n+1}$ is called the remainder term in the Taylor expansion. It is usually denoted by R_n. This form of the remainder is the simplest and is known as the Lagrange's form. There are two other forms which are more useful: Cauchy's form and the integral form of the remainder. Personally, I like the integral form as integrals are easier to estimate!

40. Exercise: Existence of some interesting C^∞ functions.

    (a) Consider $f : \mathbb{R} \to \mathbb{R}$ defined by
    $$f(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \exp(-1/t) & \text{for } t > 0. \end{cases}$$
    f is differentiable on $\mathbb{R}^n \setminus \{0\}$.
    Show that $f^{(n)}(0)$ exists for all $n \in \mathbb{N}$ and hence f is infinitely differentiable on all of $\mathbb{R}$.

    (b) Let $f : \mathbb{R} \to \mathbb{R}$ defined by
    $$f(t) = \begin{cases} e^{-\frac{1}{t^2}} & t > 0 \\ 0 & t \leq 0 \end{cases}.$$
    Show that f is infinitely differentiable.

    (c) Let f be as in Ex. 40b. Let $\varepsilon > 0$ be given. Define $g_\varepsilon(t) := f(t)/(f(t) + f(\varepsilon - t))$ for $t \in \mathbb{R}$. Then g_ε is differentiable, $0 \leq g_\varepsilon \leq 1$, $g_\varepsilon(t) = 0$ iff $t \leq 0$ and $g_\varepsilon(t) = 1$ iff $t \geq \varepsilon$.

41. The next few items (Items 42–48 deal with the other forms of the remainder term in the Taylor expansion and their nontrivial applications. If time is short, they may be omitted in a first course and one may jump to Item 49.

42. (Taylor's Theorem with Remainder) Let n and p be natural numbers. Assume that $f\colon [a,b] \to \mathbb{R}$ is such that $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}(x)$ exists on (a,b). Then there exists $c \in (a,b)$ such that
$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n, \quad (24)$$
where
$$R_n = \frac{(b-c)^{n-p}(b-a)^p}{p(n-1)!}f^{(n)}(c). \quad (25)$$
In particular, when $p = n$, we get Lagrange's form of the remainder
$$R_n = \frac{(b-a)^n}{n!}f^{(n)}(c), \quad (26)$$
and when $p = 1$, we get Cauchy's form of the remainder
$$R_n = (b-a)\frac{(b-c)^{n-1}}{(n-1)!}f^{(n)}(c), \text{ where } c = a + \theta(b-a),\ 0 < \theta < 1 \quad (27)$$
$$= \frac{(b-a)^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(a+\theta(b-a)), \text{ where } 0 < \theta := \frac{(c-a)}{(b-a)} < 1. \quad (28)$$

Proof. The proof is an obvious modification the earlier proof.

Consider

$$F(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!}f''(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x).$$

We have for $x \in (a,b)$,

$$F'(x) = \frac{-(b-x)^{n-1}f^{(n)}(x)}{(n-1)!}. \tag{29}$$

We now define

$$g(x) = F(x) - \left(\frac{b-x}{b-a}\right)^p F(a). \tag{30}$$

The g is continuous on $[a,b]$, differentiable on (a,b) and $g(a) = 0 = g(b)$. Hence by Rolle's theorem, there exists $c \in (a,b)$ such that $g'(c) = 0$. Using the definition of g in (30) we get

$$0 = g'(c) = F'(c) + \frac{p(b-c)^{p-1}}{(b-a)^p}F(a). \tag{31}$$

Using (29) in (31) and simplifying we get

$$\frac{(b-c)^{n-1}}{(n-1)!}f^{(n)}(c) = \frac{p(b-c)^{p-1}}{(b-a)^p}F(a). \tag{32}$$

That is, $F(a) = \frac{(b-c)^n - p(b-a)^p}{p(n-1)!}f^{(n)}(c)$. This is what we set out to prove.

Lagrange's form of the remainder (26) and Cauchy's form (27) are obvious. If we write $c = a + \theta(b-a)$ for some $\theta \in (0,1)$, Cauchy's form (28) of the remainder is obtained from (25). □

43. **Binomial Series.** We now show how both forms of the remainder are required to prove the convergence of the binomial series.

Let $m \in \mathbb{R}$. Define

$$\binom{m}{0} = 1 \text{ and } \binom{m}{k} := \frac{m(m-1)\cdots(m-k+1)}{k!} \text{ for } k \in \mathbb{N}.$$

Then

$$(1+x)^m = \sum_{k=0}^{\infty}\binom{m}{k}x^k = 1 + mx + \frac{m(m-1)}{2!} + \cdots, \text{ for } |x| < 1.$$

Proof. If $m \in \mathbb{N}$, this is the usual binomial theorem. In this case, the series is finite and there is no restriction on x.

Let $m \notin \mathbb{N}$. Consider $f\colon (-1,\infty) \to \mathbb{R}$ defined by $f(x) = (1+x)^m$. For $x > -1$, we have

$$f'(x) = m(1+x)^{m-1}, \ \ldots, f^{(n)}(x) = m(m-1)\cdots(m-n+1)(1+x)^{m-n}.$$

If $x = 0$, the result is trivial as $1^m = 1$. Now for $x \neq 0$, by Taylor's theorem

$$f(x) = f(0) + xf'(0) + \cdots + R_n = \sum_{k=0}^{n-1} \binom{m}{k} x^k + R_n.$$

Therefore, to prove the theorem, we need to show that, for $|x| < 1$,

$$|f(x) - \sum_{k=0}^{n-1} \binom{m}{k} x^k| = |R_n| \to 0 \text{ as } n \to \infty.$$

To prove $R_n \to 0$, we use Lagrange's form for the case $0 < x < 1$.

$$|R_n| = |\frac{x^n}{n!} f^{(n)}(\theta x)| = |\binom{m}{n} x^n (1+\theta x)^{m-n}| < |\binom{m}{n} x^n|,$$

if $n > m$, since $0 < \theta < 1$. Letting $a_n := |\binom{m}{n} x^n|$, we see that $a_{n+1}/a_n = x|\frac{m-n}{n+1}| \to x$. Since $0 < x < 1$, the ratio test says that the series $\sum_n a_n$ is convergent. In particular, the n-th term $a_n \to 0$. Since $|R_n| < a_n$, it follows that $R_n \to 0$ when $0 < x < 1$.

Let us now attend to the case when $-1 < x < 0$. If we try to use the Lagrange form of the remainder we obtain the estimate

$$|R_n| < |\binom{m}{n} x^n (1+\theta x)^{m-n}| < |\binom{m}{n} x^n (1-\theta)^{m-n}|$$

if $n > m$. This is not helpful as $(1-\theta)^{m-n}$ shoots to infinity if θ goes near 1. Let us now try Cauchy's form.

$$\begin{aligned} |R_n| &= |mx^n \binom{m-1}{n-1} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} (1+\theta x)^{m-1}| \\ &\leq |mx^n \binom{m-1}{n-1} (1+\theta x)^{m-1}|. \end{aligned} \tag{33}$$

Now, $0 < 1 + x < 1 + \theta x < 1$. Hence $|(1+\theta x)^{m-1}| < C$ for some $C > 0$. Note that C is independent of n but dependent on x.

It follows that $|x^n \binom{m-1}{n-1}| \to 0$ so that $R_n \to 0$. This completes the proof of the theorem. □

44. As another application of Taylor's theorem with Lagrange form of the remainder, we now establish the sum of the standard alternating series is $\log 2$:

$$\log 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}.$$

Consider the function $f\colon (-1, \infty) \to \mathbb{R}$ defined by $f(x) := \log(1 + x)$. By a simple induction argument, we see that

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}.$$

Hence the Taylor series of f around 0 is

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$

We now wish to show that the series is convergent at $x = 1$. This means that we need to show that the sum of the series at $x = 1$ is convergent. We therefore take $a = 0$, $b = 1$ in Taylor's theorem and show that the remainder term (in Lagrange's form) $R_n \to 0$. For each $n \in \mathbb{N}$, there exists $c_n \in (0, 1)$ such that

$$R_n := \frac{f^{(n)}(c_n)}{n!} = \frac{(-1)^{n-1}(n-1)!}{n!(1+c_n)^n}.$$

We have an obvious estimate:

$$|R_n| = |\frac{(-1)^{n-1}(n-1)!}{n!(1+c_n)^n}| \leq |\frac{1}{n(1+c_n)^n}| \leq \frac{1}{n}.$$

Hence $\log 2 = f(1) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} 1^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$. □

45. **Taylor's Theorem with Integral Form of the Remainder.** Let f be function on an interval J with $f^{(n)}$ continuous on J. Let $a, b \in J$. Then

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n-1}(a)}{(n-1)!}(b-a)^{n-1} + R_n \tag{34}$$

where

$$R_n = \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt. \tag{35}$$

Proof. We begin with

$$f(b) = f(a) + \int_a^b f'(t)\, dt.$$

We apply integration by parts formula $\int_a^b u dv = uv|_a^b - \int u' dv$ to the integral where $u(t) = f'(t)$ and $v = -(b-t)$. (Note the *non-obvious* choice of v!) We get

$$\int_a^b f'(t) dt = -f'(t)(b-t)\Big|_a^b + \int_a^b f''(t)(b-t)\, dt.$$

Hence we get

$$f(b) = f(a) + f'(a)(b-a) + \int_a^b f''(t)(b-t)\, dt.$$

We again apply integration by parts to the integral where $u(t) = f''(t)$ and $v(t) = -(b-t)^2/2$. We obtain

$$\int_a^b f''(t)(b-t)\, dt = f''(t)\frac{(b-t)^2}{2}\Big|_a^b + \int_a^b f^{(3)}(t)((b-t)^2/2)\, dt.$$

Hence

$$f(b) = f(a) + f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2} + \int_a^b f^{(3)}(t)\frac{(b-t)^2}{2}\,dt.$$

Assume that the formula for R_k is true:

$$R_k = \int_a^b \frac{(b-t)^{k-1}}{(k-1)!} f^{(k)}(t)\,dt.$$

We let

$$u(t) = f^{(k)}(t) \text{ and } v(t) = -(b-t)^{k-1}$$

and apply integration by parts. We get

$$\int_a^b f^{(k)}(t)\frac{(b-t)^{k-1}}{(k-1)!}\,dt = f^{(k)}(t)\frac{(b-a)^k}{k!} + \int_a^b \frac{(b-t)^k}{k!} f^{(k+1)}(t)\,dt.$$

By induction the formula for R_n is obtained. □

46. Recall the mean value theorem for the Riemann integral: *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then there exists $c \in (a,b)$ such that*

$$(b-a)f(c) = \int_a^b f(t)\,dt, \text{ that is, } f(c) = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

□

47. We use the last item to deduce Cauchy's form (27) of the remainder in the Taylor's theorem. Applying the mean value theorem for integrals to (35), we conclude that there exists $c \in (a,b)$ such that

$$R_n = (b-a)\frac{(b-c)^{n-1}}{(n-1)!} f^{(n)}(c),$$

which is Cauchy's form (27) of the remainder.

48. We now apply the integral form of the remainder to deduce Theorem 43 on binomial series.

Assume that m is not a non-negative integer. Then $a_n := \binom{m}{n} \neq 0$. Since

$$\frac{a_{n+1}}{a_n} = \frac{m-n}{n+1} \to 1,$$

the binomial series

$$(1+x)^m = 1 + \sum_{n=1}^{\infty} \binom{m}{n} \frac{x^n}{n!}$$

has radius of convergence 1. Similarly, the series $\sum_n n\binom{m}{n}x^n$ is convergent for $|x| < 1$. Hence

$$n\binom{m}{n}x^n \to 0 \text{ for } |x| < 1. \tag{36}$$

We now estimate the remainder term using (36). We have, for $0 < |x| < 1$,

$$\begin{aligned} R_n &= \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} n! \binom{m}{n} (1+t)^{m-n}\, dt \\ &= \int_0^x n \binom{m}{n} \left(\frac{x-t}{1+t}\right)^{n-1} (1+t)^{m-1}\, dt. \end{aligned} \tag{37}$$

We claim that

$$|\frac{x-t}{1+t}| \le |x| \text{ for } -1 < x \le t \le 0 \text{ or } 0 \le t \le x < 1.$$

Write $t = sx$ for some $0 \le s \le 1$. Then

$$|\frac{x-t}{1+t}| = |\frac{x-sx}{1+st}| = |x||\frac{1-s}{1+ts}| \le |x|,$$

since $1 + sx \ge 1 - s$. Thus the integrand in (37) is bounded by

$$|n\binom{m}{n}\left(\frac{x-t}{1+t}\right)^{n-1}(1+t)^{n-1}| \le n|\binom{m}{n}||x|^{n-1}(1+t)^{m-1}.$$

Therefore, we obtain

What is C here?

$$|R_n(x)| \le n|\binom{m}{n}||x|^{n-1}\int_{-|x|}^{|x|}(1+t)^{m-1} \le Cn|\binom{m}{n}||x|^{n-1},$$

which goes to 0 in view of (36).

This completes the proof of the fact that the binomial series converges to $(1+x)^m$. □

49. **Typical Applications.** Write the n-th degree Taylor expansion and analyze the error.
    (a) e^x.
    (b) $\sin x$, etc.
    (c) The function in Item 31.

50. **Inverse function theorem.** *Let $f\colon I := (a,b) \to \mathbb{R}$ be continuously differentiable with $f'(x) \neq 0$ for all x. Then (i) f is strictly monotone. (ii) $f(I) = J$ is an interval and (ii) $g := f^{-1}$ is (continuous and) differentiable on the interval $J := f((a,b))$ and we have*

$$g'(f(x)) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))} \text{ for all } x = g(y) \in [a,b].$$

Proof. Since f' is continuous on I, by the intermediate value theorem exactly one of the following holds: either $f' > 0$ or $f' < 0$ on I. Hence f is strictly monotone on I. By Item 34, J is an interval. Note that by Item 35, the inverse function g is continuous on J. Fix $c \in (a,b)$. Let $d := f(c)$. Then

$$\frac{g(y)-g(d)}{y-d} = \frac{x-c}{f(x)-f(c)} = \frac{1}{\frac{f(x)-f(c)}{x-c}}. \tag{38}$$

Since g is continuous, if (y_n) is a sequence in J converging to d, then $x_n := g(y_n) \to c$. Hence

$$\lim_{n\to\infty} \frac{g(y_n) - g(d)}{y_n - d} = \lim_{n\to\infty} \frac{1}{\frac{f(x_n)-f(c)}{x-c}} = \frac{1}{f'(c)}.$$

Since this is true for any sequence (y_n) in J converging to d, we conclude that

$$\lim_{y\to d} \frac{g(y) - g(d)}{y - d} = \frac{1}{f'(c)}.$$

One may also take $\lim_{y\to d}$ in (38) and observe that $y \to d$ iff $x \to c$, thanks to the continuity of f and g. ☐

51. Let $f\colon J \to \mathbb{R}$ be a differentiable function on an interval J. A point $c \in J$ is a point of inflection of f if $f(x) - f(c) - f'c)(x - c)$ changes sign as x increases through c in an interval containing c. Geometrically this means that the graph as a curve crosses the tangent at the point of inflection. We let $Tx) := f(x) - f(c) - f'c)(x - c)$ denote the tangent to the graph of f at $(c, f(c))$.

52. **Maxima and Minima.** Let $n \geq 2$, $r > 0$. Let $f^{(n)}$ be continuous on $[a - r, a + r]$. Assume that $f^{(k)}(a) = 0$ for $1 \leq k \leq n - 1$, but $f^{(n)}(a) \neq 0$.
(i) If n is even, then a is a local extremum. It is a minimum if $f^{(n)} > 0$ and a local maximum if $f^{(n)} < 0$.

Proof. By Taylor's theorem we have

$$f(x) = f(a) + \frac{(x - a)^n}{n!} f^{(n)}(c), \text{ for some } c \text{ between } a \text{ and } x. \tag{39}$$

(i) We have $(x - a)^n \geq 0$ for all x. Now, $f^{(n)}(a) < 0$ implies that $f^{(n)}(c) < 0$ for x near a by the continuity of $f^{(n)}$. Hence (39) implies that $f(x) \leq f(a)$ for all such x. Thus a is a local maximum. The other case is similar.

(ii) $n \geq 3$ odd implies that

$$f(x) - f(a) - f'(a)(x - a) = \frac{(x - a)^n}{n!} f^{(n)}(c).$$

If $f^{(n)}(a) < 0$, then $f(x) - T(x)$ changes from positive to negative as x increases through a. Hence a is a point of inflection. Other case is also similar. ☐

53. Let J be intervals in $\mathbb{R}$. A function $f\colon J \to \mathbb{R}$ is said to be *convex* if for all $t \in [0, 1]$ and for all $x, y \in J$, the following inequality holds:

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y). \tag{40}$$

Recall that the line joining $(y, f(y))$ and $(x, f(x))$ in $\mathbb{R}^2$ is given by

$$t(x, f(x)) + (1 - t)(y, f(y)) = (tx + (1 - t)y, tf(x) + (1 - t)f(y)), \qquad 0 \leq t \leq 1.$$

Then the y-coordinates of points on this line are given by $tf(x) + (1 - t)f(y)$. Thus the geometric meaning of the definition is that the line segment joining $(x, f(x))$ and $(y, f(y))$ is never below the graph of the function.

If for $0 < t < 1$, strict inequality holds in (40), then the function is said to be *strictly convex.*

We say that f is concave if the reverse inequality holds in (40).

54. **Derivative test for convexity.** Assume that $f\colon [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b). If f' is increasing on (a,b), then $f$is convex on $[a,b]$. In particular, if f'' exists and is nonnegative on (a,b), f is convex.

Proof. Let $z = (1-t)x + ty$ for some $x, y \in J := [a,b]$, $0 \le t \le 1$. Write $f(z) = (1-t)f(z) + tf(z)$. It suffices to show that

$$(1-t)(f(z) - f(x)) \le t(f(y) - f(z)). \tag{41}$$

By the MVT, there exist $x < r < z$ and $z < s < y$ such that

$$f(z) - f(x) = f'(r)(z-x) \;\text{ and }\; f(y) - f(z) = f'(s)(y-z).$$

Note that z divides the segment $[x,y]$ in the ratio of $(1-t) : t$ and hence $(1-t)(z-x) = t(y-x)$. Since $f'(r) \le f'(s)$, the inequality (41) follows. □

55. Examples of convex functions.
(i) e^x is (strictly) convex on $\mathbb{R}$.
(ii) x^α is convex on $(0,\infty)$ for $\alpha \ge 1$.
(iii) $-x^\alpha$ is strictly convex on $(0,\infty)$ for $0 < \alpha < 1$.
(iv)) $x \log x$ is strictly convex on $(0, ?)$.
(v) $f(x) = x^4$ is strictly convex but $f''(0) = 0$.
(vi) $f(x) = x + (1/x)$ is convex on $(0,\infty)$.
(vii) $f(x) = 1/x$ is convex on $(0,\infty)$.

56. If $f : (a,b) \to \mathbb{R}$ is convex and $c \in (a,b)$ is a local minimum, then c is a minimum for f on (a,b). That is, local minima of convex functions are global minima.

This result is one of the reasons why convex functions are very useful in applications.

Proof. If c were not a point of global minimum, there exists a $d \in (a,b)$ ($d \le c$ or $d \ge c$ with $f(d) < f(c)$. Consider the curve $\gamma(t) = (1-t)c + td$. Then $\gamma(0) = c$, $\gamma(1) = d$ and

$$\begin{aligned} f(\gamma(t)) &\le (1-t)f(c) + tf(d) \\ &< (1-t)f(c) + tf(c), \qquad t \ne 0 \\ &= f(c) \qquad \text{for all } t \in (0, 1\ ] \end{aligned} \tag{42}$$

But for t sufficiently small, $\gamma(t) \in (c-\varepsilon, c+\varepsilon)$ so that

$$f(\gamma(t)) \ge f(c) \quad \text{for } 0 < t < \varepsilon.$$

which contradicts (42). □

5 Infinite Series

1. When we write $1/3 = 0.3333\ldots$, what do we mean by it?

2. Given a sequence (a_n) of real/complex numbers, a formal sum of the form $\sum_{n=1}^{\infty} a_n$ (or $\sum a_n$, for short) is called an infinite series.

   For any $n \in \mathbb{N}$, the finite sum $s_n := a_1 + \cdots + a_n$ is called (n-th) partial sum of the series $\sum a_n$.

   We say that the infinite series $\sum a_n$ is convergent if the sequence (s_n) of partial sums is convergent. In such a case, the limit $s := \lim s_n$ is called the sum of the series and we denote this fact by the **symbol** $\sum a_n = s$.

   We say that the series $\sum a_n$is divergent if the sequence of its partial sums is divergent.

   The series $\sum_n a_n$ is said to be *absolutely convergent* if the infinite series $\sum_n |a_n|$ is convergent.

   If a series is convergent but not absolutely convergent, then it is said to be *conditionally convergent.*

3. Let (a_n) be a constant sequence $a_n = c$ for all n. Then the infinite series $\sum a_n$ is convergent iff $c = 0$. For, the partial sum is $s_n = nc$. (s_n) is convergent iff $c = 0$. (Why? Use Archimedean property.)

4. Let a_n be nonnegative reals and assume that $\sum a_n$ is convergent. Then its sum s is given by $s := \sup\{s_n : n \in \mathbb{N}\}$. Hence a series of nonnegative terms is convergent iff the sequence of partial sums is bounded.

5. **Geometric Series.** This is the most important example. Let $z \in \mathbb{C}$ be such that $|z| < 1$. Consider the infinite series $\sum_{n=0}^{\infty} z^n$. We claim that the series converges to $\alpha := 1/(1-z)$ for $|z| < 1$. Its n-th partial sum s_n is given by

$$s_n := \sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}.$$

   Now, $|\alpha - s_n| = \frac{z^{n+1}}{1-z}$ which converges to 0 as $n \to \infty$. Hence the claim.

   Also note that if $|z| > 1$ then the n-term does not go to 0, so that the series cannot be convergent in this case. (See Item 10.)

6. **Telescoping series.** Let (a_n) and (b_n) be two sequences such that $a_n = b_{n+1} - b_n$. Then $\sum a_n$ converges iff $\lim b_n$ exists, in which case we have

$$\sum a_n = -b_1 + \lim b_n.$$

   A typical example: $\sum \frac{1}{n(n+1)}$.

7. $\sum_n \frac{1}{n^2}$. Observe that $\frac{1}{n^2} < \frac{1}{n(n-1)}$ for $k \geq 2$. Hence in view of Items 4 and 6, we see that the series $\sum n^{-2}$ is convergent.

8. **Harmonic Series.** The series $\sum_{n=1}^{\infty} n^{-p}$ is convergent if $p > 1$ and is divergent if $p \leq 1$.

   This series is quite often used in conjunction with the comparison test.

   Look at $p = 1$ first. Observe that $s_1 = 1$, $s_2 = 3/2$ and $s_{2^k} > 1 + \frac{k}{2}$.

   For $0 < p < 1$, observe that $s_n \geq n \cdot n^{-p} = n^{1-p} \to \infty$.

   For $p > 1$, observe that

$$\begin{aligned}
\frac{1}{2^p} + \frac{1}{3^p} &< \frac{2}{2^p} = \frac{2}{2^p} \\
\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p} &< \frac{4}{4^p} = \left(\frac{2}{2^p}\right)^2 \\
\frac{1}{8^p} + \frac{1}{9^p} + \cdots + \frac{1}{15^p} &< \frac{8}{8^p} = \left(\frac{2}{2^p}\right)^3.
\end{aligned}$$

   In general, we have

$$\frac{2^k}{2^{p(k+1)}} < \sum_{2^k+1}^{2^{k+1}} \frac{1}{n^p} < \frac{2^k}{2^{pk}}. \tag{43}$$

   Now the geometric series $\sum_k 2^k/2^{p(k+1)}$ is divergent if $p \leq 1$ and the geometric series $\sum_k 2^k/2^{pk}$ is convergent if $p > 1$.

9. **Cauchy criterion for the convergence of an infinite series.** The series $\sum a_n$ converges iff for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n, m \geq N \implies |s_n - s_m| < \varepsilon.$$

   This is quite useful when we want to show that a series is convergent without knowing its 'sum'. See Item 13.

