Summary of Real Analysis 2 – Semester 2 (2009-10)

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Riemann Integral on $\mathbb{R}^n$

1. An interval (or a rectangle) J in $\mathbb{R}^n$ is a set of the from $[a_1, b_1] \times \cdots \times [a_n, b_n]$ where $a - k, b_k \in \mathbb{R}$ with $a_k \leq b_k$ for $1 \leq k \leq n$. The 'length', 'area' , or the 'volume' $|J|$ of J is defined as $|J| := \prod_{k=1}^n (b_k - a_k)$.

2. Any partition P of an interval $J \subset \mathbb{R}^n$ is a 'product' of partitions P_k of the component intervals $[a_k, b_k]$. Any 'subinterval' of the partition P of J is of the form $I_1 \times \cdots \times I_n$ where I_k is a subinterval of the partition P_k of $[a_k, b_k]$. If $P_k = J_{k1} \cup \cdots \cup J_{km_k}$ is the partition of $[a_k, b_k]$, then a component subinterval of the partition P of J is of the form $J_{1r_1} \times J_{2r_2} \times \cdots J_{nr_n}$ where $1 \leq r_k \leq m_k$ for $1 \leq k \leq n$. (Draw pictures when $n = 2$. This will help you understand and not be overwhelmed by the excessive notation.)

   The mesh of P is the largest side of all component sub-intervals, that is, the largest of the meshes of P_k.

3. We defined Riemann integrability of a bounded function $f \colon J \to \mathbb{R}$ as in the one dimensional case. Given a partition P of J, we let $M_I(f)$ to be the supremum of f on the subinterval I of the partition P. Similarly, $m_I(f)$ is the infimum of f on the subinterval I of the partition P. The upper and lower (Darboux) sums are defined as follows:

$$U(f, P) := \sum_I M_I(f)|I| \text{ and } L(f, P) := \sum_I m_I(f)|I|$$

4. One similarly defines the upper integral and the lower integral of f as follows.

$$U(f) := \inf\{U(f, P) : P \text{ a partition of } J\} \text{ and } L(f) := \sup\{L(f, P) : P \text{ a partition of } J\}.$$

   We say that f is (Darboux) integrable on J if $U(f) = L(f)$. The common value is called the integral of f on J and is denoted by $\int_J f(x)\, dx$ or simply $\int_J f$.

5. Let $f \colon J \to \mathbb{R}$ be a bounded function, say, $m \leq f \leq M$ on J. Let P be a partition of J. Then the following are easy to verify.

   (a) $m|J| \leq L(f, P) \leq U(f, P) \leq M|J|$.

(b) $m|J| \leq L(f)) \leq M|J|$.

(c) $m|J| \leq U(f)) \leq M|J|$.

6. Refinements of a partition P are defined as in the one dimensional case. A partition Q of J is a refinement of P is every component subinterval of Q is a subinterval of a component of P. In other words, if $P = (P_1, \ldots, P_n)$ and $Q = (Q_1, \ldots, Q_n)$, then Q_k is a refinement of P_k for each k.

7. The following results are proved in the same way as in the one dimensional case and pose no new problems.

   (a) If Q is a refinement of P, then $L(f, Q) \geq L(f, P)$ and $U(f, Q) \leq U(f, P)$.

   (b) For *any* two partitions P_1, P_2, we have $L(f, P_1) \leq U(f, P_2)$.

   (c) $L(f) \leq U(f)$.

   (d) f is integrable iff for any given $\varepsilon > 0$, there exists a partition P of J such that $U(f, P) - L(f, P) < \varepsilon$.

8. Using the uniform continuity, we show that a continuous function $f\colon J \to \mathbb{R}$ is integrable.

9. The following facts are proved as in the one dimensional case.

   (a) The set of integrable functions on J is a real vector space and the integral $f\colon \mapsto \int_J f$ is linear.

   (b) The integral is monotone: if $f \leq g$ on J, then $\int_J f \leq \int_J g$.