10. If $\sum_n a_n$ converges, then $a_n \to 0$. *Hint:* The sequence (s_n) of partial sums is convergent and, in particular, $|s_n - s_m| \to 0$ as $m, n \to \infty$.

    The converse is not true, look at $\sum \frac{1}{n}$. (Item 8).

11. If $\sum_n a_n = s$, then $\sum_{n=N+1}^{\infty} a_n = s - \sum_{k=1}^{N} a_k$.

    Let s_n denote the partial sums of $\sum a_k$. Let $\sigma_n := \sum_{N+1}^{N+n} a_k$. Let $S := a_1 + \cdots + a_N$. Then $\sigma_n = s_{N+n} - S$.

12. **Algebra of convergent series.** Sum of 2 convergent infinite series and the 'scalar multiple' of a convergent infinite series are convergent. We also know their sums.

13. If $\sum a_n$ is absolutely convergent then $\sum_n a_n$ is convergent.

    To prove: let s_n and σ_n denote the partial sums of $\sum a_n$ and $\sum |a_n|$ respectively. Then, for $n > m$,

$$|s_n - s_m| = |\sum_{k=m+1}^{n} a_k| \leq \sum_{k=m+1}^{n} |a_k| = \sigma_n - \sigma_m,$$

    which converges to 0, as (σ_n) is convergent. Hence (s_n) is Cauchy in $\mathbb{C}$.

14. **Comparison Test.** Let $\sum a_n$ and $\sum_n b_n$ be series of positive reals. Assume that $a_n \leq b_n$ for all n. Then (i) if $\sum b_n$ is convergent, then so is $\sum a_n$, and (ii) if $\sum a_n$ is divergent, so is $\sum b_n$.

    An extension of comparison test. Let $\sum_n a_n$ be a series of positive reals. Assume that $\sum_n a_n$ is convergent and that $|b_n| \leq a_n$ for all n. Then $\sum_n b_n$ is absolutely convergent and hence convergent.

    If $\sum a_n$ is divergent, note that its partial sums form an increasing unbounded sequence. In the case of extension of the comparison test, compare the series $\sum_n |b_n|$ and $\sum_n a_n$ to conclude that $\sum_n b_n$ is absolutely convergent. □

15. Exercises for comparison test.

    (a) Let $b_n > 0$ and $a_n/b_n \to \ell > 0$. Then either both $\sum a_n$ and $\sum b_n$ converge or both diverge.

    (b) Let $a_n > 0$ and $b_n > 0$. Assume that $(a_{n+1}/a_n) \leq (b_{n+1}/b_n$ for all n. Show that (i) if $\sum b_n$ converges, then $\sum a_n$ converges and (ii) if $b_n \to 0$ so does a_n.

16. The geometric series and the comparison test along with the integral test are the most basic tricks in dealing with infinite series.

17. **d'Alembert's Ratio Test.** Let $\sum_n c_n$ be a series of positive reals. Assume that

$$\lim_n c_{n+1}/c_n = r.$$

    Then the series $\sum_n c_n$ is
    (a) convergent if $0 \leq r < 1$,
    (b) divergent if $r > 1$.

    The test is inconclusive if $r = 1$.

    If $r < 1$, choose an s such that $r < s < 1$. Then $c_{n+1} \leq sc_n$ for all $n \geq N$. Hence $c_{N+k} \leq s^k c_N$, for $k \in \mathbb{N}$. The convergence of $\sum c_n$ follows. If $r > 1$ then $c_n \geq c_N$ for all $n \geq N$ and hence $\sum c_n$ is divergent as the n-th term does not go to 0. The failure of the test when $r = 1$ follows from looking at the examples $\sum_n 1/n$ and $\sum_n 1/n^2$. □

18. **Cauchy' Root Test.** Let $\sum_n a_n$ be a series of positive reals. Assume that $\lim_n a_n^{1/n} = a$. Then the series $\sum_n a_n$ is convergent if $0 \leq a < 1$, divergent if $a > 1$ and the test is inconclusive $a = 1$.

    If $a < 1$, then choose α such that $a < \alpha < 1$. Then $a_n < \alpha^n$ for $n \geq N$. Hence by comparing with the geometric series $\sum_{n \geq N} \alpha^n$, the convergence of $\sum_n a_n$ follows. If $a > 1$, then $a_n \geq 1$ for all large n and hence n-th term does not approach zero. The examples $\sum_n 1/n$ and $\sum 1/n^2$ illustrate the failure of the test when $r = 1$. □

19. If $f\colon [a, b] \to \mathbb{R}$ is continuous with $\alpha \leq f(x) \leq \beta$ for $x \in [a, b]$, then

$$\alpha(b - a) \leq \int_a^b f(x)\, dx \leq \beta(b - a).$$

    We motivated this inequality geometrically by considering a nonnegative function f and using the geometric interpretation of the definite integral.

20. **Integral Test.** Assume that $f\colon [1,\infty] \to ([0,\infty)$ is continuous and decreasing. Let $a_n := f(n)$. and $b_n := \int_1^n f(t)\, dt$. Then
(i) $\sum a_n$ converges if (b_n) converges.
(ii) $\sum a_n$ diverges if (b_n) diverges.

Proof. Observe that for $n \geq 2$, we have $a_n \leq \int_{n-1}^n f(t)\, dt \leq a_{n-1}$ so that

$$\sum_{k=2}^{n} a_k \leq \int_1^n f(t)\, dt \leq \sum_{k=1}^{n-1} a_k.$$

If the sequence (b_n) converges, then (b_n) is a bounded increasing sequence. $\sum_{k=2}^n a_k \leq b_n$. Hence (s_n) is convergent.

If the integral diverges, then $b_n \to \infty$. Since $b_n \leq \sum_{k=1}^{n-1} a_k$, the divergence of the series follows. ☐

21. Typical applications of the integral test.

    (a) The p-series $\sum_n n^{-p}$ converges if $p > 1$ and diverges if $p \leq 1$.

    (b) The series $\sum \frac{1}{(n+2)\log(n+2)}$ diverges.

The next two items will not be discussed in the class. They are here for record's sake.

22. Exercises:

    (a) If (a_n) and (b_n) are sequences of positive terms such that $a_n/b_n \to \ell$. Prove that $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

    (b) As an application of the last item, discuss the convergence of
    (a) $\sum 1/2n$, (b) $\sum 1/(2n-1)$ and (c) $\sum 2/(n^2+3)$.

    (c) Let $a_n > 0$, $a_n \searrow 0$. and $\sum a_n$ is convergent. Prove that $na_n \to 0$. *Hint:* Consider $a_{n+1} + \cdots + a_{2n}$.

    (d) Assume that $\sum a_n$ is absolutely convergent and (b_n) is bounded. Show that $\sum a_n b_n$ is convergent. *Hint:* Use Cauchy criterion.

    (e) Let $\sum a_n$ be a convergent series of positive terms. Show that $\sum a_n^2$ is convergent. More generally, show that $\sum a_n^p$ is convergent for $p > 1$.

    (f) Let $\sum a_n$ and $\sum b_n$ be convergent series of positive terms. Show that $\sum \sqrt{a_n b_n}$ is convergent. *Hint:* Observe that $\sqrt{a_n b_n} \leq a_n + b_n$ for all n,

    (g) Give an example of a convergent series $\sum a_n$ such that the series $\sum a_n^2$ is divergent.

    (h) Give an example of a divergent series $\sum a_n$ such that the series $\sum a_n^2$ is convergent.

    (i) Let (a_n) be a real sequence. Show that $\sum (a_n - a_{n+1})$ is convergent iff (a_n) is convergent. If the series converges, what is its sum?

    (j) When does a series of the form $a + (a+b) + (a+2b) + \cdots$ converge?

    (k) Prove that if $\sum |a_n|$ is convergent, then $|\sum a_n| \leq \sum |a_n|$.

    (l) Prove that if $|x| < 1$,

$$1 + x^2 + x + x^4 + x^6 + x^3 + x^8 + x^{10} + x^5 + \cdots = \frac{1}{1-x}.$$

(m) Prove that if a convergent series in which only a finite number of terms are negative is absolutely convergent.

(n) If $(n^2 a_n)$ is convergent, then $\sum a_n$ is absolutely convergent.

23. ***Cauchy Condensation Test.** If (a_n) is a decreasing sequence of nonnegative terms, then $\sum a_n$ and $\sum_k 2^k a_{2^k}$ are either both convergent or both divergent.

Proof. Observe

$$\begin{aligned} a_3 + a_4 &\geq 2a_4 \\ &\vdots \\ a_{2^{n-1}+1} + a_{2^{n-1}+2} + \cdots + a_{2^n} &\geq 2^{n-1} a_{2^n}. \end{aligned}$$

Adding these inequalities, we get

$$\sum_{k=1}^{2^n} a_k > \sum_{k=3}^{2^n} a_k > \sum_{k=1}^{n} 2^{k-1} a_{2^k}. \tag{44}$$

If $\sum_{k=1}^{\infty} 2^k a_{2^k}$ diverges so does $\sum_{k=1}^{\infty} 2^{k-1} a_{2^k}$. (44) shows that $\sum a_k$ diverges.

Note that

$$\begin{aligned} a_2 + a_3 &\leq 2a_2 \\ &\vdots \\ a_{2^{n-1}} + a_{2^{n-1}+1} + \cdots + a_{2^n-1} &\leq 2^{n-1} a_{2^n-1}. \end{aligned}$$

Adding these inequalities, we get

$$a_2 + a_3 + \cdots + a_{2^n-1} \leq \sum_{k=1}^{n-1} 2^k a_{2^k}. \tag{45}$$

If $\sum_{k=1}^{\infty} 2^k a_{2^k}$ is convergent, then arguing as above, we conclude that the series $\sum a_k$ is convergent.

24. * A typical application of the condensation test: The series $\sum \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

25. * **Abel-Pringsheim.** If $\sum_n a_n$ is a series of nonnegative terms with (a_n) decreasing, then $na_n \to 0$.

Proof. Choose N such that $|s_n - s_m| < \varepsilon$ for $n, n \geq N$. For $k \geq N$,

$$ka_{2k} \leq a_{k+1} + \cdots + a_{2k} = s_{2k} - s_k.$$

Similarly, $(k+1)a_{2k+1} \leq s_{2k+1} - s_k < \varepsilon$. Hence $2ka_{2k} < 2\varepsilon$ and $(2k+1)a_{2k+1} \leq 2(k+1)a_{2k+1} < 2\varepsilon$. ☐

26. * One may also deduce Abel-Prinsheim from the condensation test. For $\sum 2^k a_{2k}$ is convergent. Note that $2^n a_{2^n} \to 0$. Given k choose n such that $2^n \leq k \leq 2^{n+1}$. Then $ka_k < 2^{n+1} a_{2^n} = 2(2^n a_{2^n} to 0$.

27. * The series $\sum \frac{1}{an+b}$, $a > 0$, $b \geq 0$ is divergent. Note that (a_n) is decreasing. If it is convergent, then $na_n \to 0$ by Abel-Pringsheim.

28. * **Theorem (Abel).** Let $\sum a_n$ and $\sum b_n$ be convergent, say, with sums A and B respectively. Assume that their Cauchy product $\sum c_n$ is also convergent to C. Then $C = AB$. See Item ?***

29. * The tests we have seen so far (except the alternating series test) are for absolute convergence. There are infinite series which are convergent but not absolutely convergent. These are often quite subtle to handle. (See Items 221 and 224.) We give some tests which are useful to deal with such series. The basic tool for these tests is the following Abel's summation formula.

30. * **Abel's summation by parts formula.** Let (a_n) and (b_n) be two sequences of complex numbers. Define $A_n := a_1 + \cdots + a_n$. We then have the identity

$$\sum_{k=1}^{n} a_k b_k = A_n b_{n+1} - \sum_{k=1}^{n} A_k (b_{k+1} - b_k). \tag{46}$$

It follows that $\sum_k a_k b_k$ converges if (i) the series $\sum A_k(b_{k+1} - b_k)$ and (ii) the sequence $(A_n b_{n+1})$ converges.

Sketch of a proof. Let $A_0 = 0$. We have

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (A_k - A_{k-1}) b_k = \sum_{k=1}^{n} A_k b_k - \sum_{k=1}^{n} A_k b_{k+1} + A_n b_{n+1}.$$

(46) follows from this. The last conclusion is an easy consequence of (46).

31. ***Abel's Lemma.** Keep the notation of the last item. Assume further that (i) $m \leq A_n \leq M$ for $n \in \mathbb{N}$ and (ii) (b_n) is a decreasing sequence. Then

$$mb_1 \leq \sum_{k=1}^{n} a_k b_k \leq Mb_1. \tag{47}$$

Proof. (47) follows from the summation formula (46), the fact $b_k - b_{k+1} \geq 0$ and telescoping:

$$\begin{aligned} \sum_{k=1}^{n} a_k b_k &\leq M \sum_{k=1}^{n} (b_k - b_{k+1}) + M b_{n+1} \\ &= M \left[(b_1 - b_2) + (b_2 - b_3) + \cdots + (b_n - b_{n+1}) + b_{n+1}\right] \\ &= M b_1. \end{aligned}$$

The left side inequality in (47) is proved in a similar way. □

32. ***Dedekind's test.** Let (a_n), (b_n) be two complex sequences. Let $A_n := (a_1 + \cdots + a_n)$. Assume that (i) (A_n) is bounded, (ii) $\sum |b_{n+1} - b_n|$ is convergent and $b_n \to 0$. Then $\sum a_n b_n$ is convergent. In fact, $\sum a_n b_n = \sum A_n (b_{n+1} - b_n)$.

Proof. We use Abel's summation formula (46).

33. * **Dirichlet's Test.** The series $\sum a_k b_k$ is convergent if the sequence (A_n) where $A_n := \sum_{k=1}^n a_k$ is bounded and (b_k) is decreasing to zero.

Sketch of a proof. Assume that $|A_n| \leq M$ for all n. We have $A_n b_{n+1} \to 0$. In view of the last item, it suffices to prove that $\sum_k A_k(b_{k+1} - b_k)$ is convergent. Since $b_k \searrow 0$, we have

$$|A_k(b_{k+1} - b_k)| \leq M(b_k - b_{k+1}).$$

The series $\sum_k (b_k - b_{k+1})$ is telescoping. We conclude that $\sum_k A_k(b_{k+1} - b_k)$ is absolutely convergent. □

34. * An important example. Let $b_n \searrow 0$. Then $\sum b_n \sin nx$ is convergent for all $x \in \mathbb{R}$ and $\sum b_n \cos nx$ s convergent for all $x \in \mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\}$.

Consider the geometric series:

$$\sum_{k=1}^n e^{ikx} = e^{ix}\frac{1 - e^{inx}}{1 - e^{ix}} = e^{i\frac{(n+1)x}{2}} \frac{\sin(nx/2)}{\sin(x/2)}.$$

Taking real and imaginary parts, we get

$$\sum_{k=1}^n \cos kx = \frac{\sin \frac{nx}{2} \cos(n+1)\frac{x}{2}}{\sin \frac{x}{2}} \tag{48}$$

$$\sum_{k=1}^n \sin kx = \frac{\sin \frac{nx}{2} \sin(n+1)\frac{x}{2}}{\sin \frac{x}{2}}. \tag{49}$$

We thus have the easy estimates

$$|\sum_{k=1}^n \cos kx| \leq \frac{1}{|\sin \frac{x}{2}|} \text{ and } |\sum_{k=1}^n \sin kx| \leq \frac{1}{|\sin \frac{x}{2}|} \text{ if } \sin \frac{x}{2} \neq 0.$$

Now it is easy complete the proof. □

35. * **Abel's Test.** Assume that the series $\sum a_k$ is convergent and the sequence (b_k) is monotone and bounded. Then the series $\sum a_k b_k$ is convergent.

Proof is quite similar to that of Dirichlet's test. □

36. **Leibniz Test or Alternating Series Test.** Let (t_n) be a a real monotone sequence converging to zero. Then $\sum (-1)^{n-1} t_n$ is convergent and we have

$$t_1 - t_2 \leq \sum (-1)^{n-1} t_n \leq t_1.$$

Proof. Clearly $s_{2n} = (t_1 - t_2) + \cdots (t_{2n-1} - t_{2n})$ is increasing. Also,

$$s_{2n} = t_1 - (t_2 - t_3) - \cdots - (t_{2n-2} - t_{2n-1}) - t_{2n} \leq t_1.$$

Hence the sequence (s_{2n}) is a bounded increasing sequence and hence is convergent, say, to $s \in \mathbb{R}$. We claim that $s_n \to s$. Given $\varepsilon > 0$, find $N \in \mathbb{N}$ such that

$$n \geq N \implies |s_{2n} - s| < \varepsilon/2 \text{ and } |t_{2n+1}| < \varepsilon/2.$$

For $n \geq N$, we have

$$|s_{2n+1} - s| \leq |s_{2n} + t_{2n+1} - s| \leq |s_{2n} - s| + |t_{2n+1}| < \varepsilon.$$

□

A typical example: $\sum \frac{(-1)^n}{n}$.

37. Let $\sum a_b$ and a bijection $f : \mathbb{N} \to \mathbb{N}$ be given. Define $b_n := a_{f(n)}$. Then the new series $\sum b_n$ is said to be a rearrangement of $\sum a_n$.

38. Rearranged series $\sum b_n$ may converge to a sum different from that of $\sum a_n$.

    Consider the standard alternating series $\sum(-1)^{n+1}n^{-1}$. We know that it s convergent, say, to a sum s. From Item 36 we also know $s \geq t_1 - t_2 = 1/2$. Hence $s \neq 0$. We rearrange the series to get a new series $\sum b_n$ which converges to $s/2$!

    Rearrange the given series in such a way that two negative terms follow a positive term:

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \cdots + \frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n} + \cdots .$$

    (Can you write an explicit formula for b_n?) Let t_n denote the n-th partial sum of this rearranged series. We have

$$t_{3n} = \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n}\right) + \cdots .$$

    In each of the block of three terms (in the brackets), subtract the second term from the first to get

$$t_{3n} = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{4n-2} - \frac{1}{4n}\right) + \cdots = \frac{s_{2n}}{2}.$$

    Thus $t_{3n} \to s/2$. Also, $t_{3n+1} = t_{3n} + \frac{1}{2n+1} \to s/2$ etc. Hence we conclude that $t_n \to s/2$. □

39. We need to be careful when dealing with infinite series. Mindless algebraic/formal manipulations may lead to absurdities. Let

$$s = 1 - 1 + 1 - 1 + \cdots + (-1)^{n+1} + \cdots$$

    (Note that s has no meaning, if we apply our knowledge of infinite series!) Then

$$-s = -1 + 1 - 1 + 1 + \cdots = 1 + (-1 + 1 + \cdots) - 1 = s - 1.$$

    Hence $s = 1/2$. On the other hand

$$s = (1 - 1) + (1 - 1) + \cdots = 0.$$

    Hence $0 = 1/2$!

40. **Riemann's Theorem.** A conditionally convergent series can be made to converge to any arbitrary real number or even made to diverge by a suitable rearrangement of its terms. □

    For a proof, refer to Apostol Theorem 8.33 (page 197).

41. Given a real series $\sum a_n$ we let $a_n^+ := \begin{cases} a_n & \text{if } a_n > 0 \\ 0 & \text{otherwise} \end{cases}$ and $a_n^- := \begin{cases} -a_n & \text{if } a_n < 0 \\ 0 & \text{otherwise} \end{cases}$.

We call the series $\sum a_n^+$ (respectively, $\sum a_n^-$) as the positive part or the series of positive terms (respectively, the negative part or the series of negative terms) of the given series $\sum a_n$. Note that both these series have nonnegative terms.

42. If $\sum a_n$ is conditionally convergent, then the series of its positive terms and the series of negative terms are both divergent.

Proof. Let s_n denote the n-th partial sum, α_n the sum of the positive terms in s_n and $-\beta_n$, the sum of the negative terms in s_n. Then $\alpha_n \geq 0$ and $\beta_n \geq 0$. Also, we have

$$\sigma_n := \sum_{k=1}^{n} |a_k| = \alpha_n + \beta_n, \text{ and } s_n = \alpha_n - \beta_n.$$

Let $s_n \to s$. Observe that (α_n) and (β_n) are increasing. By hypothesis, $\sigma_n \to \infty$ (why?), $s_n \to s$. Note that

$$\alpha_n = \frac{\sigma_n + s_n}{2} \text{ and } \beta_n = \frac{\sigma_n - s_n}{2}.$$

Now it is easy to complete the proof. □

43. The proof above shows the following. If $\sum a_n$ is a series of real numbers, then $\sum a_n$ converges iff $\sum a_n^+$ and $\sum a_n^-$ converge, in which case we have $s = \alpha - \beta$. (Here $\sum a_n^+ = \alpha$ and $\sum a_n^- = \beta$.)

44. **Rearrangement of terms.** If $\sum a_n$ is absolutely convergent and $\sum b_n$ is a rearrangement of $\sum a_n$, then $\sum b_n$ is convergent and we have $\sum a_n = \sum b_n$.

Proof. Let t_n denote the n-th partial sum of the series $\sum b_n$. Let $\sum a_n = s$. We claim that $t_n \to s$. Let $\varepsilon > 0$ be given. Choose $n_0 \in \mathbb{N}$ such that

$$(n \geq n_0 \implies |s_n - s| < \varepsilon) \text{ and } \sum_{n_0+1}^{\infty} |a_n| < \varepsilon.$$

Choose $N \in \mathbb{N}$ such that $\{a_1, \ldots, a_{n_0}\} \subset \{b_1, \ldots, b_N\}$. (That is, choose N so that $\{1, \ldots, n_0\} \subseteq \{f(1), \ldots, f(N)\}$.) We then have for $n \geq N$,

$$|t_n - s_{n_0}| \leq \sum_{n_0+1}^{\infty} |a_k| < \varepsilon.$$

It follows that for $n \geq N$, we have $|t_n - s| \leq |t_n - s_{n_0}| + |s_{n_0} - s| < 2\varepsilon$.

45. Rearrangement of series of positive terms does not affect the convergence and the sum. (Why?)

46. Given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, a natural way of defining their product would be $\sum c_n$ where $c_n := a_n b_n$. If we take $\sum a_n = \sum (-1)^{n+1}/\sqrt{n} = \sum b_n$, then they are convergent and the resulting product series is divergent.

47. Given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, their *Cauchy product* is a series $\sum_{n=1}^{\infty} c_n$ where $c_n := \sum_{k=0}^{n} a_k b_{n-k}$. It is motivated by the product of polynomials and power series. For instance, if we let $p(z) := \sum_{k=0}^{m} a_k z^k$ and $q(z) := \sum_{k=0}^{n} b_k z^k$, then the product of polynomials is given by $pq(z) = \sum_{r=0}^{m+n} c_r z^r$, where $c_r := \sum_{k+l=r} a_k b_l$.

48. In general the Cauchy product of two convergent series may not be convergent. Consider the series $\sum a_n$ and $\sum b_n$ where $a_n = \frac{(-1)^n}{\sqrt{n+1}}$. Then the c_n, the n-th term of their Cauchy product is

$$c_n = (-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(n-k+1)(k+1)}}.$$

For $k \leq n$, we have

$$(n-k+1)(k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2 \leq \left(\frac{n}{2}+1\right)^2.$$

Hence $|c_n| \geq \frac{2(n+1)}{n+2} \to 2$.

49. **Mertens' Theorem.** *Let $\sum_{n=0}^{\infty} a_n$ be absolutely convergent and $\sum_{n=0}^{\infty} b_n$ be convergent. Define $c_n := \sum_{k=0}^{n} a_k b_{n-k}$. If $A := \sum_n a_n$ and $B := \sum_n b_n$, then $\sum_n c_n$ is convergent and we have $\sum_n c_n = AB$.*

Proof. Using an obvious notation, we let A_n, B_n and C_n denote the partial sums of the three series. Let $D_n := B - B_n$.

$$\begin{aligned} C_n &= \sum_{k=0}^{n} c_k \\ &= \sum_{k=0}^{n} \sum_{r=0}^{k} a_r b_{k-r} \\ &= \sum_{r+s \leq n} a_r b_s \\ &= a_0(b_0 + b_1 + \cdots + b_n) + a_1(b_0 + b_1 + \cdots + b_{n-1}) + \cdots + a_n b_0 \end{aligned}$$

Hence, we have

$$\begin{aligned} C_n &= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \\ &= a_0(-B + B_n) + a_1(-B + B_{n-1}) + \cdots + a_n(-B + B_0) + B\left(\sum_{k=0}^{n} a_k\right) \\ &= A_n B - R_n, \end{aligned} \tag{50}$$

where $R_n := a_0 D_n + a_1 D_{n-1} + \cdots + a_n D_0$. Let $\alpha := \sum_n |a_n|$. Since $D_n \to 0$, (D_n) is bounded, say by D: $|D_n| \leq D$ for all n. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{n \geq N} |a_n| < \varepsilon$ and $|D_n| \leq \varepsilon$. For all $n \geq 2N$, we have

$$\begin{aligned} |R_n| &\leq (|a_0| + \cdots + |a_{n-N}|)\varepsilon + (|a_{n-N+1}| + \cdots + |a_n|)D \\ &\leq (\alpha + D)\varepsilon. \end{aligned}$$

Hence $R_n \to 0$. Since $A_n B \to AB$, the result follows from (50). □

50. We know that $\sum_{n=0}^{\infty} z^n = 1/(1-z)$ for $|z| < 1$. If we take $a_n = z^n = b_n$ in the theorem, we get $\sum_{n=1}^{\infty} nz^{n-1} = 1/(1-z)^2$ for $|z| < 1$.

51. **Existence of Decimal Expansion.** We motivate the study of infinite series by means of the example of decimal expansions of a real number. Let $a \in \mathbb{R}$. Let $a_0 := [a]$. Write $a = a_0 + x_1$. Then $0 \leq x_1 < 1$. Observe that $a = a_0 + \frac{10x_1}{10}$. Let $a_1 = [10x_1]$. Then $0 \leq a_1 \leq 9$. Also, $10x_1 = a_1 + x_2 = a_1 + \frac{10x_2}{10}$ with $0 \leq x_2 < 1$. Hence

$$a = a_0 + \frac{a_1}{10} + \frac{10x_2}{10^2} \text{ with } 0 \leq 10x_2 < 10.$$

Inductively, we obtain

$$0 \leq a - \left(a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + + \cdots + \frac{a_n}{10^n}\right) = \frac{10x_{n+1}}{10^{n+1}} < \frac{1}{10^n}.$$

Hence $a = \sum_{n=0}^{\infty} \frac{a_n}{10^n}$ which is usually denoted by $a = a_0.a_1a_2 \cdots a_n \cdots$.

6 Uniform Convergence.

1. Let $f_n \colon X \to \mathbb{R}$ be a sequence of functions from a set X to $\mathbb{R}$. We say that f_n converges to f *pointwise* on X if for each $x \in X$ the sequence $(f_n(x))$ converges to $f(x)$ in Y. Note that this means that given $\varepsilon > 0$ there exists an $n_0(x, \varepsilon) \in \mathbb{N}$ such that for $n \geq n_0(x, \varepsilon)$ we have $|(f_n(x) - f(x)| < \varepsilon$. Thus n_0 depends not only on ε but also on x.

2. Examples. Show that the following sequences of real valued functions on $\mathbb{R}$ converge pointwise to f. Find explicitly an $n_0(x, \varepsilon)$ for each of them. We drew pictures of all these examples to get an idea of what is going on.

   (a) $f_n(x) = \frac{x}{n}$ and $f(x) = 0$ for all $x \in \mathbb{R}$.

   (b) $f_n(x) = \begin{cases} 0, & -\infty < x \leq 0 \\ nx, & 0 \leq x \leq \frac{1}{n} \\ 1, & x \geq \frac{1}{n} \end{cases}$ and $f(x) = \begin{cases} 0, & -\infty \leq x \leq 0 \\ 1, & x > 0. \end{cases}$

   (c) $f_n(x) = \begin{cases} nx, & \text{if } 0 \leq x \leq \frac{1}{n} \\ n(\frac{2}{n} - x), & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0, & \text{if } \frac{2}{n} \leq x \leq 1 \end{cases}$ and $f(x) = 0$ for all $x \in \mathbb{R}$.

   (d) $f_n(x) = \begin{cases} 1, & -n \leq x \leq n \\ 0, & |x| > n \end{cases}$ and $f(x) = 1$ for all $x \in \mathbb{R}$.

3. A sequence $f_n \colon X \to \mathbb{R}$ is said to be *converge uniformly on* X to f if given $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X$ and $n \geq n_0$. If $f_n \to f$ uniformly on X we denote it by $f_n \rightrightarrows f$ on X.

   It is clear that (i) uniform convergence implies pointwise convergence but not conversely and (ii) uniform convergence on X implies the uniform convergence on Y where $Y \subseteq X$.

4. We interpreted the uniform convergence in a geometric way. Let $X \subset \mathbb{R}$, say, an interval. Draw the graphs of f_n and f. Put a band of width ε around the graph of f. The uniform convergence $f_n \rightrightarrows f$ is equivalent to asserting the existence of N such that the graphs of f_n over X will lie inside this band, for $n \geq N$.

5. $f_n \rightrightarrows f$ on X iff for every $\varepsilon > 0$ there is an n_0 such that

$$\sup \{|f_n(x) - f(x)| \ \mid \ x \in X\} < \varepsilon \text{ if } n \geq n_0.$$

6. We proved that none of the sequences in Ex. 2 are uniformly convergent.

7. Let $f_n(x) = x^n$ and $f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x = 1 \end{cases}$ on [0,1]. Then f_n converges to f pointwise but not uniformly. *Hint:* If N does the job for $\varepsilon = \frac{1}{2}$ we'd then have $x^N < \frac{1}{2}$ for $0 \leq x < 1$. Let $x \to 1-$. (f_n) does not converge uniformly even on $(0, 1)$.

   Or, if we choose x such that $1 > x > 1/2^{1/N}$, then $x^N > 1/2$, a contradiction.

8. Let $f_n \colon J \subseteq \mathbb{R} \to \mathbb{R}$ converge uniformly on J to f. Assume that f_n are continuous at $a \in X$. Then f is continuous at a.

   Proof. Given $\varepsilon/3 > 0$ choose N by uniform convergence. Choose δ by continuity of f_N at a for $\varepsilon/3$. Observe that for $x \in (a - \delta, a + \delta) \cap J$, we have

   Curry leaves trick.

$$|f(x) - f(a)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(a)| + |f_N(a) - f(a)| < \varepsilon.$$

□

9. Let us recast the definitions of pointwise and uniform convergence of a sequence (f_n) of functions from a set X to $\mathbb{R}$, using the quantifiers $\exists$ and $\forall$.

   $f_n \to f$ pointwise iff

$$\forall x \in X \, (\forall \varepsilon > 0 \, (\exists n_0 = n_0(x, \varepsilon) \, (\forall n \geq n_0 \, (|f_n(x) - f(x)| < \varepsilon)))) \, .$$

   $f_n \to f$ uniformly on X iff

$$\forall \varepsilon > 0 \, (\exists n_0 = n_0(\varepsilon) \, (\forall n \geq n_0 \, (\forall x \in X \, (|f_n(x) - f(x)| < \varepsilon)))) \, .$$

10. We looked at some examples:

   (a) $f_n \colon \mathbb{R} \to \mathbb{R}$ given by $f_n(x) = 0$ if $|x| \leq n$ and $f_n(x) = n$ if $|x| > n$. Then $f_n \to 0$ pointwise but not uniformly.

   (b) $f_n(x) = \frac{nx}{1+n^2x^2}$ and $f(x) = 0$ for $x \in \mathbb{R}$. After some algebraic manipulation, we saw that $f_n(x) = \frac{1}{(nx)+\frac{1}{nx}}$. This reminded us of $t + \frac{1}{t} \geq 2$ for $t > 0$ and equality iff $t = 1$. Hence we chose $x_n = 1/n$ so that $f_n(1/n) = 1/2$.

   (c) $f_n(x) = \frac{x^n}{n+x^n}$ and $f(x) = 1$ if $0 \leq x < 1$, $f(1) = 1/2$ and $f(x) = 0$ if $x > 1$ for $x \in [0, \infty)$. f_n converges to f uniformly on $[0, 1]$ but not on $[0, \infty)$. How about on $(1, \infty)$?

(d) $f_n(x) = x^2 e^{-nx}$ on $[0, \infty)$. As $f_n(x) \to 0$ as $x \to \infty$, we wanted to find the maximum value of f_n using calculus. Since the maximum value of f_n tends to zero, the convergence $f_n \to 0$ is uniform on $[0, \infty)$.

11. Let $f_n \to f$ pointwise. Let $M_n :=$ l.u.b. $\{|f_n(x) - f(x)| : x \in X\}$, if it exists. Then $f_n \rightrightarrows f$ iff $M_n \to 0$.

12. What does it mean to say that (f_n) is pointwise Cauchy and uniformly Cauchy on X?

13. **Cauchy criterion for uniform convergence.** Let $f_n \colon X \to \mathbb{R}$ be a sequence of functions from a set X to $\mathbb{R}$. Then f_n is uniformly convergent iff the sequence (f_n) is uniformly Cauchy on X, that is, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|f_m(x) - f_n(x)| < \varepsilon$ for $m, n \geq n_0$ and for all $x \in X$.

Proof. Let $f_n \rightrightarrows f$ on X. Let $\varepsilon > 0$ be given. Choose N such that

$$n \geq N \implies |f_n(x) - f(x)| < \varepsilon/2 \text{ for all } x \in X.$$

Then for $n, m \geq N$, we get

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < 2 \times \varepsilon/2 = \varepsilon \text{ for all } x \in X.$$

Let (f_n) be uniformly Cauchy on X. We need to prove that (f_n) is uniformly convergent on X. This uses our curry-leaves trick. It is clear that for each $x \in X$, the sequence $(f_n(x))$ of real numbers is Cauchy and hence by Cauchy completeness there exists $r_x \in \mathbb{R}$ such that $f_n(x) \to r_x$. We define $f \colon X \to \mathbb{R}$ by setting $f(x) = r_x$. We claim that $f_n \rightrightarrows f$ on X. curry leaves trick

Let $\varepsilon > 0$ be given. Since $f_n \to f$ pointwise on X, for a given $x \in X$, there exists $N_x = N_x(\varepsilon)$ such that

$$n \geq N_x \implies |f_n(x) - f(x)| < \varepsilon/2.$$

Also, since (f_n) is uniformly Cauchy on X, we can find N such that

$$m, n \geq N \implies |f_n(x) - f_m(x)| < \varepsilon/2 \text{ for all } x \in X.$$

We observe, for $n \geq N$

$$\begin{aligned}
|f_n(x) - f(x)| &\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \text{ for any } m \\
&\leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \text{ for } m > \max\{N, N_x\} \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

□

14. So far, we have seen four times the curry-leaves trick in our course. I suggest that you go through all of them together so that you can master the trick. List them

15. An application of the Cauchy criterion for uniform convergence (Item 13).

Let $f_n \colon J := (a, b) \to \mathbb{R}$ be differentiable. Assume that $f_n' \rightrightarrows g$ uniformly. Further assume that there exists $c \in J$ such that the real sequence $(f_n(c))$ converges. Then the sequence (f_n) converges uniformly to a continuous function $f \colon J \to \mathbb{R}$.

Proof. Fix $x \in J$. We claim that (f_n) is uniformly Cauchy. By mean value theorem we have

$$f_n(x) - f_m(x) - (f_n(c) - f_m(c)) = f_n'(t) - f_m'(t)(x - c) \text{ for some } t \text{ between } x, c. \quad (51)$$

Given $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$n \geq n_1 \implies |f_n(c) - f_m(c)| < \varepsilon/2. \quad (52)$$

Also, since $f_n' \rightrightarrows g$, the sequence (f_n') is uniformly Cauchy and hence there exists $n_2 \in \mathbb{N}$ such that

$$n \geq n_2 \implies |f_n(s) - f_m(s)| < \frac{\varepsilon}{2(b-a)} \text{ for all } s \in J. \quad (53)$$

If $N := \max\{n_1, n_2\}$, using (52) and (52) in (51) we get

$$n \geq N \implies |f_n(x) - f_m(x)| < \varepsilon, \text{ for all } x \in J.$$

That is, (f_n) is uniformly Cauchy on J and hence is uniformly convergent to a function $f\colon J \to \mathbb{R}$. Since f_n are continuous, so is f by Item 8. □

16. Exercises:

(a) Check for uniform convergence:

i. $f_n(x) = nx^n$ and $f(x) = 0$ for $x \in [0, 1)$.

ii. $f_n(x) = \frac{x}{1+nx}$ and $f(x) = 0$ for $x \geq 0$.

iii. $f_n(x) = \frac{nx}{1+n^2x^2}$ and $f(x) = 0$ for $x \in \mathbb{R}$. *Hint:* Observe that $f_n(\frac{1}{n}) = 1/2$. Exploit Item 5.

iv. $f_n(x) = nx^n(1-x)$ and $f(x) = 0$ for $x \in [0, 1]$. *Hint:* Observe that $f_n(n/n + 1) \to 1/e$.

v. $f_n(x) = n^2x^n(1-x)$ and $f(x) = 0$ for $x \in [0, 1]$. f_n takes max at $n/(n+1)$ so that $f_n(n/n+1) = \frac{n^2}{n+1}\left(\frac{n}{n+1}\right)^n \to \infty \times (1/e)$.

vi. $f_n(x) = \frac{x}{1+nx^2}$ and $f(x) = 0$ for $x \in \mathbb{R}$. *Hint:* Use calculus to compute the maximim of f_n's.

vii. $f_n(x) = \frac{x^n}{n+x^n}$ and $f(x) = 1$ if $0 \leq x < 1$, $f(1) = 1/2$ and $f(x) = 0$ if $x > 1$ for $x \in [0, \infty)$. f_n converges to f uniformly on $[0, 1]$ but not on $[0, \infty)$.

viii. $f_n(x) = x - x^n$ and $f(x) = x$ if $0 \leq x < 1$ and $f(1) = 0$.

ix. $f_n(x) = (1-x)x^n$ converges to 0 on [0,1].

x. $f_n(x) = \sin\frac{x}{n}$ is convergent to 0 on $[0, 1]$.

xi. $f_n(x) = \begin{cases} nx, & 0 \leq x \leq \frac{1}{n} \\ 2 - nx, & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0, & \frac{2}{n} < x, \end{cases}$ for all $x \geq 0$.

xii. $f_n(x) = xe^{-nx}$ on $[0, \infty)$. *Hint:* Find the maximum value of xe^{-nx}.

xiii. $f_n(x) = x^2e^{-nx}$ on $[0, \infty)$.

xiv. $f_n(x) := \frac{nx}{n+x}$ for $x \geq 0$.

xv. $f_n(x) = \frac{nx}{1+n^2x^2}$ for $x \geq 0$.

xvi. $f_n(x) = \begin{cases} n^2, & 0 \le x \le 1/n \\ n^2 - n^3(x - 1/n), & 1/n \le x \le 2/n \\ 0, & \text{otherwise.} \end{cases}$

(b) Consider $f_n(x) = x$ and $g_n(x) = 1/n$ for $x \in \mathbb{R}$. f_n and g_n are obviously uniformly convergent but their product is not.

(c) Let $f_n(x) = x\left(1 - \frac{1}{n}\right)$ and $g_n(x) = 1/x^2$ for $x \in (0,1)$. Show that (f_n) and (g_n) are uniformly convergent on $(0,1)$ but their product is not.

(d) Let $g_n(x) = \frac{\sin nx}{nx}$ on $(0,1)$. Is the sequence convergent? If so, what is the limit? Is the sequence uniformly convergent?

(e) Complete the sentence: The sequence of functions h_n on $[0, A]$ defined by $h_n(x) = \frac{nx^3}{1+nx}$ converges to .... Is the convergence uniform?

(f) On $[0,1]$, define $f_n(x) = x^n(1-x^n)$ and $g_n(x) = nxe^{-nx^2}$. Discuss their convergence on [0,1]. *Hint:* Show that f_n has a maximum $\frac{1}{4}$ at $(\frac{1}{2})^{\frac{1}{n}}$ and that g_n has a maximum $\sqrt{\frac{n}{2e}}$ at $\sqrt{\frac{1}{2n}}$. Show also that $g_n(\frac{1}{n}) = e^{-\frac{1}{n}} \ge e^{-1}$.

(g) Let $\{\lambda_n\}$ be a sequence consisting of all rational numbers. Define

$$f_n(x) = \begin{cases} 1, & \text{if } x = \lambda_n \\ 0, & \text{otherwise.} \end{cases}$$

Then f_n converges pointwise to $f = 0$, but not uniformly on *any* interval of $\mathbb{R}$.

(h) Let $f_n(x) := nx^n$, $x \in [0,1)$. Show that the sequence (f_n) converges pointwise but not uniformly on $[0,1)$.

(i) Let $f_n(x) := \frac{x}{1+nx^2}$, $x \in \mathbb{R}$. Use Calculus to show that $f_n \rightrightarrows 0$ on $\mathbb{R}$.

(j) Let $f_n(x) := n^2x^n(1-x)$, $x \in [0,1]$. Show that $f_n \to 0$ pointwise but not uniformly on $[0,1]$. *Hint:* f_n takes a maximum value at $n/(n+1)$.

(k) Let $f_n(x) := nx^n(1-x)$, $x \in [0,1]$. Show that $f_n \to 0$ pointwise but not uniformly on $[0,1]$. *Hint:* f_n takes a maximum value at $n/(n+1)$.

(l) Let $f_n(x) := \frac{1+2\cos^2 nx}{\sqrt{n}}$, $x \in \mathbb{R}$. Show that f_n converges uniformly on $\mathbb{R}$.

(m) Let $f_n(x) = \frac{1}{1+x^n}$, $x \in [0,\infty)$. Show that $f_n \to f$ pointwise but not uniformly on the domain where $f(x) = \begin{cases} 1 & 0 \le x < 1 \\ 1/2 & x = 1 \\ 0 & x > 0 \end{cases}$.

(n) Discuss the pointwise and uniform convergence of the sequence (f_n) where $f_n(x) = \frac{x^n}{1+x^n}$, $x \in [0,\infty)$.

(o) Let $f_n(x) = \frac{x^n}{n+x^n}$, $x \in [0,\infty)$. Show that $f_n \to f$ pointwise but not uniformly on the domain where $f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x > 0 \end{cases}$. Show that $f_n \rightrightarrows f$ on $[0,1]$.

(p) Let $f_n(x) = x - x^n$, $x \in [0,1]$. Show that $f_n \to f$ pointwise but not uniformly on the domain where $f(x) = \begin{cases} x & 0 \le x < 1 \\ 0 & x = 1 \end{cases}$.

(q) $f_n(x) := (x/n)^n$ for $x \in [0, \infty)$.

(r) $f_n(x) := \frac{nx}{2n+x}$ for $x \in [0, \infty)$. Show that f_n converge uniformly on $[0, R]$ for any $R > 0$ but not on $[0, \infty)$.

(s) Let $f_n(x) := \frac{x^n}{1+x^{2n}}$ for $x \in [0, 1)$. Show that f_n is pointwise convergent on $[0, 1)$ but is not uniformly convergent on $[0, 1)$. *Hint:* Note that $\lim_{x \to 1-} f_n(x) = 1/2$. Hence $f_n(x_n) > 1/4$ at some x_n.

(t) Let $f_n \colon [-1, 1] \to \mathbb{R}$ be defined by $f_n(x) := \frac{x}{1+nx^2}$. Show that $f_n \rightrightarrows 0$ on $[-1, 1]$.

(u) Let $f_n(x) = \int_0^x e^{int}\, dt$ for $x \in \mathbb{R}$. The sequence $f_n \rightrightarrows 0$ on $\mathbb{R}$.

(v) Let f_n be R-integrable and $g_n(x) := \int_a^x f_n(t)\, dt$ and f_n converge uniformly on $[a, b]$. Show that g_n converges uniformly on $[a, b]$.

(w) Show that $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = \frac{x^n}{1+x^{2n}}$ converges uniformly on $[a, b]$ iff $[a, b]$ does not contain either of ± 1.

(x) Let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = \frac{nx}{1+n^2 xp}$, for $p > 0$. Find for what values of p the sequence f_n converges uniformly to the limit.

17. Let $f_n \colon (a, b) \to \mathbb{R}$ be differentiable. Assume that there exists $f, g \colon (a, b) \to \mathbb{R}$ such that $f_n \rightrightarrows f$ and $f'_n \rightrightarrows g$ on (a, b). Then f is differentiable and $f' = g$ on (a, b).

Proof. Fix $c \in (a, b)$. Consider $g_n := f_{n1}$ in the notation of Item 6, that is,

$$g_n(x) = \begin{cases} \frac{f_n(x) - f_n(c)}{x - c} & \text{for } x \neq c \\ f'_n(c) & \text{for } x = c. \end{cases}$$

Then g_n are continuous and they converge pointwise to $\varphi(x) = \frac{f(x)-f(c)}{x-c}$ for $x \neq c$ and $\varphi(c) = g(c)$.

We claim that g_n are uniformly Cauchy on (a, b) and hence uniformly convergent to a continuous function $\psi \colon (a, b) \to \mathbb{R}$. For, by the mean value theorem, we have, for some ξ between $x \neq c$ and c,

$$g_n(x) - g_m(x) = \frac{f'_n(\xi) - f'_m(\xi)(x - c)}{x - c} = f'_n(\xi) - f'_m(\xi).$$

Since (f'_n) converge uniformly on $J = (a, b)$, it is uniformly Cauchy and hence (g_n) is uniformly Cauchy on J. It follows from Item 13 that g_n converge uniformly to a function, say, ψ. The function ψ is continuous by Item 8. By the uniqueness of the pointwise limits, we see that $\varphi(x) = \psi(x)$ for $x \in J$. Hence φ is continuous or it is the "f_1" (in the notation of Theorem 6 on page 37) for the function f at c. Thus, f is differentiable at c with $f'(c) = \varphi(c) = g(c)$. □

18. Let $f_n \colon [a, b] \to \mathbb{R}$. Assume that there is some $x_0 \in [a, b]$ such that $(f_n(x_0))$ converges and that f'_n exist and converges uniformly to g on $[a, b]$. Then f_n converges uniformly to some f on $[a, b]$ such that $f' = g$ on $[a, b]$.

This is an immediate consequence of the last two items. □

19. Let (f_n) be a sequence of continuous functions on $[a,b]$. Assume that $f_n \to f$ uniformly on $[a,b]$. Then f is continuous and hence R-integrable on $[a,b]$. Furthermore $\int_a^b f(x)\,dx = \lim \int_a^b f_n(x)\,dx$. That is,

$$\lim_n \int_a^b f_n(t)\,dt = \int_a^b \lim_n f_n(t)\,dt.$$

Observe that

$$\begin{aligned} |\int_a^b f(x)\,dx - \int_a^b f_n(x)\,dx| &\leq \int_a^b |f(x) - f_n(x)|\,dx \\ &\leq (b-a) \times \text{l.u.b. } \{|f(x) - f_n(x)| : x \in [a,b]\}. \end{aligned}$$

20. Let $f_n \colon [a,b] \to \mathbb{R}$ be sequence of continuously differentiable functions converging uniformly to f on $[a,b]$. Assume that f_n' converge uniformly on $[a,b]$ to g. Then $g = f'$ on $[a,b]$. *Hint:* f is continuous on $[a,b]$. Write $f_n(x) = f_n(a) + \int_a^x f_n'(t)\,dt$ and take limits to conclude that $f(x) - f(a) = \int_a^x g(t)\,dt$. Apply fundamental theorem of calculus. □

21. Let (f_n) be a sequence of Riemann integrable functions on $[a,b]$. Assume that $f_n \to f$ uniformly on $[a,b]$. Then f is R-integrable on $[a,b]$ and $\int_a^b f(x)\,dx = \lim \int_a^b f_n(x)\,dx$. That is, $\lim \int_a^b f_n(x)\,dx = \int_a^b \lim f_n(x)\,dx$.