   (c) If $m \leq f \leq M$ on J and if $g\colon [m, M] \to \mathbb{R}$ is continuous, then $g \circ f$ is integrable on J.

   (d) If f is integrable on J, then so is $|f|$ and we have $|\int_j f| \leq \int_J |f|$.

   (e) If f and g are integrable on J,, then so are f^2 and fg.

   (f) If $f_k \to f$ uniformly on J and if each f_k is integrable on J, then so is f and we have $\int_J f = \lim \int_J f_k$.

10. For physical applications, it is important to know the original definition of Riemann integrability of a function. Let $f\colon J \to \mathbb{R}$ be bounded. Let P be partition of J. If I is a subinterval of partition, let $t_I \in I$ be a any point. Let $\mathbf{t} := \{t_I : I \text{ a subinterval of the partition } P\}$. Then the pair $(P, \mathbf{t})$ is called the tagged partition of J and t_I are called the tags.

    The Riemann sum corresponding to the tagged partition is defined as

    $$S(f, P, \mathbf{t}) := \sum_I f(t_I)|I|.$$

    We say that f is Riemann integrable on J if there exists $A \in \mathbb{R}$ such that for a given $\varepsilon > 0$, there exists $\delta > 0$ such that for any partition P of J whose mesh is less than δ and for *any* set $\mathbf{t}$ of tags, we have

    $$|S(f, P, \mathbf{t}) - A| < \varepsilon.$$

11. **Theorem.** A bounded function $f\colon J \to \mathbb{R}$ is integrable iff it is Riemann integrable.

12. We proved this in the one dimensional case, later adapted the argument to prove the n-dimensional case. We went through the proof as in a textbook and simplified the argument on the way.

Theorem 1 (Fubini)**.** *Let $J_1 \subset \mathbb{R}^m$ and $J_2 \subset \mathbb{R}^n$ be (closed and bounded) intervals. Let $J := J_1 \times J_2$ and $f\colon J \to \mathbb{R}$ be integrable. Let $f_x(y) := f(x, y)$ for $(x, y) \in J_1 \times J_2$. Then $g(x) := L(f_x, J_2)$, the lower integral of f_x on J_2 and $h(x) := U(f_x, J_2)$, the upper integral of f_x on J_2 are integrable on J_1 and we have*

$$\int_J f(x,y)\,dx\,dy = \int_{J_1} L(f_x, J_2) = \int_{J_1} \left(\underline{\int}_{J_2} f(x,y)\,dy \right) dx \tag{1}$$

$$= \int_{J_1} U(f_x, J_2) = \int_{J_2} \left(\overline{\int}_{J_1} f(x,y)\,dy \right) dx. \tag{2}$$

In particular, if f is continuous on J, then we have

$$\int_J f(x,y)\,dx\,dy = \int_{J_1} \left(\int_{J_2} f(x,y)\,dy \right) dx = \int_{J_2} \left(\int_{J_1} f(x,y)\,dx \right) dy. \tag{3}$$

Proof. Given $\varepsilon > 0$, since f is integrable on J there exists a partition $P = (P_1, P_2)$ of J such that $U(f, P) - L(f, P) < \varepsilon$. Let $R = R_1 \times R_2$ be an arbitrary subinterval of the partition. Then we have, forr any fixed $x \in J_1$,

$$\begin{aligned} U(f,P) &= \sum_R M_f(R)|R| \\ &= \sum_{R_1} \sum_{R_2} M_{R_1 \times R_2}(f)|R_2||R_1| \\ &\geq \sum_{R_1} \left(\sum_{R_2} M_{R_2}(f_x)|R_2| \right) |R_1| \\ &= \sum_{R_1} U(f_x, P_2)|R_1| \\ &\geq \sum_{R_1} U(f_x, J_2)|R_1|. \end{aligned}$$

In particular, we have for any $x \in R_1$, $U(f, P) \geq \sum_{R_1} h(x)|R_1|$ so that by taking supremum over $x \in R_1$, we obtain

$$U(f,P) \geq \sum_{R_1} M_{R_1}(h)|R_1|.$$

Hence $U(f, P) \geq U(h, P_1)$.