    Proof. Given $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\|f - f_n\| < \frac{\varepsilon}{4(b-a)}$ for $n \geq N$. (Here $\|f - f_n\| := \sup\{|f_n(x) - f(x)| \mid x \in [a,b]\}$.) Since f_N is R-integrable, there is a partition $P := \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ such that $U(f_N, P) - L(f_N, P) < \frac{\varepsilon}{2}$. Note that $M_j(f) \leq M_j(f_N) + \varepsilon/4(b-a)$, where $M_j(f) := \sup\{f(x) \mid x \in [x_{j-1}, x_j]\}$. Hence $U(f,P) \leq U(f_N, P) + \varepsilon/4$ and $L(f,P) + \varepsilon/4 \geq L(f_N, P)$. Hence $U(f,P) - L(f,P) < \varepsilon$. Hence f is R-integrable in $[a,b]$. The rest of the proof follows the hint of the previous analogous exercise. □

22. Exercises:

    (a) Let $f_n(x) := \begin{cases} 1, & \text{if } x = r_1, \dots, r_n, \\ 0, & \text{otherwise} \end{cases}$ for $x \in [0,1]$ and where $\{r_n\}$ is an enumeration of all the rationals in $[0,1]$. Then f_n is R-integrable, $f_n \to f$ pointwise but f is not R-integrable.

    (b) Let f_n, f be as in 4) of Item 2. Compute $\lim_n \int_0^1 f_n(t)\,dt$ and $\int_0^1 \lim f_n(t)\,dt$.

    (c) Let $f_n \colon [0,1] \to \mathbb{R}$ be given by $f_n(x) = nxe^{-nx^2}$. Then $f_n \to 0$ pointwise. Compute $\lim_n \int_0^1 f_n(t)\,dt$ and $\int_0^1 \lim f_n(t)\,dt$.

    (d) $f_n(x) = nx(1-x^2)^n$. Find the pointwise limit f of f_n. Does $\int_0^1 f_n(x)\,dx \to \int_0^1 f(x)\,dx$?

    (e) Let $f_n(x) = \frac{n^2x^2}{1+n^3x^3}$ on $[0,1]$. Show that f_n does not satisfy the conditions of Item 18, but that the derivative of the limit function exists on $[0,1]$ and is equal to the limit of the derivatives.

(f) If $f_n(x) = \frac{x}{1+n^2x^2}$ on $[-1,1]$, show that f_n is uniformly convergent, and that the limit function is differentiable, but $f' \neq \lim f_n'$ on $[-1,1]$.

(g) Let $f_n : [0,2] \to \mathbb{R}$ be defined by $(1+x^n)^{\frac{1}{n}}$. Show that f_n is differentiable on $[0,2]$ and converges uniformly to a limit function which is not differentiable at 1.

(h) Let $f(x) = |x|$ for $x \in \mathbb{R}$. We replace part of the graph of f on the interval $[-1/n, 1/n]$ by a part of the parabola that has correct values and the correct derivatives (so that the tangents match) at the end point $\pm(1/n)$. Let

$$f_n(x) := \begin{cases} \frac{nx^2}{2} + \frac{1}{2n}, & -1/n \leq x \leq 1/n \\ |x| & |x| > 1/n. \end{cases}$$

Show that $f_n \to f$ pointwise. Is the convergence uniform? Note that f_n are differentiable but f is not.

(i) Let $f_n(x) := x^n(x-2)$, $x \in [0,1]$. Show that $f_n \to g$ where $g(x) = 0$ for $0 \leq x < 1$ and $g(1) = -1$. Can g be the derivative of any function? *Hint:* Darboux!

23. Theoretical Exercises:

(a) Let $f_n \colon J \subseteq \mathbb{R} \to \mathbb{R}$ converge uniformly on J to f. Assume that f_n are uniformly continuous at $a \in X$. Then show that f is uniformly continuous at a.

(b) (Dini) Let $f_n \colon [a,b] \to \mathbb{R}$ be monotone. Assume that the sequence (f_n) converges pointwise to a *continuous* function f. Then $f_n \to f$ uniformly on $[a,b]$. *Hint:* Use the uniform continuity of f to partition the interval $[a,b]$ into $a = a_0 < a_1 < \cdots < a_N = b$. Choose n_0 such that $|f_n(a_i) - f(a_i)| < \varepsilon$ for $n \geq n_0$ and for each i.

(c) Let $f(x) = \sqrt{x}$ on $[0,1]$. Let $f_0 = 0$ and $f_{n+1}(x) := f_n(x) + \left[x - (f_n(x))^2\right]/2$ for $n \geq 0$. Show that i) f_n is a polynomial, ii) $0 \leq f_n \leq f$, iii) $f_n \to f$ pointwise and iv) $f_n \to f$ uniformly on $[0,1]$.

(d) Let $f_n \colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} 0, & x \leq n \\ x - n, & n \leq x \leq n+1 \\ 1, & x \geq n+1 \end{cases}.$$

Show that $f_n \geq f_{n+1}$, $f_n \to 0$ pointwise but not uniformly. Compare this with Item 23b

(e) Let $f \colon [0,1] \to \mathbb{R}$ be continuous. Consider the partition $\{0, 1/n, \ldots, \frac{n-1}{n}, 1\}$ of $[0,1]$. Define

$$f_n(t) := \begin{cases} f(k/n), & (k-1)/n \leq t \leq k/n \\ f(1/n), & t = 0 \end{cases}$$

for $1 \leq k \leq n$. Then f_n is a step function taking the value $f(k/n)$ on $(k-1/n, k/n]$. Show that $f_n \to f$ uniformly. *Hint:* Use the uniform continuity of f: Given ε choose δ and then N so that $1/N < \delta$. Then l.u.b. $\{|f(x) - f_N(x)| : x \in [0,1]\} < \varepsilon$.

(f) Let $f \colon \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Let $f_n(x) = f(x + \frac{1}{n})$. Show that $f_n \to f$ uniformly on $\mathbb{R}$.

(g) Let (f_n) be a sequence of real valued functions converging uniformly on X. Let $|f_n(x)| \le M$ for all $n \in \mathbb{N}$ and $x \in X$. Assume that $g\colon [-M, M] \to \mathbb{R}$ be continuous. Show that $(g \circ f_n)$ is uniformly convergent on X.

(h) Let $\phi : [0,1] \to \mathbb{R}$ be continuous. Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = x^n \phi(x)$. Prove that f_n converges uniformly on $[0,1]$ iff $\phi(1) = 0$.

(i) Let $f_n\colon X \to \mathbb{R}$. We say that the sequence (f_n) is uniformly bounded on X if there exists $M > 0$ such that

$$|f_n(x)| \le M \text{ for all } x \in X \text{ and } n \in \mathbb{N}.$$

Assume that f_n's are bounded and that $f_n \rightrightarrows f$ on X. Show that the sequence (f_n) is uniformly bounded and that f is bounded.

(j) Assume that $f_n\colon J \subset \mathbb{R} \to \mathbb{R}$ are continuous and $f_n \to f$ uniformly on $\mathbb{Q} \cap J$. Show that $f_n \rightrightarrows f$ on J.

(k) Let f be a continuously differentiable function on $\mathbb{R}$. Let $f_n(x) := n[f(x + \frac{1}{n}) - f(x)]$. Then $f_n \to f'$ uniformly on compact subsets of $\mathbb{R}$.

24. **Weierstrass approximation theorem.** Let $f\colon [0,1] \to \mathbb{R}$ be continuous. Given $\varepsilon > 0$, there exists a real polynomial function $P = P(x)$ such that

$$|f(x) - P(x)| < \varepsilon \text{ for all } x \in [0,1].$$

In particular, there exists a sequence P_n of polynomial functions such that $P_n \rightrightarrows f$ on $[0,1]$.

Proof. We need the identity

$$\frac{x(1-x)}{n} = \sum_{k=0}^{n} x^k (1-x)^{n-k} \left(x - \frac{k}{n}\right)^2. \tag{54}$$

Consider

$$1 = (x + (1-x))^n = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}. \tag{55}$$

Differentiate both sides of (55) and simplify to obtain

$$0 = \sum_{k=0}^{n} x^{k-1} (1-x)^{n-k-1} (k - nx). \tag{56}$$

Multiply both sides of (56) by $x(1-x)$, to obtain

$$0 = \sum_{k=0}^{n} x^k (1-x)^{n-k} (k - nx). \tag{57}$$

Differentiate both sides of (57) and multiply through by $x(1-x)$. On simplification, we get

$$0 = -nx(1-x) + \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (k - nx)^2. \tag{58}$$

Dividing both sides by n^2, we obtain (54).

We now define Bernstein polynomial B_n by

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k(1-x)^{n-k}.$$

Then by (55),

$$B_n(x) - f(x) = \sum_{k=0}^{n} \binom{n}{k} \left(f\left(\frac{k}{n}\right) - f(x) \right) x^k(1-x)^{n-k}. \tag{59}$$

Let $\varepsilon > 0$ be given. By uniform continuity of f on $[0,1]$, there exists $\delta > 0$ such that

$$x, y \in [0,1] \text{ and} |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon/4.$$

Let $M > 0$ be such that $|f(x)| \leq M$ for $x \in [0,1]$. Choose $N \in \mathbb{N}$ such that $N > \frac{M}{\varepsilon\delta^2}$.

Let $x \in [0,1]$ and $0 \leq k \leq n$. We can write $\{0,1,2,\ldots,n\} = A \cup B$ where

$$A := \left\{ k : |x - \frac{k}{n}| < \delta \right\} \text{ and } B := \left\{ k : |x - \frac{k}{n}| \geq \delta \right\}.$$

Case 1. $k \in A$. Then we have $|f(x) - f(k/n)| < \varepsilon/4$. Summing over those $k \in A$, we have by (55), that

$$\sum \binom{n}{k} |f\left(\frac{k}{n}\right) - f(x)| x^k(1-x)^{n-k} \leq \frac{\varepsilon}{4}. \tag{60}$$

Case 2. $k \in B$. We have, summing over $k \in B$

$$\begin{aligned}
&\sum_{k \in B} \binom{n}{k} \left(|f(k/n)| + |f(x)|\right) x^k(1-x)^{n-k} \\
&\leq \ 2M \sum \binom{n}{k} \left(x - \frac{k}{n}\right)^2 \left(x - \frac{k}{n}\right)^{-2} x^k(1-x)^{n-k} \\
&\leq \ 2M\delta^{-2} \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} \left(x - \frac{k}{n}\right)^2 \\
&= \ 2M\delta^{-2} \frac{x(1-x)}{n}, \text{ by (54)}, \\
&\leq \ 2\varepsilon x(1-x), \text{ since } n > \frac{M}{\varepsilon\delta^2}, \\
&\leq \ \varepsilon/2, \text{ since } x(1-x) \leq 1/4,
\end{aligned} \tag{61}$$

for, $4x(1-x) - 1 = -(2x-1)^2 \leq 0$ (or by 2nd derivative test).

It follows from (59)–(61) that

$$|B_n(x) - f(x)| \leq \frac{3\varepsilon}{4} < \varepsilon,$$

for $x \in [0,1]$ and $n \geq N$. $\square$

25. Weierstrass theorem remains true if $[0,1]$ is replaced by any closed and bounded interval $[a,b]$.

For, consider the map $h\colon [0,1] \to [a,b]$ defined by $h(t) := a + t(b-a)$. Then h a continuous bijection. Also, $t := h^{-1}(x) = (x-a)/(b-a)$ is continuous. Given a continuous function $g\colon [a,b] \to \mathbb{R}$, the function $f := g \circ h\colon [0,1] \to \mathbb{R}$ is continuous. For $\varepsilon > 0$, let P be a polynomial such that $|f(t) - P(t)| < \varepsilon$ for all $t \in [0,1]$. Then $Q(x) := P \circ h^{-1}(t) = P(\frac{x-a}{b-a})$ is a polynomial in x. Observe that, for all $x = h(t) \in [a,b]$,

$$|g(x) - Q(x)| = |f \circ h^{-1}(x) - P \circ h^{-1}(x)| = |f(t) - P(t)| < \varepsilon.$$

26. **Probabilistic reason underlying the Bernstein polynomial.** The proof of Weierstrass approximation theorem using Bernstein polynomials, has its origin in probability. Imagine a loaded or biased coin which turns 'heads' with probability t, $0 \le t \le 1$. If a player tosses the coin n times, the probability of getting the 'heads' k times is given by $\binom{n}{k} t^k (1-t)^{n-k}$.

Now suppose that a continuous function f, considered as a 'payoff', assigns a 'prize' as follows: the player will get $f\left(\frac{k}{n}\right)$ rupees if he gets exactly k heads out of n tosses. Then the 'expected value' E_n, (also known as the 'mean') the player is likely to get out of a game of n tosses is

$$E_n = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} t^k (1-t)^{n-k}.$$

Note that E_n is the n-th Bernstein polynomial of f. It is thus the average/mean value of a game of n tosses.

It is reasonable to expect that if n is very large, the head will turn up approximately nt times. This implies that the 'prize' $f\left(\frac{tn}{n}\right) = f(t)$ and $E_n(t)$ are likely to be very close to each other. That is, we expect that $|f(t) - E_n(t)| \to 0$.

Our treatment of the sum over B is motivated by the proof of Chebyshev's inequality used in the proof of weak law of large numbers.

27. A standard application. Let $f\colon [0,1] \to \mathbb{R}$ be continuous. Assume that $\int_0^1 f(x)x^n\, dx = 0$ for $n \in \mathbb{Z}_+$. Then $f = 0$. *Hint:* Figure this out: $0 = \int_0^1 f(x)P_n(x)\, dx \to \int_0^1 f^2(x)\, dx$!

28. Let $f_n\colon X \to \mathbb{R}$ be a sequence of functions from a set X to $\mathbb{R}$. We say the series $\sum f_n$ is uniformly convergent (respectively, pointwise convergent) on X if the sequence (s_n) of partial sums $s_n := \sum_{k=1}^{n} f_k$ is uniformly convergent (respectively pointwise convergent) on X. If f is the uniform limit of (s_n) we write $\sum f_n = f$ uniformly on X.

We say the series $\sum f_n$ is absolutely convergent on X if the sequence $(\sigma_n(x))$ of partial sums $\sigma_n(x) := \sum_{k=1}^{n} |f_k(x)|$ is convergent for each $x \in X$.

29. Let $X = [0,1]$ and $f_n(x) := x^n$ on X. Then the infinite series of functions $\sum_{n=0}^{\infty} f_n(x)x^n$ is pointwise convergent to $1/(1-x)$ for $x \in [0,1)$ and not convergent when $x = 1$. The infinite series is not uniformly convergent on $[0,1)$. (Why?)

30. Formulate the analogue of Cauchy criterion for the uniform convergence of an infinite series of $\mathbb{R}$ (or $\mathbb{C}$) valued functions.

31. **Weierstrass M-test.** Let f_n be a sequence of real (or complex) valued functions on a set X. Assume that there exist $M_n \geq 0$ such that $|f_n(x)| \leq M_n$ for all $n \in \mathbb{N}$ and $x \in X$ and that $\sum_n M_n < \infty$. Then the series $\sum_n f_n$ is absolutely and uniformly convergent on X.

Proof. Apply the Cauchy criterion to s_n. □

32. Typical applications:

(a) Fix $0 < r < 1$. The series $\sum_{n=1}^{\infty} r^n \cos nt$ and $\sum_{n=1}^{\infty} r^n \sin nt$ are uniformly convergent on $\mathbb{R}$.

(b) The series $\sum_{n=1}^{\infty} \frac{x}{1+nx^2}$ is uniformly convergent on any interval $[a,b]$.
Using $(a+b)/2 \geq \sqrt{ab}$ for positive a, b, we obtain

$$\frac{\frac{1}{x} + nx}{2} \geq \sqrt{\frac{1}{x} nx} = \sqrt{n}, \text{ for } x > 0.$$

Hence $|\frac{x}{n(1+nx^2)}| \leq \frac{1}{2n^{3/2}}$. (One may also use calculus to obtain such an estimate.) M-test can be applied now. What happens if $x \leq 0$?

33. Exercises:

(a) Show that if $w_n(x) = (-1)^n x^n (1-x)$ on (0,1), then $\sum w_n$ is uniformly convergent.

(b) If $u_n(x) = x^n(1-x)$ on $[0,1]$, then does the series $\sum u_n$ converge? Is the convergence uniform?

(c) If $v_n(x) = x^n(1-x)^2$ on $[0,1]$, does $\sum v_n$ converge? Is the convergence uniform?

(d) Show that $\sum x^n(1-x^n)$ converges pointwise but not uniformly on $[0,1]$. What is the sum?

(e) Prove that $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is continuous on $\mathbb{R}$.

(f) Prove that $\sum_{n=1}^{\infty} \frac{1}{1+x^n}$ is continuous for $x > 1$. *Hint:* $x^n \geq 2$ for large n. Hence

$$\frac{1}{1+x^n} \leq \frac{1}{x^n - 1} \leq \frac{2}{x^n}, \text{ for } n \gg 0.$$

(g) Prove that $\sum_{n=1}^{\infty} e^{-nx} \sin nx$ is continuous for $x > 0$.
The next two exercises are analogues of Item 17 and Item 21 for series.

(h) Let $f_n\colon (a,b) \to \mathbb{R}$ be differentiable. Let $x_0 \in (a,b)$ be such that $\sum f_n(x_0)$ converges. Assume further that there is $g\colon (a,b) \to \mathbb{R}$ such that $\sum f_n' = g$ uniformly on (a,b). Then

i. There is an $f\colon (a,b) \to \mathbb{R}$ such that $\sum f_n = f$ uniformly on (a,b).
ii. $f'(x)$ exists for all $x \in (a,b)$ and we have $\sum f_n' = f'$ uniformly on (a,b).

(i) Let $f_n, f\colon [a,b] \to \mathbb{R}$ be such that $\sum f_n = f$ uniformly on $[a,b]$. Assume that each f_n is R-integrable.

i. f is R-integrable.
ii. $\int_a^b f(t)\,dt = \sum \int_a^b f_n(t)\,dt$.

(j) Justify:

$$\frac{d}{dx}\sum_{n=1}^{\infty}\frac{\sin nx}{n^3} = \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}.$$

34. * We have analogoues of Dirichlet's and Abel's test for uniform convergence too. We state the results, as the proofs follow from the observation that the estimates obtained in the earlier results are 'uniform'.

35. ***Dirichlet's Test.** Let (f_n) and (g_n) be two sequences of real valued functions on a set X. Let $F_n(x) := \sum_{k=1}^{n} f_k(x)$. Assume that
    (i) (F_n) is uniformly bounded on X, that is, there exists $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in X$ and $n \in \mathbb{N}$.
    (ii) (g_n) is monotone decreasing to 0 pointwise on X.
    Then the series $\sum_{n=1}^{\infty} f_n g_n$ is uniformly convergent on X. □

36. * **Abel's Test.** Let (f_n) and (g_n) be two sequences of real valued functions on a set X. Assume that
    (i) $\sum_n f_n$ is uniformly convergent on X.
    (ii) There exists $M > 0$ such that $|g_n(x)| \leq M$ for all $x \in X$ and $n \in \mathbb{N}$.
    (iii) $(g_n(x))$ is monotone for each $x \in X$.
    Then the series $\sum_{n=1}^{\infty} f_n g_n$ is uniformly convergent on X. □

37. * Consider the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ and $\sum_{n=1}^{\infty} \frac{\cos nx}{n^p}$. If $p > 1$, we can apply M-test to conclude that they are uniformly convergent on $\mathbb{R}$.

    When $0 < p \leq 1$, we can use Dirichlet's test to show that both series converge uniformly on $[\delta, 2\pi - \delta]$ for $0 < \delta < 2\pi$. See Item 34.

38. A *power series* is an expression of the form $\sum_{k=0}^{\infty} a_k (x-a)^k$ where $a_k, a, z \in \mathbb{R}$. We do not assume that the series converges.

39. Consider the three power series:
    (1) $\sum_{n=1}^{\infty} n^n x^n$,
    (2) $\sum_{n=0}^{\infty} x^n$ and
    (3) $\sum_{n=0}^{\infty} (x^n/n!)$.
    We claim that if $x \neq 0$ then the first series does not converge. For, if $x \neq 0$, choose $N \in \mathbb{N}$ so that $1/N < |x|$. Then for all $n \geq N$, we have $|(nx)^n| > 1$ and hence the series is not convergent. We have already seen that the second series converges absolutely for all x with $|x| < 1$ whereas the third series converges absolutely for all $x \in \mathbb{R}$.

40. Let $\sum_{n=0}^{\infty} a_n (z-a)^n$ be a power series. There is a unique extended real number R, $0 \leq R \leq \infty$, such that the following hold:

    (i) for all x with $|x - a| < R$, the series $\sum_{n=0}^{\infty} a_n (z-a)^n$ converges absolutely and uniformly, say, to a function f, on $(-R, R)$ for any $0 < r < R$,

    (ii) if $0 < R \leq \infty$, then f is continuous, differentiable on $(-R, R)$ with derivative $f'(x) = \sum_n n a_n x^{n-1}$,

    (iii) Term-wise integration is also valid, that is, $\int_x^y f(t)\,dt = \sum_n a_n \int_x^y (t-a)^n\,dt$ for $-R < x < y < R$.

    (iv) for all x with $|x - a| > R$, the series $\sum_{n=0}^{\infty} a_n (z-a)^n$ diverges.

Proof. Assume $a = 0$. Let $E := \{|z| : \sum_{n=0}^{\infty} a_n z^n \text{ is convergent.}\}$ and $R :=$ l.u.b. E, if E is bounded above, otherwise $r = \infty$. Then $\sum_{n=0}^{\infty} a_n z^n$ is divergent if $|z| > R$, by very definition. Hence (iii) is proved.

If $R > 0$ choose r such that $0 < r < R$. Since R is the least upper bound for E and $r < R$, there exists $z_0 \in E$ such that $|z_0| > r$ and $\sum a_n z_0^n$ is convergent. Hence $\{a_n z_0^n\}$ is bounded, say, by M:

$$|a_n z_0^n| \le M \text{ for all } n.$$

Now, if $|z| \le r$, then

$$|a_n z^n| \le |a_n| r^n \le |a_n z_0^n| (r/|z_0|)^n \le M (r/|z_0|)^n.$$

But the ("essentially geometric") series $M \sum (r/|z_0|)^n$ is convergent. By Weierstrass M-test, the series $\sum_{n=0}^{\infty} a_n z^n$ is uniformly and absolutely convergent for z with $|z| \le r$.

f is continuous at any x with $|x| < R$. For if r is chosen so that $|x| < r < R$, then the series $\sum_n a_n x^n$ is uniformly convergent on $(-r, r)$ and hence the sum, namely, f is continuous on $(-r, r)$, in particular at x.

We claim that f is differentiable. First of all note that the term-wise differentiated series $\sum_n n a_n x^{n-1}$ is uniformly convergent on any $(-r, r)$, $0 < r < R$. For, arguing as in the case of uniform convergence, we have an estimate of the form

$$\begin{aligned}
\sum_{k=0}^{\infty} |k c_k x^{k-1}| &\le \sum_{k=0}^{\infty} k |c_k| r^{k-1} \\
&= \sum_{k=0}^{\infty} k |c_k x_0^k| \frac{r^{k-1}}{|x_0|^{k-1}} |x_0|^{-1} \\
&\le (M/r) \sum_{k=0}^{\infty} k t^{k-1}, \text{ where } t = (r/|x_0|).
\end{aligned}$$

The series $\sum k t^{k-1}$ is convergent by ratio test. By Weierstrass M-test, the term-wise differentiated series is uniformly convergent on $(-r, r)$. Now we can appeal to Item 17 to conclude the desired result.

The proof of (iii) is similar and left to the reader. □

41. The extended real number R of the theorem in Item 40 is called the *radius of convergence* of the power series $\sum_{n=0}^{\infty} a_n (z-a)^n$. The open interval $(-R, R)$ is called the interval of convergence of the power series.

    There are 'formulas' to find the radius of convergence R. See Items 44 and 13.

42. It is important to note that if R is the radius of convergence of a power series $\sum_{n=0}^{\infty} c_n x^n$, it is uniformly convergent *only* on the subintervals $(-r, r)$ for $0 < r < R$. The theorem does not claim that it is uniformly convergent on $(-R, R)$. For example, consider the power series $\sum_{n=0}^{\infty} x^n$. Its radius of convergence is $R = 1$. (Why?) We have seen in Item 29 that it is not uniformly convergent on $[0, 1)$.

43. The following are well-known power series:
(i) $\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$.
(ii) $\sin(x) := \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n-1}}{(2n-1)!}$.
(iii) $\cos(x) := \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$.

The following theorem is quite useful in practice, as it gives two simple formulae to determine the radius of convergence of a power series.