In a similar way, we obtain $L(f, P) \leq L(g, P_1)$. We now observe that

$$L(f,P) \leq L(g,P_1) \leq L(h,P_1) \leq U(h,P_1) \leq U(f,P).$$

Since the first and fifth terms are ε-close, it follows that $U(h, P_1) - L(h, P_1) < \varepsilon$, that is, h is integrable on J_1. Consequently, (2) follows. □

Theorem 2. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ be open sets. Let $\varphi\colon U \to V$ be a C^1-diffeomorphism, that is, φ is bijective, C^1 and its inverse φ^- is also C^1. Let $j(x) := |\det(D\varphi(x))|$ for $x \in U$. Let $f\colon V \to \mathbb{R}$ be a continuous function. Then*

$$\int_V f(y)\,dy = \int_U f \circ \varphi(x) j(x)\,dx. \tag{4}$$

Proof. As we have seen earlier, if φ is the restriction of a nonsingular linear map A, then (4) is true. We shall make use of this in the sequel.

For estimating purpose, it is easier to use the following norms:

$$\|x\| := \|x\|_\infty \equiv \max\{|x_i| : 1 \le i \le n\} \text{ and } \|A\| := \max\left\{\sum_{k=1}^n |a_{ik}| : 1 \le i \le n\right\}. \tag{5}$$

It is then easy to check that $\|Ax\| \le \|A\|\,\|x\|$. The advantage of this norm lies in the fact that a cube (centered at a and side-length $2r$) can be described as: $Q(a,r) := \{z \in \mathbb{R}^n : \|z - a\| \le r\}$.[1]

As usual, we write $|E|$ to denote the n-dimensional volume of $E \subset \mathbb{R}^n$. Write $\varphi = (\varphi_1, \ldots, \varphi_n)$. Let $x \in Q(a,r)$. Then by mean value theorem applied to φ_i, we get

$$\varphi_i(x) - \varphi_i(a) = \sum_{k=1}^n \frac{\partial \varphi_i}{\partial x_k}(z_i)(x_k - a_k), \tag{6}$$

for some z_i in the line segment joining a and z. In view of our definitions of the norm on $\mathbb{R}^n$, we see that

$$|\varphi_i(x) - \varphi_i(a)| \le \sum_{k=1}^n |\frac{\partial \varphi_i}{\partial x_k}(z_i)||(x_k - a_k)|. \tag{7}$$

Now in view of our definition of the norm of a matrix, we obtain

$$\|\varphi(x) - \varphi(a)\| \le r \max\{\|D\varphi(x)\| : x \in Q\}. \tag{8}$$

In particular, the image $\varphi(Q)$ is completely contained in the image of the cube defined by $\{z : \|z - \varphi(a)\| \le r \max\{j(x) : x \in Q\}$. Hence it follows that

$$|\varphi(Q)| \le (\max\{|Df(x)| : x \in Q\})^n\,|Q|. \tag{9}$$

Let A be any nonsingular linear map. We apply (9) to the map $A^{-1}\varphi$ and make use of the earlier observation that $|A^{-1}(E)| = |\det(A^{-1})||E|$ for $E \subset \mathbb{R}^n$. We obtain

$$|\det(A^{-1})||\varphi(Q)| \le \left(\max\{A^{-1}Df(x) : x \in Q\}\right)^n |Q|. \tag{10}$$

Hence we get, since $\det(A^{-1}) = \det(A)^{-1}$,

$$|\varphi(Q)| \le |\det(A)| \left(\max\{|A^{-1}Df(x)| : x \in Q\}\right)^n |Q|. \tag{11}$$

[1] See, in particular, the way the inequalities (7) and (8) are derived.