44. * Let $\sum_{n=0}^{\infty} a_n(z-a)^n$ be given. Assume that one of the following limits exists as an extended real number.
(1) $\lim |a_{n+1}/a_n| = \rho$
(2) $\lim |a_n|^{1/n} = \rho$.
Then the radius of convergence of the power series is given by $R = \rho^{-1}$.

Proof. (1) follows from the ratio test and (2) from the root test. For instance, if (1) holds and if z is fixed, then

$$\left|\frac{a_{n+1}(z-a)^{n+1}}{a_n(z-a)^n}\right| = |z-a|\left|\frac{a_{n+1}}{a_n}\right| \to |z-a|\rho.$$

By the ratio test, we know that the *numerical* series $\sum_n a_n(z-a)^n$ is convergent if $|z-a|\rho < 1$. Thus the radius of convergence of the series is at least $1/\rho$. By the same ratio test, we know that if $|z-a|\rho > 1$, then the series is divergent. Hence we conclude that the radius of convergence of the given series is $1/\rho$.

The proof of (ii) is similar to that of (i) and is left to the reader. □

45. If a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ has a positive radius of convergence $0 < R \leq \infty$, then its sum defines a function, say f, on the interval $(a-R, a+R)$. The function f is infinitely differentiable on this interval. Also, the Taylor series of f in powers of $(x-a)$ is the original power series. We also have $c_n := \frac{f^{(n)}(a)}{n!}$.

46. Find an "explicit formula" for the function represented by the power series $\sum_{k=1}^{\infty} kx^k$ in its interval of convergence.

Let $f(x) := \sum_{k=1}^{\infty} kx^k$. The interval of convergence is $(-1,1)$. By term-wise integration, we see that

$$\int_0^x \frac{f(t)}{t}\,dt = \sum_{k=1}^{\infty} k \int_0^x t^{k-1}\,dt = \sum_{k=1}^{\infty} x^k = \frac{x}{1-x},\ x \in (-1,1).$$

Hence by the fundamental theorem of calculus, we obtain

$$\frac{f(x)}{x} = \left(\frac{x}{1-x}\right)' = \frac{1}{(1-x)^2}.$$

Therefore, $f(x) = \frac{x}{(1-x)^2}$ on $(-1,1)$.

47. * **Abel's Limit Theorem.** *If* $\sum_{n=0}^{\infty} a_n$ *converges, then the power series* $\sum_{n=0}^{\infty} a_n x^n$ *converges uniformly on [0,1].*

Proof. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$N \leq m \leq n \implies |\sum_{m+1}^{n} a_k| < \varepsilon.$$

If $0 \leq x \leq 1$, we apply (47) on page 60 to $(a_k)_{k=m+1}^{\infty}$ and $(x^k)_{k=m+1}^{\infty}$ to obtain

$$-\varepsilon x^{m+1} < a_{m+1}x^{m+1} + \cdots + a_n x^n < \varepsilon x^{m+1}.$$

That is,

$$|\sum_{k=m+1}^{n} a_k x^k| < \varepsilon x^{m+1} \leq \varepsilon, \text{ for } N \leq m \leq n \text{ and } x \in [0,1].$$

If we let $f_n(x) := \sum_{k=0}^{n} a_k x^k$, it follows from the last inequality that the sequence (f_n) is uniformly Cauchy on $[0,1]$. □

48. * It may be a good exercise to write a 'self-contained' proof of the last theorem which incorporates the partial summation trick and which does not use it.

49. * Find an explicit expression for the function represented by the power series $\sum_{k=0}^{\infty} \frac{x^k}{k+1}$.
Clearly $R = 1$. Also, observe that at $R = -1$,the series is convergent by the alternating series test. By term-wise differentiation, we get

$$(xf(x))' = \sum_{k=0}^{\infty} \left(\frac{x^{k+1}}{k+1}\right)' = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \ x \in (-1,1).$$

By the fundamental theorem of calculus, we obtain

$$xf(x) = \int_0^x \frac{dt}{1-t} = -\log(1-x), \ x \in (-1,1).$$

Note that we can appeal to Abel's theorem (Item 47) to conclude that

$$f(x) = \begin{cases} -\frac{\log(1-x)}{x} & x \in [-1,1), x \neq 0 \\ 1 & x = 0. \end{cases}$$

50. * An application of Abel's Limit Theorem. Let $\sum a_n$ and $\sum b_n$ be convergent, say, with sums A and B respectively. Assume that their Cauchy product $\sum c_n$ is also convergent to C. Then $C = AB$.

Proof. Consider the power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$. From hypothesis, their radii of convergence is at least 1. For $|x| < 1$, both the series are absolutely convergent, and hence by Merten's theorem, we have

$$\sum_{n=0}^{\infty} c_n x^n = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right).$$

In this we let $x \to 1-$ and apply Abel's theorem to conclude the result. □

51. * The sum of the standard alternating series is $\log 2$! Consider $f(x) = \log(1+x)$. Then the Taylor series of f at $x = 0$ is given by

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots$$

Clearly the power series is convergent at $x = 1$ by the alternating series test. Hence by Abel's limit theorem, we see that the power series is uniformly convergent on $(-1, 1]$ and hence the sum of the series at $x = 1$ is $f(1) = \log 2$. □

52. Two standard applications of the M-test are:
(i) Construction of an everywhere continuous and nowhere differentiable function on $\mathbb{R}$.
(ii) Construction of a space filling curve, that is, a continuous map from the unit interval $[0, 1]$ onto the unit square $[0, 1] \times [0, 1]$.

We shall not deal with them in our course. The reader may consult Appendix E in Bartle and Sherbert.

7 Limit Inferior and Limit Superior

1. **Limit Inferior and Limit Superior.** Given a bounded sequence (a_n) of real numbers, let $A_n := \{x_k : k \geq n\}$. Consider the numbers

$$s_n := \inf\{a_k : k \geq n\} \equiv \inf A_n \text{ and } t_n := \sup\{a_k : k \geq n\} \equiv \sup A_n.$$

If $|x_k| \leq M$ for all n, then $-M \leq s_n \leq t_n \leq M$ for all n. The sequence (s_n) is an increasing sequence of reals bounded above while (t_n) is a decreasing sequence of reals bounded below. Let

$$\liminf a_n := \lim s_n \equiv \text{l.u.b. } \{s_n\} \text{ and } \limsup a_n := \lim t_n \equiv \text{g.l.b. } \{t_n\}.$$

In case, the sequence (a_n) is not bounded above, then its $\limsup$ is defined to be $+\infty$. Similarly, the $\liminf$ of a sequence not bounded below is defined to be $-\infty$.

2. Let (x_n) be the sequence where $x_n = (-1)^{n+1}$. Then $\liminf x_n = -1$ and $\limsup x_n = 1$.

3. For any bounded sequence (x_n), we have $\liminf x_n \leq \limsup x_n$. *Hint:* $s_n \leq t_n$.

4. Let (a_n) be a bounded sequence of real numbers with $t := \limsup a_n$. Let $\varepsilon > 0$. Then
(a) There exists $N \in \mathbb{N}$ such that $a_n < t + \varepsilon$ for $n \geq N$.
(b) $t - \varepsilon < a_n$ for infinitely many n.
(c) In particular, there exists infinitely many $r \in \mathbb{N}$ such that $t - \varepsilon < a_r < t + \varepsilon$.

Proof. Let $A_k := \{x_n : n \geq k\}$.

(a) Note that $\limsup a_n = \inf t_n$ in the notation used above. Since $t + \varepsilon$ is greater than the greatest lower bound of (t_n), $t + \varepsilon$ is not a lower bound for t_n's. Hence there exists $N \in \mathbb{N}$ such that $t + \varepsilon > t_N$. Since t_N is the least upper bound for $\{x_n : n \geq N\}$, it follows that $t + \varepsilon > x_n$ for all $n \geq N$.

(b) $t - \varepsilon$ is less than the greatest lower bound of t_n's and hence is certainly a lower bound for t_n's. Hence, for any $k \in \mathbb{N}$, $t - \varepsilon$ is less than t_k, the least upper bound of

$\{a_n : n \geq k\}$. Therefore, $t - \varepsilon$ is not an upper bound for $\{a_n : n \geq k\}$. Thus, there exists n_k such that $a_{n_k} > t - \varepsilon$. For $k = 1$, let n_1 be such that $a_{n_1} > t - \varepsilon$. Since $t - \varepsilon$ is not an upper bound of A_{n_1+1} there exists $n_2 \geq n_1 + 1 > n_1$ such that $t - \varepsilon < a_{n_2}$. Proceeding this way, we get a subsequence (a_{n_k}) such that $t - \varepsilon < a_{n_k}$ for all $k \in \mathbb{N}$. □

5. Analogous results for lim inf: Let (a_n) be a bounded sequence of real numbers with $s := \liminf a_n$. Let $\varepsilon > 0$. Then
   (a) There exists $N \in \mathbb{N}$ such that $a_n > t - \varepsilon$ for $n \geq N$.
   (b) $t + \varepsilon > a_n$ for infinitely many n.
   (c) In particular, there exists infinitely many $r \in \mathbb{N}$ such that $s - \varepsilon < a_r < s + \varepsilon$.

6. Understand the last two results by applying them to the sequence with $x_n = (-1)^{n+1}$.

7. A sequence (x_n) in $\mathbb{R}$ is convergent iff (i) its bounded and (ii) $\limsup x_n = \liminf x_n$, in which case $\lim x_n = \limsup x_n = \liminf x_n$.

   Proof. Assume that $x_n \to x$. Then (x_n) is bounded. Then $s = \liminf x_n$ and $t = \limsup x_n$ exist. We need to show that $s = t$. Note that $s \leq t$. Let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies x - \varepsilon < x_n < x + \varepsilon.$$

   In particular, $x - \varepsilon < s_N := \inf\{x_n : n \geq N\}$ and $t_N := \sup\{x_n : n \geq N\} < x + \varepsilon$. But we have

$$s_N \leq \liminf x_n \leq \limsup x_n \leq t_N.$$

   Hence it follows that

$$x - \varepsilon < s_N \leq s \leq t \leq t_N < x + \varepsilon.$$

   Thus, $|s - t| \leq 2\varepsilon$. This being true for all $\varepsilon > 0$, we deduce that $s = t$. Also, $x, s \in (x - \varepsilon, x + \varepsilon)$ for each $\varepsilon > 0$. Hence $x = s = t$.

   Let $s = t$ and $\varepsilon > 0$ be given. Using Items 5 and 4, we see that there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies s - \varepsilon < x_n \text{ and } x_n < s + \varepsilon.$$

   □

8. A traditional proof of the Cauchy completeness of $\mathbb{R}$ runs as follows.

   Proof. Let (x_n) be a Cauchy sequence of real numbers. Then it is bounded and hence $s = \liminf x_n$ and $t = \limsup x_n$ exist as real numbers. It suffices to show that $s = t$. Since $s \leq t$ always, we need only show that $t \leq s$, that is, $t \leq s + \varepsilon$ for any give $\varepsilon > 0$. Since (x_n) is Cauchy there exists $N \in \mathbb{N}$ such that

$$m, n \geq N \implies |x_n - x_m| < \varepsilon/2, \text{ in particular, } |x_n - x_N| < \varepsilon/2.$$

   It follows that for $n \geq N$,

$$x_N - \varepsilon/2 \leq \text{g.l.b. } \{x_n : n \geq N\} \leq \text{l.u.b. } \{x_n : n \geq N\} \leq x_N + \varepsilon/2.$$

Hence, we obtain

$$t_n := \text{l.u.b. } \{x_n : n \geq N\} \leq \text{g.l.b. } \{x_n : n \geq N\} + \varepsilon = s_n + \varepsilon, \text{ for } n \geq N.$$

Taking limits, we get $\lim t_n \leq \lim s_n + \varepsilon$. □

9. Let (x_n) be a bounded sequence of real numbers. Let

$$S := \{x \in \mathbb{R} : x \text{ is the limit of a subsequence of } (x_n)\}.$$

Then

$$\limsup x_n, \liminf x_n \in S \subseteq [\liminf x_n, \limsup x_n]$$

Proof. First, notice that S is nonempty, since by Bolzano-Weierstrass theorem there exists a convergent subsequence. Also, if $-M \leq x_n \leq M$, then the limit x of any convergent subsequence will also satisfy $-M \leq x \leq M$. Hence S is bounded.

Let $s = \liminf x_n$. Then by Item 5 there exists infinitely many n such that $s - \varepsilon < x_n < s + \varepsilon$. Hence for each $\varepsilon = 1/k$, we can find $n_k > n_{k-1}$ such that $s - \frac{1}{k} < x_{n_k} < s + \frac{1}{k}$. It follows that $x_{n_k} \to s$ and hence $s \in S$. One shows similarly that $t \in S$.

Let $x \in S$. Let $x_{n_k} \to x$. We shall show $x \leq t + \varepsilon$ for any $\varepsilon > 0$. By Item 4, there exists N such that $n \geq N$ implies $x_n < t + \varepsilon$. Hence there exists k_0 such that if $k \geq k_0$, then $x_{n_k} < t + \varepsilon$. It follows that the limit x of the sequence (x_{n_k}) is at most $t + \varepsilon$. □

10. Note that the last item gives another proof of Cauchy completeness of $\mathbb{R}$. For, $\limsup x_n$ is the limit of a convergent subsequence, see Item 48!

11. Note that Item 9 implies the following.

$$\limsup x_n = \text{l.u.b. } S = \max S \quad \text{and} \quad \liminf x_n = \text{g.l.b. } S = \min S.$$

In fact, in some treatments, $\limsup x_n$ (respectively, $\liminf x_n$) is defined as l.u.b. S (respectively, g.l.b. S). I believe that our approach is easy to understand and allows us to deal with these concepts more easily.

12. Exercises on limit superior and inferior.

    (a) Consider $(x_n) := (1/2, 2/3, 1/3, 3/4, 1/4, 4/5, \ldots, 1/n, n/(n+1), \ldots)$. Then $\limsup = 1$ and $\liminf x_n = 0$.

    (b) Find the limsup and liminf of the sequences whose n-th term is given by:

        i. $x_n = (-1)^n + 1/n$
        ii. $x_n = 1/n + (-1)^n/n^2$
        iii. $x_n = (1 + 1/n)^n$
        iv. $x_n = \sin(n\pi/2)$.

There is a formula for the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ in terms of the coefficients a_n.

13. ***Hadamard formula for the radius of convergence** The radius of convergence ρ of $\sum_{n=0}^{\infty} c_n(z-a)^n$ is given by

$$\frac{1}{\rho} = \limsup |c_n|^{1/n} \text{ and } \rho = \liminf |c_n|^{-1/n}.$$

Proof. Let $\frac{1}{\beta} := \limsup |c_n|^{1/n}$. We wish to show that $\rho = \beta$.

If z is given such that $|z-a| < \beta$, choose μ such that $|z-a| < \mu < \beta$. Then $\frac{1}{\mu} > \frac{1}{\beta}$ and hence there exists N (by the last lemma) such that $|c_n|^{1/n} < \frac{1}{\mu}$ for all $n \geq N$. It follows that $|c_n|\mu^n < 1$ for $n \geq N$. Hence $(|c_n|\mu^n)$ is bounded, say, by M. Hence, $|c_n| \leq M\mu^{-n}$ for all n. Consequently,

$$|c_n(z-a)^n| \leq M\mu^{-n}|z-a|^n = M\left(\frac{|z-a|}{\mu}\right)^n.$$

Since $\frac{|z-a|}{\mu} < 1$, the convergence of $\sum c_n(z-a)^n$ follows.

Let $|z-a| > \beta$ so that $\frac{1}{|z-a|} < \frac{1}{\beta}$. Then $\frac{1}{|z-a|} < |c_n|^{1/n}$ for infinitely many n. Hence $|c_n||z-a|^n \geq 1$ for infinitely many n so that the series $\sum c_n(z-a)^n$ is divergent. We therefore conclude that $\rho = \beta$.

The other formula for the radius of convergence is proved similarly. □

14. * Exercise: Find the radius of convergence of the power series $\sum_n a_n z^n$, whose n-th coefficient a_n is given below:

(1)	$1/(n^2+1)$	(2)	$2^n - 1$
(3)	$e^{in\pi}/n$	(4)	$(1+i)^n$
(5)	$n^k, \quad k$ fixed	(6)	n^n
(7)	$n^k/n!$	(8)	$(1+\dfrac{2}{n^2})^{n^2}$
(9)	$(\log n^n)/n!$	(10)	$\dfrac{2^n n!}{(2n)!}$
(11)	$(\dfrac{n}{\log n})^n$	(12)	$\dfrac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}}$
(13)	$n^2\dfrac{1}{2^{2n}}$	(14)	$\dfrac{1}{5}n^4(n^3+1)$
(15)	$\dfrac{(2n)!}{n!}$	(16)	$\dfrac{n!}{(2n)!}$
(17)	$\dfrac{n!}{n^n}$	(18)	$\dfrac{n^{2n}}{(n!)^2}$
(19)	$[1+(-1)^n 3]^n$	(20)	$(a^n+b^n), a > b > 0$
(21)	$\dfrac{1}{n^p}, \quad p > 0$	(22)	$4^{n(-1)^n}$.

8 Metric Spaces

1. Let X be a nonempty set and $d\colon X \times X \to \mathbb{R}$ be a function with the following properties:
   (i) $d(x,y) \geq 0$ for all $x, y \in X$ and $d(x,y) = 0$ iff $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.
(iii) $d(x, y) \leq d(x, z)+d(z, y)$ for all $x, y, z \in X$. (This is known as triangle inequality)

Then d is called a *metric* or a distance function on X. The pair (X, d) is called a metric space.

2. Examples:

   (a) The most important one is, of course, $X = \mathbb{R}$ with $d(x, y) := |x - y|$.

   (b) On $X = \mathbb{R}^n$, we define $d_1(x, y) := \sum_{k=1}^{n} |x_k - y_k|$.

   (c) On $X = \mathbb{R}^n$, we define $d_\infty(x, y) := \max\{|x_k - y_k| : 1 \leq k \leq n\}$.

   (d) The standard Euclidean distance in $\mathbb{R}^2$, defined by

   $$d_2((x_1, y_1), (x_2, y_2)) := \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

   is a metric on $\mathbb{R}^2$.

   (e) Let $X := C[a, b]$ denote the set of real valued continuous functions on the closes and bounded interval $[a, b]$. If we set $d_\infty(f, g) := \sup\{|f(x) - g(x)| : x \in [a, b]\}$, then d is well-defined and is a metric on X.

   (f) Let A be any nonempty set and let $X := B(A, \mathbb{R})$ be the set of bounded real valued functions on A. Then

   $$d_\infty(f, g) := \sup\{|f(a) - g(a)| : a \in A\}$$

   defines a metric on X. Compare this with the last item.

   (g) On any (nonempty) set X, we define $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$. Such a metric is called a discrete metric and (X, d) is called discrete metric space.

   (h) Let (X, d) be a metric space. If $A \subset X$, then the restriction of d to $A \times A$ is a metric, say, ρ on A: $\rho(a_1, a_2) = d(a_1, a_2)$ for $a_1, a_2 \in A$. The metric ρ is called the induced metric on the subset A. It is customary to denote ρ also by d. (We observed that this may not be the most natural way of finding distance between two points, if we are constrained to live only on A!)

3. Let $A = [0, 1] \subset \mathbb{R}$ and $f(x) = x$ and $g(x) = x^2$. What is $d_\infty(f, g)$?

4. Let (X, d) be a metric space. Let $p \in X$ and $r > 0$. Consider the set

   $$B(p, r) := \{y \in X : d(x, y) < r\}.$$

   Then $B(p, r)$ is called an open ball with centre p and radius r.

5. Draw the pictures of the open balls $B(p, 1)$ where $p = (0, 0) \in \mathbb{R}^2$ and the metric is d_1, d_2 and d_∞.

6. Open balls in a discrete metric space and in $\mathbb{Z} \subset \mathbb{R}$:

   (a) Let (X, d) be a metric space with at least two points and $p \in X$. What is $B(p, r)$ if $0 < r < 1$, $r = 1$ and $r > 1$?

   (b) Let $A = \mathbb{Z}$. Let d be the metric induced by the standard metric on $\mathbb{R}$. What are $B(k, r)$ for $k \in \mathbb{Z}$ and $r > 0$? Answer: $B(k, r) := \{k + t : |t| \leq [r]\}$.

7. Given a sequence (x_n) in a metric space, we say that it converges (in the metric d) to an $x \in X$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n \geq N \implies d(x_n, x) < \varepsilon, \quad \text{that is,} \quad x_n \in B(x, \varepsilon).$$

   We say that x is a limit of the sequence (x_n).

8. Examples of Convergent Sequences.

   (a) Let $p_n := (x_n, y_n) \in \mathbb{R}^2$. Then the sequence (p_n) converges to $p = (x, y) \in \mathbb{R}^2$ in any of the metrics d_1, d_2, d_∞ if and only if $x_n \to x$ and $y_n \to y$.

   (b) A sequence (f_n) in $C[a, b]$ converges to an $f \in C[a, b]$ iff $f_n \rightrightarrows f$ on $[a, b]$. Similarly, f_n converges to f in the metric space $(B(A, \mathbb{R}), d_\infty)$ iff $f_n \rightrightarrows f$ on A.

   (c) A sequence (x_n) in a discrete metric space (X, d) converges iff it is eventually a constant sequence.

   (d) Let $X = R$ and $A := \mathbb{Z}$. Let d denote the standard metric o $\mathbb{R}$ as well as the induced metric on A. Then a sequence in (A, d) is convergent iff it is eventually constant.

9. Let $A = (0, 1) \subset \mathbb{R}$. We restrict the standard metric d on $\mathbb{R}$ to A. The sequence $(1/n)$ in A is not convergent to any point in A.

10. If a sequence (x_n) in a metric space (X, d) converges to $x, y \in X$, then $x = y$, that is, limit of a convergent sequence in a metric space is unique.

11. Let $f: (X, d) \to (Y, d)$ be a function and $a \in X$. We say that f is continuous at a if for any sequence (x_n) in X converging (in the metric on X) to a, we have $f(x_n) \to f(a)$ in the metric on Y.

    It is straightforward to give ε-δ definition of continuity.

12. One can mimic the proof of Item 9 to show that the two definitions of continuity are equivalent. In the proof there, one needs to interpret $|x - y|$ as $d(x, y)$, the metric on $\mathbb{R}$.

13. Let (X, d) be a metric space. A subset $K \subset X$ is said to be *compact* if every sequence (x_n) in K has a convergent subsequence (x_{n_k}) which converges to an x **in** K. We say that X is a compact metric space if X is compact as a subset.

14. Examples of compact and noncompact sets.

    (a) $R := [a, b] \subset \mathbb{R}$ is a compact subset of $\mathbb{R}$ with the standard metric.

    (b) The rectangle $[a, b] \times [c, d] \subset \mathbb{R}^2$ is compact in $(\mathbb{R}^2, d)$ where d could be any of the three metrics seen earlier.

    Let $(x_n, y_n) \in \mathbb{R}$. Let (x_{n_k}) be a convergent subsequence with limit $x \in [a, b]$. Then (y_{n_k}) has a convergent subsequence $(y_{n_{k_r}})$ converging to $y \in [c, d]$. Then the subsequence $(x_{n_{k_r}}, y_{n_{k_r}})$ converges to $(x, y) \in R$, by Item 8a.

    The common mistake students do is to take converging subsequences of (x_n) and (y_n). The result may not be subsequence. For, $((-1)^n, (-1)^n)$, the x_{n_k} may have n_k's even and y_{n_k}'s may have n_k's odd!

Let $x_n = (-1)^{n+1}$ and $y_n = r$ where $0 \le r < 3$ is the remainder when n is divided by 3. Thus

$$x = (1, -1, 1, -1 \ldots,) \text{ and } y = (1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, \ldots).$$

Suppose the convergent subsequence of (x_n) is the sequence (x_{2n-1}) of odd terms. Then a convergent subsequence of the second sequence is given by (y_{1+6n}). Hence a convergent subsequence of (x_n, y_n) is (x_{6n+1}, y_{6n+1}). Can you give another such?

(c) A discrete metric space is compact iff it is finite.

(d) $\mathbb{R}$ with the standard metric is not compact. For, consider the sequence (x_n) with $x_n = n$.

(e) $(0, 1)$ with the metric induce from $\mathbb{R}$ is not compact. For, consider the sequence (x_n) with $x_n = 1/n$.

The next item allows us to 'characterize' compact subsets of the metric space $(C[a, b], d_\infty)$.

15. **Equicontinuity and Arzela-Ascoli's theorem.** This is best learnt in the context of compact metric spaces.

(a) Let $\mathcal{F} := \{f_i : i \in I\}$ be a family of functions defined on a set $J \subset \mathbb{R}$. We say that $\mathcal{F}$ is *equicontinuous* on J if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in J \text{ and } |x - y| < \delta \implies |f_i(x) - f_i(y)| < \varepsilon \text{ for all } i \in I.$$

Note that any $f_i \in \mathcal{F}$ is uniformly continuous on J.
We say that $\mathcal{F}$ is *uniformly bounded* on J if there exists $M > 0$ such that

$$|f_i(x)| \le M \text{ for all } x \in J \text{ and for all } i \in I.$$

(b) Let $\mathcal{F}$ be a family of differentiable functions on $J := [a, b]$ such that there exists $M > 0$ such that

$$\text{for all } f \in \mathcal{F}, |f'(x)| \le M \text{ for all } x \in J \text{ and } |f(a)| \le M.$$

Then $\mathcal{F}$ is equicontinuous and uniformly bounded on J. *Hint:* Mean value theorem.

(c) Arzela-Ascoli Theorem. Let $\mathcal{F}$ be a family of functions on $J := [a, b]$. Assume that $\mathcal{F}$ is equicontinuous and uniformly bounded on J. Then given any sequence (f_n) in $\mathcal{F}$, there exists a subsequence (f_{n_k}) which is uniformly convergent to a function $f \colon J \to \mathbb{R}$.
The proof is omitted. Note the similarity of this result with the Bolzano-Weierstrass theorem which says that any sequence in a bounded subset $A \subset \mathbb{R}$ has a a subsequence which is convergent (not necessarily to an element of A).

16. **Dedekind cuts.** Let $a \in \mathbb{R}$. Let $\alpha := \{x \in \mathbb{Q} : x < a\}$. Observe the following properties of α:

(a) α is neither empty nor all of $\mathbb{Q}$.

(b) If $x \in \alpha$ and if $s \in \mathbb{Q}$ satisfies $s < x$, then $s \in \alpha$.

(c) There is no maximum or largest element in α.

A set α of rational numbers is called a Dedekind cut if it satisfies the properties (a)-(c) above.

17. Let $\mathcal{C}$ denote the set of Dedekind cuts of $\mathbb{Q}$. It is easy to see (using the LUB property of $\mathbb{R}$) that the map $\alpha \mapsto$ l.u.b. α is a bijection of $\mathcal{C}$ with $\mathbb{R}$.

18. It is the ingenious idea of Dedekind to use this observation to build/construct the real number system $\mathbb{R}$ via Dedekind cuts. He defined $\mathbb{R}$ as the set of Dedekind cuts of $\mathbb{Q}$, defined the algebraic operations, order relations and showed that $\mathbb{R}$, constructed this way, is a field which is order complete (that is, enjoys the LUB property). For details, refer to Rudin's book listed in References.

We started the course with the properties of the real number system and end the course with 'a definition' of real number system!

9 Riemann Integration

Unless specified otherwise, we let $J = [a, b]$ denote a closed and bounded interval, $f, g\colon [a, b] \to \mathbb{R}$ bounded functions. If f is given, and $J_i = [t_i, t_{i+1}]$ is a subinterval of J, we let

$$m_i(f) = \text{g.l.b. } \{f(t) : t_i \le t \le t_{i+1}\} \text{ and } M_i(f) = \text{l.u.b. } \{f(t) : t_i \le t \le t_{i+1}\}.$$

If f is understood or clear from the context, we simply denote these by m_i and M_i. We also let $m = \text{g.l.b. } \{f(x) : x \in [a, b]$ and $M = \text{l.u.b. } \{f(x) : x \in [a, b]$.

1. Integration is much much earlier than differentiation. The main idea of integration is to assign a real number A, called 'area', to region R bounded by the curves $x = a$, $x = b$, $y = 0$ and $y = f(x)$, where we assume that f is nonnegative. The number A is called the are of the region R, and called the integral of f over $[a, b]$ and denoted by the "symbol" $\int_a^b f(x)\, dx$

2. The most basic area is that of a rectangle. The area of the rectangle whose sides are ℓ and b is $\ell \cdot b$. Our definition of an integral should be such that if $f(x) = c$, a constant, then $\int_a^b f(x)\, dx = c(b - a)$.

   We use this basic notion of area as the building blocks to assign an area to the regions under the graphs of bounded functions. To understand the concepts and results of this section, it is suggested that the reader may assume that f is nonnegative and draw pictures whenever possible.

3. A partition or subdivision P of an interval $[a, b]$ is a finite set $\{x_0, x_1, \dots, x_n\}$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. The points x_i are called the nodes of P.

   Examples: (i) $P = \{a = x_0, x_1 = b\}$, (ii) For any $n \in \mathbb{N}$, let $x_i = a + \frac{i}{n}(b - a)$ for $0 \le i \le n$. Then $\{x_0, \dots, x_n\}$ is a partition, say, P_n.

   Given two partitions P and Q of $[a, b]$, we say that Q is a refinement of P is $P \subset Q$. In the example (ii) above, the partition $P_{2^{k+1}}$ is a refinement of P_{2^k}.

4. Given $f\colon [a, b] \to \mathbb{R}$ and a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$, we let

$$\begin{aligned} L(f, P) &:= \sum_{i=0}^{n} m_i(f)(x_{i+1} - x_i) \\ U(f, P) &:= \sum_{i=0}^{n} M_i(f)(x_{i+1} - x_i). \end{aligned}$$

   Observe that, if $f \ge 0$, $L(f, P)$ (respectively, $U(f, P)$) is the sum of areas of rectangles 'inscribed' inside (respectively, circumscribing) the region bounded by the graph.

   $L(f, P)$ and $U(f, P)$ are called the lower and the upper Darboux sums of f with respect to the partition P. They 'approximate' the area under the graph 'from below' and 'from above.'

   It is clear that $m(b - a) \le L(f, P) \le U(f, P) \le M(b - a)$.

5. Given a partition $P = \{x_0, \dots, x_n\}$, we insert a new node, say, t such that $x_i < t < x_{i+1}$ for some i and get a new partition Q. Then drawing pictures of a nonnegative function, it is clear $L(f,Q) \geq L(f,P)$ and $U(f,Q) \leq U(f,P)$. Thus, Q "produces" better approximation to the area bound by the graph. This suggests that to get the 'real' area we should look at

$$L(f) \equiv \underline{\int_a^b} f(x)\,dx \quad := \quad \text{l.u.b. } \{L(f,P) : P \text{ is a partition of } [a,b]\}$$

$$U(f) \equiv \overline{\int_a^b} f(x)\,dx \quad := \quad \text{g.l.b. } \{U(f,P) : P \text{ is a partition of } [a,b]\}.$$

These numbers exist (why?) and are called the lower and upper integral of f on $[a,b]$. They may be understood as the area of the region under the graph as approximated from below and from above respectively.

6. We say that f is Darboux integrable (or simply integrable) on $[a,b]$ if the upper and lower integrals coincide. (This intuitively says that we require that the area should be 'approximable' both from below and from above.) If f is integrable, the common value of the upper and lower integrals are denoted by the symbol $\int_a^b f(x)\,dx$.

7. Example. Let $f\colon [0,1] \to \mathbb{R}$ be defined by $f(x) = 1$ if $0 \leq x < 1$ and $f(1) = 10^9$. We claim that f is integrable on $[0,1]$ and $\int_0^1 = 1$. Let $P = \{x_0, \dots, x_n\}$ be a partition of $[0,1]$. Note that $m_i(f) = M_i(f) = 1$ for $0 \leq i < n-1$ and $M_n(f) = 10^9$ and $m_n(f) = 1$. Hence we find

$$L(f,P) = 1 \text{ and } U(f,P) = x_{n-1} + 10^9(1 - x_{n-1}).$$

It follows that l.u.b. $\{L(f,P) : P\} = 1$ and g.l.b. $\{U(f,P) : P\} = 1$.

8. **Theorem.** Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Let P and Q be partitions of $[a,b]$. Then
(i) If Q is a refinement of P, then $L(f,P) \leq L(f,Q)$ and $U(f,P) \geq U(f,Q)$.
(ii) $L(f,P) \leq U(f,Q)$ for *any* two partitions P and Q.
(iii) The lower integral of f is less than or equal to the upper integral.

Proof. (i) It is enough to to prove it when $Q = P \cup \{c\}$, that is, Q contains exactly one extra node. (Draw pictures, you will understand (i) immediately.) Let $x_i < c < x_{i+1}$. All the $j \neq i$-th terms in $L(f,P)$ and $U(f,P)$ will be present in $L(f,Q)$ and $U(f,Q)$. Corresponding to the term $m_i(f)(x_{i+1} - x_i)$, we have two terms in $L(f,Q)$:

$$\text{g.l.b. } \{f(x) : x \in [x_i, c]\}(c - x_i) + \text{g.l.b. } \{f(x) : x \in [c, x_{i+1}\}(x_{i+1} - c)$$

Note that $m_i(f) = \text{g.l.b. } \{f(x) : x \in [x_i, x_{i+1}]\} \leq \text{g.l.b. } \{f(x) : x \in [x_i, c]\}(c - x_i)$ etc. Hence

$$\begin{aligned}
&\text{g.l.b. } \{f(x) : x \in [x_i, c]\}(c - x_i) + \text{g.l.b. } \{f(x) : x \in [c, x_{i+1}]\}(x_{i+1} - c) \\
&\quad \leq \quad m_i(f)(c - x_i) + m_i(f)(x_{i+1} - c) \\
&\quad = \quad m_i(f)(x_{i+1} - x_i).
\end{aligned}$$

It follows that

$$L(f,P) = \left(\sum_{j \neq i} m_j (x_{j+1} - x_j) \right) + m_i(x_{i+1} - x_i) \leq L(f,Q).$$

Similarly, we obtain

$$\begin{aligned} &\text{l.u.b. } \{f(x) : x \in [x_i, c]\}(c - x_i) + \text{l.u.b. } \{f(x) : x \in [c, x_{i+1}]\}(x_{i+1} - c) \\ \leq \quad & M_i(f)(x_{i+1} - x_i), \end{aligned}$$

and conclude that $U(f,Q) \leq U(f,P)$.

(ii) is easy. Let $P' = P \cup Q$. Then by (i),

$$L(f,P) \leq L(f,P') \leq U(f,P') \leq U(f,Q).$$

(iii) It follows from (ii) that each $U(f,Q)$ is an upper bound for the set $\{L(f,P) : P \text{ a partition of } [a,b]\}$. Hence its lub, namely, the lower integral, will be at most $U(f,Q)$. Thus, the lower integral is a lower bound for the set $\{U(f,Q) : Q \text{ a partition of } [a,b]\}$. Hence its glb, namely, the upper integral is greater than or equal to the lower integral.

□

9. A bounded function $f\colon [a,b] \to \mathbb{R}$ is integrable iff for each $\varepsilon > 0$ there exists a partition P such that $U(f,P) - L(f,P) < \varepsilon$.

   Proof. Let I_1 and I_2 be the lower and upper integrals. Since $I_1 \leq I_2$, enough to show that $I_1 \geq I_2 - \varepsilon$ for any $\varepsilon > 0$. Given $\varepsilon > 0$, let P be as in the hypothesis. Observe that

   $$I_1 \geq L(f,P) \geq U(f,P) - \varepsilon \geq I_2 - \varepsilon.$$

   As you see this is just playing with the definitions of the lower and upper integrals and that of GLB an LUB. So, we leave the converse to you. □

10. Applications:

    (a) Let $f(x) = x^2$ on $[0,1]$. Let $\varepsilon > 0$ be given. Choose a partition P such that $\max\{x_{i+1} - x_i : 1 \leq 0 \leq n-1\} < \varepsilon$. Note that since f is increasing and continuous

    $$m_i(f) = f(x_i) = x_i^2 \text{ and } M_i(f) = f(x_{i+1}) = x_{i+1}^2.$$

    It follows that

    $$\begin{aligned} U(f,P) - L(f,P) \quad &= \quad \sum_{i=1}^{n} [(x_i - x_{i-1})(x_i + x_{i-1})](x_i - x_{i-1}) \\ &< \quad \sum_{i=1}^{n} [\frac{\varepsilon}{2} \times 2](x_i - x_{i-1}), \text{ since } 0 \leq x_i, x_{i+1} \leq 1 \\ &= \quad \varepsilon \sum_i (x_i - x_{i-1}) = \varepsilon. \end{aligned}$$

    Hence f is integrable.

(b) Consider $f\colon [-1,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} a, & -1 \le x < 0 \\ 0, & x = 0 \\ b, & 0 < x \le 1. \end{cases}$$

Assume that $a < b$. We claim that f is integrable. Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $\frac{b-a}{2N} < \varepsilon$. Consider the partition $P = \{0, \frac{-1}{N}, \frac{1}{N}, 1\}$. Then $U(f,P) - L(f,P) = (b-a)/2N < \varepsilon$.

Divide and conquer

(c) The last item can be generalized. Let $P = \{a = x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. Let $\sigma\colon [a,b] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} c_1, & x \in [a, x_1); \\ c_i, & x \in [x_{i-1}, x_i), \quad 2 \le i \le n-1; \\ c_n, & x \in [x_{n-1}, x_n]. \end{cases}$$

Then σ is integrable and we have

$$\int_a^b \sigma(t)\,dt = \sum_{i=1}^n c_i(x_i - x_{i-1}).$$

(d) Recall Thomae's function f of Item 10i on page 25. We claim that f is integrable. Let $\varepsilon > 0$ be given. Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon/2$. There exist a finite number, say, N of rational numbers p/q with $q \le k$. Denote them by $\{r_j : 1 \le j \le N\}$. Let $\delta < \varepsilon/(4N)$. Choose a partition $P = \{x_0, \dots, x_n\}$ such that

$$\max\{|x_{i+1} - x_i| : 0 \le i \le n-1\} < \delta.$$

Let $A := \{i : r_j \in [x_i, x_{i+1}],$ for some $j\}$, and $B := \{1, \dots, n\} \setminus A$. Note that the number of elements in A will be at most $2N$. (Why $2N$? Some r_j could be the left and the right end point of 'consecutive' subintervals!) For $i \in A$, we have $M_i - m_i \le 1$. For $j \in B$, we have $M_j - m_j < 1/k$. Hence

Divide and conquer

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i \in A} (M_i - m_i)(x_{i+1} - x_i) + \sum_{j \in B} (M_j - m_j)(x_{j+1} - x_j) \\ &\le (2N)\delta + \sum_{j \in B} \frac{1}{N}(x_{j+1} - x_j) \\ &\le (2N)\delta + \frac{1}{k} \\ &< \varepsilon. \end{aligned}$$

(e) Dirichlet's function. Let $f\colon [0,1] \to \mathbb{R}$ be defined by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ otherwise. Then for any partition P, we see that $m_i(f) = 0$ and $M_i(f) = 1$ so that $U(f,P) - L(f,P) = 1$ always. Hence f is not integrable.

11. Let $f\colon [a,b] \to \mathbb{R}$ be bounded and monotone. Then f is integrable.

Assume that f is increasing. Given $\varepsilon > 0$, choose N so that $\frac{(b-a)(f(b)-f(a))}{N} < \varepsilon$. Let $x_i := a + i\frac{b-a}{N}$, $0 \le i \le N$. Note that $m_i = f(x_i)$ and $M_i = f(x_{i+1})$. Hence,

$$\begin{aligned} & U(f,P) - L(f,P) \\ &= \frac{b-a}{N}\left([f(x_1) - f(x_0)] + +[f(x_2) - f(x_1)] + \cdots + [f(x_N) - f(x_{N-1})]\right) \\ &= \frac{b-a}{N}\left(f(b) - f(a)\right). \end{aligned}$$

12. Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then f is integrable.

This is the first time where we are seriously using the concept of uniform continuity.

Given $\varepsilon > 0$, choose δ using the uniform continuity of f for $\frac{\varepsilon}{b-a}$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \delta$. Let $x_i := a + i\frac{b-a}{N}$, $0 \le i \le N$. Let $P := \{x_i : 0 \le i \le N\}$. The continuous function f attains its maximum and minimum on $[x_i, x_{i+1}]$, say, at $t_i, s_i \in [x_i, x_{i+1}]$. Since $|t_i - s_i| < 1/N < \delta$, it follows that $M_i - m_i < \varepsilon/(b-a)$ for $0 \le i \le N-1$. Therefore,

$$\begin{aligned} U(f,P) - L(f,P) &< \sum_{i=0}^{N-1} \frac{\varepsilon}{b-a}\frac{b-a}{N} \\ &= \varepsilon. \end{aligned}$$

13. Exercise: $f\colon [a,b] \to \mathbb{R}$ is bounded, continuous on $[a,b]$ except at $c \in [a,b]$, then f is integrable. (Divide and conquer strategy, see Item 10b.) Divide and conquer

14. Consider $f\colon [0,1] \to \mathbb{R}$ defined by $f(x) = x^2$. We have already seen that f is integrable. We now compute its integral.

Let $P = \{0 = x_0, x_1, \ldots, x_n = 1\}$ be any partition. Consider $g(x) = x^3/3$. Then $g' = f$. By MVT, $g(x_i) - g(x_{i-1}) = f(t_i)(x_i - x_{i-1})$ for some $t_i \in [x_{i-1}, x_i]$. Hence

$$\sum_i f(t_i)(x_i - x_{i-1}) = \sum_i (g(x_i) - g(x_{i-1})) = g(1) - g(0) = 1/3. \tag{62}$$

Since $m_i \le f(t_i) \le M_i$, we see that

$$L(f,P) \le \sum_i f(t_i)(x_i - x_{i-1}) \le U(f,P). \tag{63}$$

It follows from (62)–(63) that $L(f,P) \le \frac{1}{3} \le U(f,P)$ for all partitions. That is, l.u.b. $\{L(f,P) : P\} \le 1/3 \le$ g.l.b. $\{U(f,P) : P\}$. Since f is integrable, the first and the third terms are equal. Hence $\int_0^1 f(x)\,dx = 1/3$. □

Properties of the Integral

15. **Theorem.** Let $f, g\colon [a,b] \to \mathbb{R}$ be integrable functions and $c \in \mathbb{R}$. Then
(i) cf is integrable and $\int_a^b (cf)(x)\,dx = c\int_a^b f(x)\,dx$.
(ii) $f+g$ is integrable and we have $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.
(iii) Let $R([a,b])$ denote the set of integrable functions on $[a,b]$. Then $R([a,b])$ is a vector space and the map $f \mapsto \int_a^b f(x)\,dx$ is a linear map.

Proof. These are very easy and the reader should prove them on his own.

First note that if $\emptyset \neq A \subset \mathbb{R}$ is a bounded set and $c \in \mathbb{R}$

$$\text{l.u.b. } cA = \begin{cases} c \times \text{l.u.b. } A & \text{if } c \geq 0 \\ c \times \text{g.l.b. } A & \text{if } c < 0 \end{cases}$$

It follows that, for any partition P,

$$U(cf, P) = \begin{cases} c \times U(f, P) & \text{if } c \geq 0 \\ c \times L(f, P) & \text{if } c < 0 \end{cases} \quad \text{and} \quad L(cf, P) = \begin{cases} c \times L(f, P) & \text{if } c \geq 0 \\ c \times U(f, P) & \text{if } c < 0 \end{cases}$$

Consequently,

$$U(cf) = \begin{cases} c \times U(f) & \text{if } c \geq 0 \\ c \times L(f) & \text{if } c < 0 \end{cases} \quad \text{and} \quad L(cf) = \begin{cases} c \times L(f) & \text{if } c \geq 0 \\ c \times U(f) & \text{if } c < 0 \end{cases}$$

Since $U(f) = L(f)$, it follows that $U(cf) = L(cf) = cU(f) = cL(f)$, which is (i).

To prove (ii), given $\varepsilon > 0$, there exist partitions P_1 and P_2 such that

$$U(f, P_1) - L(f, P_1) < \varepsilon/2 \quad \text{and} \quad U(g, P_2) - L(g, P_2) < \varepsilon/2.$$

Let $P = P_1 \cup P_2$ so that P is a refinement of P_1 and P_2. Hence

$$U(f, P) - L(f, P) < \varepsilon/2 \ \text{ and } \ U(g, P) - L(g, P) < \varepsilon/2. \tag{64}$$

Observe that for any nonempty subset $S \subset [a, b]$, we have

$$\begin{aligned} \text{l.u.b. } \{f(x) + g(x) : x \in S\} &\leq \text{l.u.b. } \{f(x) : x \in S\} + \text{l.u.b. } \{g(x) : x \in S\} \\ \text{g.l.b. } \{f(x) + g(x) : x \in S\} &\geq \text{g.l.b. } \{f(x) : x \in S\} + \text{g.l.b. } \{g(x) : x \in S\}. \end{aligned}$$

(Reason: Let F and G denote the GLB's of f and g on S. Then $f(x) + g(x) \geq F + G$ for all $x \in S$ so that $F + G$ is a lower bound for the set $\{f(x) + g(x) : x \in S\}$.) In particular, $m_i(f + g) \geq m_i(f) + m_i(g)$ and $M_i(f + g) \leq M_i(f) + m_i(g)$ for each i. It follows that

$$L(f + g, P) \geq L(f, P) + L(g, P) \ \text{ and } \ U(f + g, P) \leq U(f, P) + U(g, P). \tag{65}$$

The result follows from (64) and (65). □

16. (Monotonicity of the integral.) Let $f, g \colon [a, b] \to \mathbb{R}$ be integrable. Assume that $f(x) \leq g(x)$ for $x \in [a, b]$. Then $\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$.

    Easy. One needs only to observe that for any partition P, $M_i(f) \leq M_i(g)$ so that $U(f, P) \leq U(g, P)$. □

17. **Theorem.** Let f be integrable on $[a, b]$. Assume that $m \leq f(t) \leq M$ for $t \in [a, b]$. Let $g \colon [m, M] \to \mathbb{R}$ be continuous. Then $g \circ f$ is integrable on $[a, b]$.

Proof. We shall use s, t for elements of $[a,b]$ and x, y for elements of $[m, M]$.

Let $\varepsilon > 0$ be given. The uniform continuity of g on $[m, M]$ ensures a $\delta > 0$ such that

$$x, y \in [m, M] \ \& \ |x - y| < \delta \implies |g(x) - g(y)| < \varepsilon.$$

Let $\delta_1 := \min\{\delta, \varepsilon\}$. Choose a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \eta$, where η is to be specified later. Let $A := \{i : M_i(f) - m_i(f) < \delta_1\}$ and B its complement.

If $i \in A$, then $M_i(g \circ f) - m_i(g \circ f) \le \varepsilon$. If $j \in B$, $M_j(g \circ f) - m_j(g \circ f) \le 2C$ where C is an upper bound for $|g|$ on $[m, M]$. We have

$$\eta \ge \sum_{j \in B} (M_j(f) - m_j(f))(x_{j+1} - x_j) \ge \delta_1 \sum_{j \in B} (x_{j+1} - x_j).$$

Hence $\sum_{j \in B} (x_{j+1} - x_j) \le \eta/\delta_1$. We therefore obtain

Divide and conquer

$$\begin{aligned} U(g \circ f, P) - L(g \circ f, P) &= \sum_{i \in A} (M_i(g \circ f) - m_i(g \circ f))(x_{i+1} - x_i) \\ &\quad + \sum_{j \in B} (M_j(g \circ f) - m_j(g \circ f))(x_{j+1} - x_j) \\ &\le \delta_1 (b - a) + (2C)(\eta/\delta_1) \\ &\le \varepsilon(b - a) + (2C)(\eta/\delta_1), \quad \text{since } \delta_1 \le \varepsilon. \end{aligned}$$

If we choose $\eta = \delta_1^2$, since $\delta_1 \le \varepsilon$, it follows that,

$$U(g \circ f, P) - L(g \circ f, P) < \varepsilon(b - a + 2C).$$

That is, $g \circ f$ is integrable on $[a, b]$. □

18. If we drop that the condition that g is continuous, then $g \circ f$ may not be integrable. Consider Thomae's function. Let $g\colon [0,1] \to \mathbb{R}$ be defined by $g(x) = 1$ if $x \in (0,1]$ and $g(0) = 0$. Then g is integrable, but $g \circ f$ is Dirichlet's function and it is not integrable.

19. Exercise. Applications of Item 17. Assume that $f, g\colon [a,b] \to \mathbb{R}$ be integrable.

    (a) Show that $|f|$ is integrable.

    (b) Show that f^2 is integrable.

    (c) Show that fg is integrable.

    (d) Show that $\max\{f, g\}$ and $\min\{f, g\}$ are integrable.

20. (Basic estimate for integrals.) Let $f\colon [a,b] \to \mathbb{R}$ be integrable. Then $|f|$ is integrable and we have

$$|\int_a^b f(x)\, dx| \le \int_a^b |f(x)|\, dx. \tag{66}$$

Proof. $|f|$ is integrable by Item 19. Choose $\varepsilon = \pm 1$ so that $\varepsilon \int_a^b f(t)\, dt = |\int_a^b f(t)\, dt|$. Then

$$\begin{aligned} |\int_a^b f(t)\, dt| &= \varepsilon \int_a^b f(t)\, dt \\ &= \int_a^b \varepsilon f(t)\, dt && \text{(by linearity of the integral)} \\ &= \int_a^b |f(t)|\, dt && \text{(by monotonicity, since } \pm f \leq |f|\text{).} \end{aligned}$$

□

21. If f is integrable on $[a, b]$, then f is integrable on any subinterval $[c, d]$ of $[a, b]$. Furthermore,

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx, \tag{67}$$

for any $c \in (a, b)$. (This known as additivity of the integral as an interval function.)

Proof. We show that f is integrable on $[a, c]$ for $c \in (a, b)$. The general case is proved in a similar vein. Let $\varepsilon > 0$ be given. Let P be a partition of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$.

Let P be *any* partition of $[a.b]$. Let $Q = P \cup \{c\}$ and $P_1 := Q \cap [a, c]$. Then,

$$U(f, P_1) - L(f, P_1) \leq U(f, Q) - L(f, Q) \leq U(f, P) - L(f, P) < \varepsilon.$$

Hence f is integrable on $[a, c]$.

Let $Q_0 = P \cup \{c\}$, $Q_1 = Q \cap [a, c]$ and $Q_2 = Q \cap [c, b]$. We have

$$\begin{aligned} U(f, P) \geq U(f, Q) &= U(f, Q_1) + U(f, Q_2) \\ &\leq \overline{\int_a^c} f(x)\, dx + \overline{\int_c^b} f(x)\, dx \\ &= \int_a^c f(x)\, dx + \int_c^b f(x)\, dx, \text{ why?} \end{aligned}$$

Thus, we have proved

$$U(f, P) \leq \int_a^c f(x)\, dx + \int_c^b f(x)\, dx, \text{ for any partition} P.$$

Now complete the proof. □

22. If $f\colon [b, a] \to \mathbb{R}$ is integrable, we *define* $\int_a^b f(x), dx = -\int_b^a f(x)\, dx$ and $\int_s^s f(x)\, dx = 0$. Using this convention, we see that for any a, b, c such that f is integrable the smallest interval containing a, b, c, we have $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$.

23. Exercises:

    (a) Show that $f \in \mathcal{R}(I)$ implies that $f^2 \in \mathcal{R}(I)$.

(b) Let $f \in \mathcal{C}[a,b]$. If $\int_a^b f(x)dx = 0$, then $f(c) = 0$ for at least one $c \in [a,b]$.

(c) Let $f \in \mathcal{R}(I)$, $f \geq 0$, and $\int_I f = 0$. Then $f = 0$ at each point of continuity of f. *Hint:* There exists J containing c such that $f(x) \geq \frac{1}{2} f(c)$ if $x \in J \subset I$.

(d) If $f \in \mathcal{C}[a,b]$, $\int_a^b f(x)g(x) = 0$, for all $g \in \mathcal{C}[a,b]$, then $f \equiv 0$.

(e) Let $f > 0$ be continuous on $[a,b]$. Let $M = \max_{[a,b]} f$. Then

$$\lim_{n\to\infty} \left(\int_a^b [f(x)]^n \right)^{\frac{1}{n}} = M.$$

(f) $f \in \mathcal{R}(I)$ iff for all $\varepsilon > 0$, there exists step functions s_1, s_2 such that $s_1(x) \leq f(x) \leq s_2(x)$ and $\int_I (s_2 - s_1) < \varepsilon$.

(g) Let $f > 0$ be continuous, strictly increasing on $[a,b]$, where $0 < a < b$. Then $\int_a^b f + \int_{f(a)}^{f(b)} f^{-1} = bf(b) - af(a)$.

Use this result to evaluate the following: (i) $\int_a^b x^{\frac{1}{3}} dx$, $0 < a < b$, (ii) $\int_0^1 \sin^{-1} x dx$.

(h) Prove the following version of Young's inequality. Let f be a continuous and strictly increasing function for $x \geq 0$ with $f(0) = 0$. Let g be the inverse of f. Then for any $a, b > 0$ we have

$$ab \leq \int_0^a f(x)\,dx + \int_0^b g(y)\,dy.$$

Equality holds iff $b = f(a)$.

(i) Take $f(x) := x^\alpha$ in Young's inequality above to deduce the original Young's inequality: Let $p > 0$, $q > 0$ be such that $(1/p) + (1/q) = 1$. Young's inequality holds:

$$(x^p/p) + (y^q/q) \geq xy \text{ for all } x > 0 \text{ and } y > 0. \tag{68}$$

The equality holds iff $x^{p-1} = y$ iff $x^{1/q} = y^{1/p}$.

(j) For $x, y \in \mathbb{R}^n$ and for $1 \leq p < \infty$, let $\|x\|_p := (\sum_i |x_i|^p)^{1/p}$ and for $p = \infty$, let $\|x\|_\infty := \max\{|x_i| : 1 \leq i \leq n\}$. For $p > 1$, let q be such that $(1/p) + (1/q) = 1$. For $p = 1$ take $q = \infty$. Prove **Hölder's inequality:**

$$\sum_i |a_i||b_i| \leq \|a\|_p \|b\|_q, \text{ for all } a, b \in \mathbb{K}^n.$$

Hint: Take $x = \frac{|a_i|}{\|a\|_p}$ and $y = \frac{|b_i|}{\|b\|_q}$ in Young's inequality (68) and sum over i. When does equality occur?

Fundamental theorems of calculus

24. **First fundamental theorem of calculus.** Let $f\colon [a,b] \to \mathbb{R}$ be differentiable. Assume that f' is integrable on $[a,b]$. Then

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Proof. We adapt the proof of Item 14. Let $P = \{x_0, x_1 \ldots, x_n\}$ be any partition of $[a,b]$. By MVT, we obtain

$$f(x_i) - f(x_{i-1}) = f'(t_i)(x_i - x_{i-1}), \quad \text{for some } t_i(x_{i-1}, x_i). \tag{69}$$

It follows that

$$\sum_i f'(t_i)(x_i - x_{i-1}) = f(b) - f(a). \tag{70}$$

Arguing as in Item 14, we arrive at the result. (Do you see where we used the integrability of f'?) □

25. Exercises.

    (a) Let $f : [0,a] \to \mathbb{R}$ be given by $f(x) = x^2$. Find $\int_0^a f(x)dx$.

    (b) Show that $\int_0^a f(x)dx = \frac{a^4}{4}$ for $f(x) = x^3$.

    (c) Let $0 < a \leq 1$. Show that $\int_0^a \sin x = 1 - \cos a$.

26. Let $f\colon [a,b] \to \mathbb{R}$ be integrable. Then for any $x \in [a,b]$, we know that f is integrable on $[a,x]$. Hence we have a function $F\colon x \mapsto \int_a^x f(t)\,dt$, $x \in [a,b]$. F is called the indefinite integral of f.

27. **Theorem.** (Fundamental theorem of calculus-2) Let $f\colon [a,b] \to \mathbb{R}$ be integrable. The indefinite integral F of f is continuous (in fact, Lipchitz) on $[a,b]$ and is differentiable at x if f is continuous at $x \in [a,b]$. In fact, $F'(x) = f(x)$.

    Proof. Let M be such that $|f(x)| \leq M$ for $x \in [a,b]$. Then we have

    $$|F(x) - F(y)| = |\int_x^y f(t)\,dt| \leq M \int_x^y 1\,dt = M|x-y|.$$

    Let f be continuous at c. We shall show that F is differentiable at c and $F'(c) = f(c)$. Observe that, for $x > c$,

    $$\frac{F(x) - F(c)}{x - c} = \frac{1}{x-c}\int_c^x f(t)\,dt \quad \text{and} \quad f(c) = \frac{1}{x-c}\int_c^x f(c).$$

    Hence, we obtain

    $$|\frac{F(x) - F(c)}{x - c} - f(c)| \leq \frac{1}{x-c}\int_c^x |f(t) - f(c)|\,dt. \tag{71}$$

    Given $\varepsilon > 0$, by the continuity of f at c, we can find a $\delta > 0$ such that $|f(t) - f(c)| < \varepsilon$ for $|t - c| < \delta$. Hence for $x \in [a,b]$ such that $|x - c| < \delta$, we see that the RHS of (71) is estimated above by ε. Similar argument applies when $x < c$. □

28. We can deduce a weaker version of the first fundamental theorem Item 24 from the second fundamental theorem of calculus.

    Let $f\colon [a,b] \to \mathbb{R}$ be differentiable with f' continuous on $[a,b]$. Then $\int_a^b f'(x)\,dx = f(b) - f(a)$.

Proof. Since f' is continuous, it is integrable and its indefinite integral say, $G(x) = \int_a^x f'(t)\,dt$ exists. By the last item G is differentiable with derivative $G' = f'$. Hence the derivative of $f - G$ is zero on $[a, b]$ and hence the function $f - G$ is a constant on $[a, b]$. In particular, $f(a) - G(a) = f(b) - G(b)$, that is, $f(a) = f(b) - \int_a^b f'(x)\,dx$. □

29. **Theorem.** (Integration by parts). Let $u, v\colon [a, b] \to \mathbb{R}$ be differentiable. Assume that u', v' are integrable on $[a, b]$. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\mid_a^b - \int_a^b u'(x)v(x)\,dx. \tag{72}$$

Proof. Let $g := uv$. Then g is integrable, $g' = u'v + uv'$ is integrable. (Why? If we assume that u and v are continuously differentiable, then the integrability of g' etc are clear.) Apply the first fundamental theorem of calculus to $\int_a^b g'(x)\,dx$ to arrive at the result. □

30. **Theorem.** (Change of variables). Let I, J be closed and bounded intervals. Let $u\colon J \to \mathbb{R}$ be continuously differentiable. Let $u(J) \subset I$ and $f\colon I \to \mathbb{R}$ be continuous. Then $f \circ u$ is continuous on J and we have

$$\int_a^b f \circ u(x)u'(x)\,dx = \int_{u(a)}^{u(b)} f(y)\,dy, \quad a, b \in J. \tag{73}$$

Proof. Fix $c \in I$. Let $F(y) := \int_c^y f(t)\,dt$. Let $g(x) := F \circ u(x)$. Then by chain rule, g is differentiable and

$$g'(x) = F'(u(x))u'(x) = f(u(x))u'(x).$$

We apply the first fundamental theorem of calculus to g':

$$\begin{aligned}
\int_a^b f(u(x))u'(x)\,dx &= \int_a^b g'(x)\,dx \\
&= g(b) - g(a) \\
&= \int_c^{u(b)} f(t)\,dt - \int_c^{u(a)} f(t)\,dt \\
&= \int_{u(a)}^{u(b)} f(t)\,dt.
\end{aligned}$$

□

31. Exercises.

    (a) Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable. Let $F(x) = \int_0^{g(x)} t^2 dt$. Prove that $F'(x) = g^2(x)g'(x)$ for all $x \in \mathbb{R}$. If $G(x) = \int_{h(x)}^{g(x)} t^2 dt$, then what is $G'(x)$?

    (b) If f'' is continuous on $[a, b]$, show that $\int_a^b f''(x)dx = [bf'(b) - f(b)] - [af'(a) - f(a)]$. *Hint:* Integrate by parts!

    (c) Let $f > 0$ be continuous on $[1, \infty)$. Let $F(x) := \int_1^x f(t)dt \leq [f(x)]^2$. Prove that $f(x) \geq \frac{1}{2}(x - 1)$. *Hint:* The integral $\int_1^x F(t)^{-\frac{1}{2}} F'(t)dt$ is relevant.

(d) Let $\phi''(t)$ be continuous on $[a,b]$. If there exists $m>0$ such that $\phi'(t)\geq m$ for all $t\in[a,b]$, then by second mean value theorem, $|\int_a^b \sin\phi(t)dt|\leq \frac{4}{m}$. *Hint:* Multiply and divide the integrand by $\phi'(t)$.

Mean Value Theorems for Integrals

32. (Mean Value Theorem for Integrals). Let f be continuous on $[a,b]$. Then there exists $c\in[a,b]$ such that $\int_a^b f(x)dx = f(c)(b-a)$.

Proof. Let $m=\inf f$, $M=\sup f$ on $[a,b]$. Then $m\leq f(x)\leq M$ implies $\int_a^b m \leq \int_a^b f(x)dx \leq \int_a^b M$ or

$$m(b-a)\leq \int_a^b f(x)dx \leq M(b-a) \implies m\leq \frac{1}{b-a}\int_a^b f(x)dx \leq M.$$

That is, the number $\frac{1}{b-a}\int_a^b f(x)dx \in f([a,b])$. Hence by the intermediate value theorem for continuous functions, there exists $c\in[a,b]$ such that $\frac{1}{b-a}\int_a^b f(x)dx = f(c)$. □

33. (Weighted/First Mean Value Theorem for Integrals). Let f,g be continuous on $[a,b]$. Assume that g does not change sign on $[a,b]$. Then for some $c\in[a,b]$ we have $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$.

Proof. With out loss of generality, assume that $g\geq 0$. Therefore, $mg(x)\leq f(x)g(x)\leq Mg(x)$. Hence $m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx$. If $\int_a^b g(x)=0$, nothing to prove. Otherwise, $\int_a^b g(x)dx>0$. Therefore divide by it on both sides and apply intermediate value theorem as before. If $g\leq 0$, we apply the above argument to $-g$.

Note that the last item is a corollary of this. □

34. (Second Mean Value Theorem). Assume g is continuous on $[a,b]$ and that f' is also continuous and f' does not change sign on $[a,b]$. Then there exists $c\in[a,b]$ such that

$$\int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx.$$

Proof. Let $G(x)=\int_a^x g(t)dt$. g continuous implies $G'(x)=g(x)$. Hence integration by parts gives us:

$$\int_a^b f(x)g(x)dx = \int_a^b f(x)G'(x)dx = f(b)G(b) - \int_a^b f'(x)G(x)dx \tag{74}$$

since $G(a)=0$. By the first mean value theorem,

$$\int_a^b f'(x)G(x)dx = G(c)\int_a^b f'(x)dx = G(c)[f(b)-f(a)] \tag{75}$$

(74) and (75) imply that

$$\int_a^b f(x)g(x)dx = f(b)G(b) - G(c)(f(b)-f(a)) = f(a)G(c) + f(b)[G(b)-G(c)].$$

□

35. (Another version of the second mean value theorem). Let $f\colon [a,b] \to \mathbb{R}$ be monotone. Then there exists $c \in [a,b]$ such that

$$\int_a^b f(x)\,dx = f(a)(c-a) + f(b)(b-c).$$

Proof. Assume that f is increasing. Note that $f \in R([a,b])$ and we have $f(a)(b-a) \le \int_a^b f(x)\,dx \le f(b)(b-a)$.

Define $h(x) := f(a)(x-a) + f(b)(b-x)$. Then h is continuous. Note that $h(a) = f(b)(b-a)$ and $h(b) = f(a)(b-a)$. Thus, $\int_a^b f(x)\,dx$ is a value lying between the values $h(b)$ and $h(a)$. By the intermediate value theorem applied to h, we conclude that there exists $c \in (a,b)$ such that $h(c) = \int_a^b f(x)\,dx$. This is as required. □

36. Exercise:

(a) Use the first mean value theorem to prove that for $-1 < a < 0$ and $n \in \mathbb{N}$, $s_n := \int_a^0 \frac{x^n}{1+x} dx \to 0$ as $n \to \infty$.

(b) Use the second mean value theorem to prove that for f strictly increasing on $[a,b]$ and $g \in \mathcal{R}[a,b]$, $g(t) \ge 0$, there exists a $c \in [a,b]$ such that

$$\int_a^b f(t)g(t)dt = f(a)\int_a^c g(t)dt + f(b)\int_c^b g(t)dt.$$

Hint: $G(a) = 0$, $G(x) = \int_a^x g(t)dt$, $a < x \le b$, $G \in \mathcal{C}[a,b]$. Therefore G increases.

37. **Cauchy-Maclaurin Integral Test.** Let $f\colon [1,\infty) \to \mathbb{R}$ be positive and nonincreasing. Let $I_n := \int_1^n f(x)\,dx$ and $s_n := \sum_{k=1}^n f(k)$. Then
(i) the sequences (I_n) and (s_n) both converge or diverge.
(ii) $\sum_{k=1}^n f(k) - \int_1^n f(x)\,dx \to \ell$ where $0 \le \ell \le f(1)$.

Proof. Since f is monotone, f is integrable on $[1,n]$ for any $n \in \mathbb{N}$. Observe that $f(k) \le f(x) \le f(k-1)$ for $x \in [k-1,k]$. Hence by the monotonicity of the integral, we obtain

$$f(k-1) \ge \int_{k-1}^k f(x)\,dx \ge f(k). \tag{76}$$

Adding these inequalities and using the additivity of the integral, we get

$$\sum_{k=1}^{n-1} f(k) \ge \int_1^n f(x)\,dx \ge \sum_{k=2}^n f(k). \tag{77}$$

From (77), (i) follows from comparison test.

To prove (ii), we define $\varphi(n) := \sum_{k=1}^n f(k) - \int_1^n f(x)\,dx$. Then φ is nonincreasing (use (76) and satisfies $0 \le \varphi(n) \le f(1)$ (use (77)). (ii) follows from these observations. □

38. Euler's Constant γ. Apply the last item to conclude that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$ converges to a real number, denoted by γ where $0 < \gamma < 1$.

39. Exercise: Prove that

    (a) $\sum_1^\infty \frac{1}{1+n^2} < \frac{1}{2} + \frac{1}{4\pi}$.

    (b) If $-1 < k \leq 0$, $1^k + 2^k + \cdots + n^k - \frac{n^{k+1}}{k+1}$ is convergent.

    (c) If $-1 < k \leq 0$, $\frac{1^k+2^k+\cdots+n^k}{n^{k+1}} \to \frac{1}{k+1}$.

    (d) $k\sum_1^\infty \frac{1}{n^{1+k}} \to 1$, as $k \to 0+$.

Riemann's Original definition

40. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Let $t_i \in [x_i, x_{i+1}]$, $0 \leq i \leq n-1$. Then t_i's are called tags. Let $\mathbf{t} = \{t_i : 0 \leq i \leq n-1\}$ be the set of tags. The pair $(P, \mathbf{t})$ is called a tagged partition of $[a, b]$.

41. The Riemann sum of f for the tagged partition $(P, \mathbf{t})$ is defined to be $S(f, P, \mathbf{t}) := \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i)$.

    We say that f is Riemann integrable on $[a, b]$ if there exists $A \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists a partition P such that for any refinement Q of P and for any tag $\mathbf{t}$ of Q, we have
$$|S(f, Q, \mathbf{t}) - A| < \varepsilon.$$
    We call A the Riemann integral of f on $[a, b]$. It is easy to see that A is unique. Note that we do not demand that f is bounded on $[a, b]$.

42. If $f : [a, b] \to \mathbb{R}$ is Riemann integrable, then it is bounded on $[a, b]$.

    Proof. Let P be partition such that for $\varepsilon < 1$, we have $|S(f, P+, \mathbf{t}) - A| < 1$, for any set of tags in P. Fix a set of tags $\{a_i\}$ in P. For any set $\{x_i\}$ of tags, we have
$$|S(f, P, \{a_i\}) - S(f, P, \{x_i\})| < 2.$$
    Let us take $x_i = a_i$ for $i \geq 2$ and $x_1 = x \in [x_0, x_1]$ arbitrary. It follows that
$$|S(f, P, \{a_i\}) - S(f, P, \{x_i\})| = |f(a_1) - f(x_1)|(x_1 - x_0) < 2,$$
    so that $|f(x)| \leq \frac{2}{(x_1-x_0)} + |f(a_1)|$. That is, f is bounded on $[x_0, x_1]$. Similarly, it is bounded on each of the subintervals of the partition and hence is bounded on $[a, b]$. $\square$

43. Riemann integrable iff Darboux integrable and both the integrals are the same.

    More precisely, let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is integrable iff it is Riemann integrable, in which case we have $\int_a^b f(x)\,dx$ is the Riemann integral.

    Proof. We shall prove a simple case first. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Take any $c_i \in [\frac{i-1}{n}, \frac{i}{n}]$. Then
$$\lim_{n\to\infty} \frac{1}{n} \sum f(c_i) = \int_0^1 f(x)dx.$$

Observe that

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} f(c_i) &= \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} [(f(c_i)-f(x))+f(x)]\,dx\\ &= \int_0^1 f(x)\,dx + \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} (f(c_i)-f(x))\,dx.\end{aligned}$$

If $n \gg 1$, the terms $|f(x)-f(c_i)|$ can be estimated using uniform continuity.

Now, we deal with the general case.

Assume that f is integrable on $[a,b]$. Let $I=\int_a^b f(x)\,dx$. Let $\varepsilon>0$ be given. Since f is integrable, there exists a partition P of $[a,b]$ (by Item 9) such that

$$L(f,P) > I-\varepsilon \quad\text{and}\quad U(f,P) < I+\varepsilon. \tag{78}$$

Let $Q=\{x_0,\dots,x_n\}$ be any refinement of P. Then (78) holds true when P is replaced by Q. Note also that for any choice of $t_i \in [tx_i, x_{i+1}]$, we have

$$m_i(f) \le f(t_i) \le M_i(f).$$

It follows that

$$\begin{aligned} I-\varepsilon &< L(f,Q),\ \text{using (78) and the fact } L(f,P)\le L(f,Q)\\ &\le \sum_{i=0}^{n-1} f(t_i)(x_{i+1}-x_i)\\ &\le U(f,Q)\\ &< I+\varepsilon,\ \text{using (78) and the fact } U(f,Q)\le U(f,P).\end{aligned}$$

That is, for any tagged partition $(Q,\mathbf{t})$ with $Q\supset P$, we have

$$|\sum_{i=0}^{n-1} f(t_i)(x_{i+1}-x_i) - I| < \varepsilon.$$

Thus, f is Riemann integrable on $[a,b]$ with Riemann integral I.

Let f be Riemann integrable with Riemann integral A. Given $\varepsilon>0$, there exists a partition $P=\{x_i : 0\le i\le n\}$ such that

$$|\sum_{i=0}^{n-1} f(t_i)(x_{i+1}-x_i) - A| < \varepsilon/3,$$

for *any* set of tags $\mathbf{t}=\{t_i : 0\le i<n\}$. There exist $s_i, u_i \in [x_{i-1}, x_i]$ such that

$$f(s_i) < m_i(f) + \frac{\varepsilon}{6(b-a)} \quad\text{and}\quad f(u_i) > M_i(f) - \frac{\varepsilon}{6(b-a)}$$

so that

$$M_i(f) - m_i(f) \le [f(u_i)-f(s_i)] + \frac{\varepsilon}{3(b-a)}$$

We now obtain

$$\begin{aligned} U(f,P)-L(f,P) &= \sum_{i=0}^{n-1}[M_i(f)-m_i(f)](x_{i+1}-x_i) \\ &< \sum_{i=0}^{n-1}(f(u_i)-f(s_i))(x_{i+1}-x_i)+\frac{\varepsilon}{3(b-a)}\sum_{i=0}^{n-1}(x_{i+1}-x_i) \\ &\leq |\sum_{i=0}^{n-1} f(u_i)(x_{i+1}-x_i)-A|+|A-\sum_{i=0}^{n-1} f(s_i)(x_{i+1}-x_i)|+\frac{\varepsilon}{3} \\ &< \varepsilon. \end{aligned}$$

Hence f is integrable.

Let $I=\int_a^b f(x)\,dx$ and A be the Riemann integral of f on $[a,b]$. We need to show that $A=I$. Let $\varepsilon>0$ be given. Since f is Riemann integrable, there exists a partition P_1 such that for *any* refinement Q of P_1 we have

$$|S(f,P,\mathbf{t})-A|<\varepsilon/3, \text{ for any set of tags } \mathbf{t}. \tag{79}$$

Since f is integrable, there exists a partition P_2 such that

$$U(f,P_2)-L(f,P_2)<\varepsilon/3 \text{ so that } U(f,P_2)<L(f,P_2)+\varepsilon/3. \tag{80}$$

Let $Q=P_1\cup P_2$.

We observe that (using (80)),

$$L(f,Q)\leq S(f,Q,\mathbf{t})\leq U(f,Q)\leq L(f,Q)+\frac{\varepsilon}{3}. \tag{81}$$

Again, using (80), we have

$$L(f,Q)\leq I\leq U(f,Q)\leq L(f,Q)+\frac{\varepsilon}{3}. \tag{82}$$

Using (79)–(82), we obtain,

$$\begin{aligned} |A-I| &\leq |A-S(f,Q,\mathbf{t})|+|S(f,Q,\mathbf{t})-L(f,Q)|+|L(f,Q)-I| \\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}. \end{aligned}$$

Since $\varepsilon>0$ is arbitrary, this shows that $A=I$. □

44. Exercise: Applications to sum of an infinite series. Show that

 (a) $s_n=\sum_{r=1}^n \frac{r}{r^2+n^2}\to\log\sqrt{2}$ as $n\to\infty$.
 (b) for $a>-1$, $s_n=\frac{1^a+2^a+\cdots+n^a}{n^{1+a}}\to\frac{1}{1+a}$.
 (c) $s_n=\frac{1}{2n+1}+\frac{1}{2n+2}+\cdots+\frac{1}{3n}\to\log(3/2)$.
 (d) $n\sum_{r=1}^n\left(\frac{1}{r^2+n^2}\right)\to\frac{\pi}{4}$.
 (e) $\sum_{k=1}^n\frac{1}{(n^2+k^2)^{\frac{1}{2}}}\to\log(1+\sqrt{2})$.

45. Exercise: Obtain the limits of the sequences whose n-th term is
 (i) $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$.
 (ii) $\frac{1}{n+1}-\frac{1}{n+2}+\cdots+\frac{(-1)^{n-1}}{2n}$.

10 Improper Riemann Integrals

We extend the concept of Riemann integral to functions defined on unbounded intervals or to unbounded functions.

1. **Observation.** Let $f\colon [a,b] \to \mathbb{R}$ be integrable. Then

$$\int_a^b f(t)\,dt = \lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d f(t)\,dt\right).$$

Proof. Let $g(x) := \int_a^x f(t)\,dt$. Then g is continuous on $[a,b]$. Therefore,

$$\begin{aligned}\int_a^b f(t)\,dt &= g(b) - g(a)\\ &= \lim_{c\to a+}\left(\lim_{d\to b-} g(d) - g(c)\right)\\ &= \lim_{d\to b-}\left(\lim_{c\to a+} g(d) - g(c)\right).\end{aligned}$$

□

2. The observation of the last item motivates the following definition. Let (a,b) be a nonempty, open, possibly unbounded interval and $f\colon (a,b)\to\mathbb{R}$.

   (a) We say that f is *locally integrable* on (a,b) if f is integrable on each closed subinterval $[c,d]$ of (a,b).

   (b) We say that the improper Riemann integral of f exists on (a,b) if

$$\lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d f(t)\,dt\right)$$

   exists. The limit is denoted by $\int_a^b f(t)\,dt$.

3. The order in which limits are taken in the last definition does not matter.

Proof. Let $t_0 \in (a,b)$ be fixed. We observe

$$\begin{aligned}\lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d f(t)\,dt\right) &= \lim_{c\to a+}\left(\int_c^{t_0} f(t)\,dt + \lim_{d\to b-}\int_{t_0}^d f(t)\,dt\right)\\ &= \lim_{c\to a+}\int_c^{t_0} f(t)\,dt + \lim_{d\to b-}\int_{t_0}^d f(t)\,dt\\ &= \lim_{d\to b-}\left(\lim_{c\to a+}\int_c^d f(t)\,dt\right).\end{aligned}$$

□

4. In view of the last item we use the notation

$$\lim_{\substack{c\to a+\\ d\to b-}} \int_c^d f(t)\,dt \quad \text{to stand for} \quad \lim_{c\to a+}\left(\lim_{d\to b-}\int_c^d f(t)\,dt\right).$$

5. If we deal with intervals of the form $(a,b]$, we may simplify the notation

$$\int_a^b f(t)\,dt = \lim_{c\to a+}\int_c^b f(t)\,dt.$$

6. Examples:

   (a) The function $f(x) := x^{-1/2}$ has an improper integral on $(0,1]$.

   (b) The function $f(x) := x^{-2}$ has an improper integral on $[1,\infty)$.

7. **Theorem.** If the improper integral of f,g exist on (a,b) and $\alpha,\beta\in\mathbb{R}$, then the improper integral of $\alpha f+\beta g$ exists on (a,b) and we have

$$\int_a^b (\alpha f(t)+\beta g(t))\,dt = \alpha\int_a^b f(t)\,dt + \beta\int_a^b g(t)\,dt.$$

8. (Comparison theorem.) Let f,g be locally integrable on (a,b). Assume that $0\le f(t)\le g(t)$ for $t\in(a,b)$ and that the improper integral of g exists on (a,b). Then the improper integral of f exists on (a,b) and we have

$$\int_a^b f(t)\,dt \le \int_a^b g(t)\,dt.$$

   Proof. Fix $c\in(a,b)$. Let $F(d) := \int_c^d f(t)\,dt$ and $G(d) := \int_c^d g(t)\,dt$ for $d\in[c,b)$. We have $F(d)\le G(d)$. Since $f\ge 0$, the function F is increasing on $[c,b]$. Hence $F(b-)$ exists. Thus, the improper integral of f exists on (c,b) and we get

$$\int_c^d f(t)\,dt = F(b-) \le G(b-) = \int_c^b g(t)\,dt.$$

   A similar argument works for the case $c\to a+$ □

9. Examples.

   (a) The function $f(x) := |x^{-3/2}\sin x|$ has improper R-integral on $(0,1]$. *Hint:* Observe that $0\le f(x)\le x^{-3/2}|x| = x^{-1/2}$ on (0,1].

   (b) The function $f(x) := x^{-5/2}\log x$ has improper R-integral on $[1,\infty)$. *Hint:* Note that $0\le f(x)\le x^{-5/2}x^{1/2}$ for all $x>M$ for some $M>0$.

10. Exercise: If f is bounded and locally integrable on (a,b) and g has improper R-integral on (a,b), then $|fg|$ has an improper R-integral on (a,b).

11. Let $f\colon(a,b)\to\mathbb{R}$.

(a) We say that f is *absolutely integrable* on (a,b) if $|f|$ has an improper integral on (a,b).

(b) When do we say f is conditionally integrable on (a,b)?

12. **Theorem.** If f is absolutely integrable on (a,b), then the improper integral of f on (a,b) exists and we have

$$|\int_a^b f(t)\,dt| \le \int_a^b |f(t)|\,dt.$$

Proof. Note first of all that f is locally integrable on (a,b). (Why?) Note that $0 \le |f(x)| + f(x) \le 2|f(x)|$. Hence the improper integral of $|f| + f$ exists on (a,b) (by comparison theorem). By Item 7, the improper integral of $f = (|f|+f) - |f|$ also exists on (a,b). Also,

$$|\int_c^d f(t)\,dt| \le \int_c^d |f(t)|\,dt \text{ for all } a < c < d < b.$$

We finish the proof by taking limits as $c \to a+$ and $d \to b-$. □

13. **An important example.** We show that $f(x) = \sin x/x$ is conditionally integrable on $[1,\infty)$.

Consider $\int_1^R \frac{\sin x}{x}\,dx$. Integration by parts yields

$$\begin{aligned}\int_1^R \frac{\sin x}{x}\,dx &= -\frac{\cos x}{x}\Big]_1^R - \int_1^R \frac{\cos x}{x^2}\,dx \\ &= \cos 1 - \frac{\cos R}{R} - \int_1^R \frac{\cos x}{x^2}\,dx. \qquad (83)\end{aligned}$$

Since x^{-2} is absolutely integrable on $[1,\infty)$ and since $|\cos x| \le 1$, it follows that $\frac{\cos x}{x^2}$ is absolutely integrable on $[1,\infty)$. We obtain from (83)

$$\int_1^\infty \frac{\sin x}{x}\,dx = \cos 1 - \int_1^\infty \frac{\cos x}{x^2}\,dx.$$

We now show that $\sin x/x$ is not absolutely integrable on $[1,\infty)$. We observe that

$$\begin{aligned}\int_1^{n\pi} \frac{|\sin x|}{x}\,dx &\ge \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx \\ &\ge \sum_{k=2}^n \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x|\,dx \\ &= \sum_{k=2}^n \frac{1}{k\pi} \int_0^\pi \sin x\,dx \\ &= \sum_{k=2}^n \frac{2}{k\pi} \to \infty.\end{aligned}$$

14. (Gamma Function.) We shall show that the improper integral of the function $f(t) := t^{x-1}e^{-t}$ exists on $(0, \infty)$.

Observe that $t^{x-1}e^{-t} \le t^{x-1}$ for $t > 0$. Hence the improper integral of f exists on $(0, 1)$ for $x > 0$.

Also, since $t^{x+1}e^{-t} \to 0$ as $t \to \infty$, it follows that

$$t^{x-1}e^{-t} \le Ct^{-2} \text{ for } t \ge 1 \text{ for some } C > 0.$$

Hence the improper integral of f exists on $[1, \infty)$.

The gamma function is defined on $(0, \infty)$ by the formula

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\, dt.$$

Using integration by parts, one shows that $\Gamma(x + 1) = x\Gamma(x)$ so that $\Gamma(n) = (n - 1)!$.

15. **Exercise.**

(a) For each of the following, find the values of $p \in \mathbb{R}$ for which the improper integral exists on the specified interval I.

i. $f(x) = x^{-p}$, $I = (1, \infty)$.
ii. $f(x) = x^{-p}$, $I = (0, 1)$.
iii. $f(x) = 1/(1 + x^p)$, $I = (0, \infty)$.
iv. $f(x) = 1/(x \log^p x)$, $I = (e, \infty)$.

(b) Decide whether the improper integral of $f(x) := (2 + x^8)^{-1/4}$ exists on $(1, \infty)$. Ans: It exists.

(c) Decide whether the improper integral of $f(x) := (\pi + x^3)^{-1/4}$ exists on $(0, \infty)$. Ans: No.

(d) Decide whether the improper integral of $f(x) := \frac{e^x}{1+e^{2x}}$ exists on $\mathbb{R}$. Ans: Yes, it is $\frac{\pi}{2}$.

(e) Show that the improper integral of $f(x) := |x|^{-1/2}$ exists on $[-1, 1]$ and its improper integral is 4.

(f) Decide which of the following functions have improper integral on I:

i. $f(x) = \sin x$, $I = (0, \infty)$
ii. $f(x) = x^{-2}$, $I = [-1, 1]$.
iii. $f(x) = x^{-1}\sin(x^{-1})$, $I = (1, \infty)$.
iv. $f(x) = \log x$, $I = (0, 1)$.

(g) Assume that the improper integral of f exists on $[1, \infty)$ and that $\lim_{x\to\infty} f(x) = L$ exists. Prove that $L = 0$.

(h) True or false: $\int_0^\pi \sec^2 x\, dx = 0$?

11 Riemann-Stieltjes Integral

Most of the arguments in this section are completely similar to those in the theory of Riemann integral. Hence most of the details will be omitted.

1. Let $\alpha\colon [a,b] \to \mathbb{R}$ be an increasing function, P a partition of $[a,b]$. We let $\Delta\alpha_i = \alpha(t_i) - \alpha(t_{i-1})$ for $1 \le i \le n$. For any bounded function $f\colon [a,b] \to \mathbb{R}$, we define

$$L(f,\alpha,P) := \sum_{i=1}^{n} m_i \Delta\alpha_i \quad \text{and} \quad U(f,\alpha,P) := \sum_{i=1}^{n} M_i \Delta\alpha_i.$$

   We let

$$\underline{I}(f,\alpha) := \sup_P L(f,\alpha,P) \quad \text{and} \quad \overline{I}(f,\alpha) := \inf_P U(f,\alpha,P).$$

   If $\overline{I}(f,\alpha) = \underline{I}(f,\alpha)$, we say that f is α-integrable on $[a,b]$ and its Riemann-Stieltjes integral (RS-integral, for short) $\int_a^b f\,d\alpha$ is the common value:

$$\int_a^b f\,d\alpha := \overline{I}(f,\alpha) = \underline{I}(f,\alpha)$$

2. We let $\mathcal{R}(\alpha,[a,b])$ (or $\mathcal{R}(\alpha)$, if the interval is understood) denote the set of all RS-integrable functions on $[a,b]$ with respect to the *integrator* α.

3. If we let $\alpha(t) = t$, we see that the Riemann integral is a special case of RS-integral.

4. The list of results below and their proofs are similar to the analogous results in the theory of R-integral.

   (a) If P_2 is a refinement of P_1, then

$$L(f,\alpha,P_1) \le L(f,\alpha,P_1) \quad \text{and} \quad U(f,\alpha,P_1) \ge U(f,\alpha,P_1).$$

   (b) For any two partitions P, Q, we have

$$L(f,\alpha,P) \le U(f,\alpha,Q).$$

   (c) We have $\underline{I}(f,\alpha) \le \overline{I}(f,\alpha)$.

   (d) $f \in \mathcal{R}(\alpha)$ iff for each $\varepsilon > 0$, there exists a partition P of $[a,b]$ such that

$$U(f,\alpha,P) - L(f,\alpha,P) < \varepsilon.$$

   (e) If f is continuous on $[a,b]$, then $f \in \mathcal{R}(\alpha)$.

   (f) Let f be monotone on $[a,b]$. If we further assume that α is continuous, then $f \in \mathcal{R}(\alpha)$.

Proof. The proof as in the Riemann integral goes through if for a given $N \in \mathbb{N}$, we can choose a partition such that

$$\Delta\alpha_i = \frac{\alpha(b) - \alpha(a)}{N}.$$

Since α is now assumed to be continuous, this is possible. Why? For instance, we are seeking a point t_1 such that

$$\alpha(t_1) = \alpha(a) + \frac{\alpha(b) - \alpha(a)}{N} \leq \alpha(b).$$

Since α is continuous, by the intermediate value theorem, there exists $t_1 \in [a, b]$ as required. This process can be repeated. □

(g) Let $f \in \mathcal{R}(\alpha)$. Let $m \leq f \leq M$ on $[a, b]$. Assume that $g\colon [m, M] \to \mathbb{R}$ be continuous. Then $g \circ f \in \mathcal{R}(\alpha)$.
Proof is similar to the one in Item 17 of Section 9.

5. Properties of the RS-integrals.

   (a) (Linearity). $\mathcal{R}(\alpha)$ is a vector space over $\mathbb{R}$ and the map $f \mapsto \int_a^b f\, d\alpha$ is a linear map.

   (b) (Monotonicity). If $f \leq g$, then $\int_a^b f\, d\alpha \leq \int_a^b g\, d\alpha$.

   (c) (Integral as a function of intervals). Let $a < c < b$ and $f \in \mathcal{R}(\alpha, [a, b])$ on $[a, b]$. Then $f \in \mathcal{R}(\alpha, [a, c])$ and $f \in \mathcal{R}(\alpha, [c, b])$ and we have

$$\int_a^b f\, d\alpha = \int_a^c f\, d\alpha + \int_c^b f\, d\alpha.$$

   (d) If $f \in \mathcal{R}(\alpha_1, [a, b])$ and $f \in \mathcal{R}(\alpha_2, [a, b])$, then $f \in \mathcal{R}(\alpha_1 + \alpha_2, [a, b])$.

   (e) If $f \in \mathcal{R}(\alpha, [a, b])$ and $c > 0$, then $f \in \mathcal{R}(c\alpha, [a, b])$ and we have

$$\int_a^b f\, d(c\alpha) = c \int_a^b f\, d\alpha.$$

   The last two are 'new' results in the theory of RS-inetgrals nd are easy to prove.

6. If $f, g \in \mathcal{R}(\alpha)$, then $fg \in \mathcal{R}(\alpha)$.

7. If $f \in \mathcal{R}(\alpha)$, then $|f| \in \mathcal{R}(\alpha)$ and we have

$$|\int_a^b f\, d\alpha| \leq \int_a^b |f|\, d\alpha.$$

8. We now specialize to the case when the integrators are either step functions or continuously differentiable functions. These two class of RS-integrals occur in many parts of mathematics, physical sciences and statistics and hence of practical importance.

9. Let $H\colon \mathbb{R} \to \mathbb{R}$ be the Heaviside function defined by $H(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$. Note that H is increasing.

10. **Lemma.** Let $f\colon [a,b] \to \mathbb{R}$ be bounded and continuous at $s \in (a,b)$. Let $\alpha(t) = H(t-s)$, the Heaviside function. Then
$$\int_a^b f\,d\alpha = f(s). \tag{84}$$

   Proof. Let $P = \{a = t_0 < t_1 = s < t_2 > t_3 = b\}$ be a partition. Then $\alpha(t_1) - \alpha(t_0) = H(t_1 - s) - H(a - S) = 0$, $\alpha(t_2) - \alpha(t_1) = H(t_2 - s) - H(t_1 - s) = 1$ and $\alpha(t_3) - \alpha(t_2) = H(t_3 - s) - H(t_2 - s) = 0$. Hence it follows that $U(f,P,\alpha) = M_1$. Similarly, $L(f,P,\alpha) = m_1$. If we choose a sequence of partitions such that $t_2 \to s$, we see that $M_1, m_1 \to f(s)$, thanks to the continuity of f at s. □

   We refer the reader to Rudin ' book for the proofs of the following theorems (Theorems 6.16 and 6.16 on pages 130–132).

11. **Theorem.** Let $\sum_{n=1}^\infty c_n$ be a convergent series of nonnegative terms. Let (s_n) be a sequence of distinct points in (a,b). Let $\alpha(x) := \sum_n c_n H(x - s_n)$. Let f be continuous on $[a,b]$. Then
$$\int_a^b f\,d\alpha = \sum_n c_n f(s_n).$$
□

12. **Theorem.** Let $\alpha\colon [a,b] \to \mathbb{R}$ be differentiable and $\alpha' \in \mathcal{R}([a,b])$. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}(\alpha)$ iff $f\alpha' \in \mathcal{R}([a,b])$. In such a case, we have $\int_a^b f\,d\alpha = \int_a^b f(t)\alpha'(t)\,dt$. □

12 Functions of Bounded Variation

1. Definition; computing the arc-length as a motivation.

2. Notation $V(f,[a,b],P)$ where P is a partition of $[a,b]$. If the interval $[a,b]$ is understood, we simplify the notation as $V(f,P)$. Observe $V(f,P) \le V(f,Q)$ if Q is a refinement of P. Variation $V_a^b(f)$.

3. Examples:

   (a) Any monotone function $f\colon [a,b] \to \mathbb{R}$ is of bounded variation and we have $V_a^b(f) = |f(b) - f(a)|$.

   (b) Let $f\colon [a,b] \to \mathbb{R}$ be Lipschitz, say, $|f(x) - f(y)| \le L|x-y|$. Then f is of bounded variation on $[a,b]$.

   (c) Let $f\colon [a,b] \to \mathbb{R}$ be continuously differentiable (that is, $f'(x)$ exists on $[a,b]$ and f' is continuous on $[a,b]$). Then f is of bounded variation. In fact, we have
$$V_a^b(f) := \int_a^b |f'(x)|\,dx$$

(d) Let $f\colon [a,b] \to \mathbb{R}$ be a function. Let $a = t_1 < \cdots < t_n = b$ be a partition of $[a,b]$. Assume that f is a constant on each of the subintervals, say, $f(t) = c_i$ on (t_i, t_{i+1}). Such an f is called a step function. Such functions are of bounded variation and $V_a^b(f)$ is the sum of $|f(t_i) - f(t_i+)|$ and $|f(t_i) - f(t_i-)|$.

(e) Let $\gamma\colon [a,b] \to \mathbb{R}^2$ be a parametrized curve. Write $\gamma(t) := (x(t), y(t))$. Given a partition P of $[a,b]$ we define $V(\gamma, P) := \sum_i \|\gamma(t_i) - \gamma(t_{i-1})\|$. (Recall the motivation.) We say that the curve γ is *rectifiable* if $\sup_P V(\gamma, P)$ is finite. Observe that γ is rectifiable iff both x and y are functions of bounded variation on $[a,b]$.

(f) The function $f(x) = x\sin(1/x)$ for $0 < x \leq 1$and $f(0) = 0$ is continuous, bounded but is not of bounded variation. (Draw the graph of this function to see why.)

4. If $f\colon [a,b] \to \mathbb{R}$ is of bounded variation, then $|f(x)| \leq |f(a)| + V_a^b(f)$, that is, f is bounded.

5. If f, g are of bounded variation on $[a,b]$ so are $f+g$ and $f-g$.

6. $V_a^b(f) = 0$ iff f is a constant.

7. $V_a^b(cf) = |c|\, V_a^b(f)$.

8. Total variation of $f\colon [a,b] \to \mathbb{R}$, a function of bounded variation is the function F defined as follows:

$$F(x) := \sup_P V(f, P),$$

where the supremum is taken over all partitions P of the interval $[a,x]$.

9. **Proposition.** Let $f\colon [a,b] \to \mathbb{R}$ be of bounded variation and F its total variation. Then
   (i) $|f(x) - f(y)| \leq |F(x) - F(y)|$ for $a \leq x < y \leq b$.
   (ii) F and $F - f$ are increasing on $[a,b]$ and
   (iii) $V_a^b(f) \leq V_a^b(F)$.

10. **Jordan's Theorem.** Let $f\colon [a,b] \to \mathbb{R}$. Then f is of bounded variation iff there exist increasing functions F and G such that $f = F - G$. *Hint:* Take $G = F - f$ where F is the total variation of f. □

Reference for this topic: Tom Apostol's Mathematical Analysis, Chapter 6 (especially Sections 6.1 – 6.7)

Topics for Students' Seminar.

1. Decimal expansion and other expansion to the base of a positive integer $a \geq 2$;
2. The constant e, Euler's constant γ; Sequences defined recursively;
3. Extension theorem for uniformly continuous functions; application to the definition of x^a;
4. Inverse function theorem; its application to $x \mapsto x^{1/n}$;
5. Cauchy's generalized mean value theorem and its applications to L'Hospital's rules;
6. Inequalities: AM-GM; Bernoulli; Cauchy-Schwarz; Hölder; Minkowski;
7. Convex functions; applications to inequalities;
8. Various forms of the remainder term in Taylor's theorem; their applications to binomial and logarithmic series;
9. Existence of C^∞ functions with 'compact support';
10. Riemann's theorem on rearrangement of series;
11. Various tests for non-absolute convergence: Abel's summation formula, Tests of Dedekind, Abel, Dirichlet with examples; both numerical series and series of functions to be dealt with;
12. Everywhere continuous nowhere differentiable functions;
13. Definition of $\log(x) := \int_1^x \frac{dt}{t}$, its properties, exponential function and the definition of x^a;
14. Weierstrass approximation theorem: Bernstein's proof, Landau kernel proof;
15. Sharp forms of ratio and root tests in terms of $\limsup$ and $\liminf$;
16. Thomae's function; discussion on points of continuity and integrability on $[0, 1]$;
17. Hadamard's formula for the radius of convergence of a power series;
18. Cantor's Construction of real numbers using Cauchy sequences;
19. Dedekind's construction of real numbers via 'cuts';
20. Functions of bounded variation and rectifiability;
21. Peano curves; main purpose here is to make the students see applications of p-adic (binary, ternary) expansions.

Things to be included in this set of notes: improper integrals; more details on functions of bounded variation;