The idea now is the subdivide the cube Q into smaller ones and replace A by $Df(a_k)$, a_k being the centres of these subcubes. Let $\{Q_k : 1 \leq k \leq N\}$ be a 'partition' [2] of Q into subcubes. Assume that the sides are of length at most δ. Let a_k be the center of Q_k. If we apply (11) to each of these cubes and if we replace A by $A_k := D\varphi(a_k)$, we obtain

$$|\varphi(Q)| \leq \sum_{k=1}^{N} \det(A_k) \left(\max\{|A_k^{-1} Df(x)| : x \in Q_k\}\right)^n |Q_k|. \tag{12}$$

Since $x \mapsto D\varphi(x)$ is (uniformly) continuous on (the compact set) Q, we see that $D\varphi(x) \to D\varphi(a_k)$ if $x \to a$ in Q_k. Hence the matrix $D\varphi(a_k)^{-1} D\varphi(x)$ goes to the identity as $x \to a_k$. In particular, the determinant $|D\varphi(a_k)^{-1} D\varphi(x)| \to 1$ as $x \to a$. Given $\varepsilon > 0$, we may choose $\delta > 0$ in such a way that $|D\varphi(z)^{-1} Df(a_k)|^n \leq 1 + \varepsilon$. This yields

$$|\varphi(Q)| \leq (1+\varepsilon) \sum_{k=1}^{N} |A_k||Q_k| = \sum_{k=1}^{N} |j(a_k||Q_k|. \tag{13}$$

The right side term of (13) is a Riemann sum which approaches the Riemann integral $\int_Q j(x)\,dx$, as the mesh $\delta \to 0$. Hence we arrive at the following inequality:

$$|\varphi(Q)| \leq \int_Q j(x)\,dx. \tag{14}$$

Now let $f \colon V \to \mathbb{R}$ be any *non-negative* continuous function such that its support

$$\text{Support}(f) := \text{Closure of } \{y \in V : f(y) \neq 0\} \subset L,$$

is a compact subset of V. Note that the support of $f \circ \varphi$ is $\varphi^{-1}(\text{Support}(f))$ is also compact, since φ is a homeomorphism.

Let $\mathcal{Q}$ be a covering of U by means of a finite collection of cubes whose mesh is δ. Let $\{Q_k : 1 \leq k \leq N\}$ be those cubes of $\mathcal{Q}$ which are contained in the support of f. Let S_N denote the complement of their union in U so that $U = S_N \cup \left(\cup_{k=1}^N Q_k\right)$. We set $b_k := \varphi(a_k)$ and $\alpha_k := f(\varphi(a_k))$, $1 \leq k \leq N$. From (14), we get

$$\sum_{k=1}^{N} \alpha_k \int_{\varphi(Q_k)} dy \leq \sum_{k=1}^{N} \alpha_k \int_{Q_k} j(x)\,dx. \tag{15}$$

We expect the term on the left side 'approximates' $\int_V f(y)\,dy$ and the one on the right 'approximates' $\int_U (f \circ \varphi) j(x)\,dx$.

Let $E := \int_V f(y)\,dy - \int_U (f \circ \varphi)(x) j(x)\,dx$. We then have, in view of (15)

$$E \leq \left(\int_V f(y)\,dy - \sum_{k=1}^{N} \alpha_k \int_{\varphi(Q_k)}\right) + \left(\sum_{k=1}^{N} \alpha_k \int_{Q_k} j(x)\,dx - \int_U (f \circ \varphi)(x) j(x)\,dx\right). \tag{16}$$

[2] A partition of Q is a collection of subcubes such that their union is Q and any two distinct subcubes meet at most along their sides.

Since $\varphi(U) = \cup_{k=1}^N \varphi(Q_k) \cup \varphi(S_n)$,[3] we have

$$E \leq \int_{\varphi(S_N)} f(y)\,dy + \sum_{k=1}^N \int_{\varphi(Q_k)} (f - \alpha_k)\,dy + \sum_{k=1}^N \int_{Q_k} (\alpha_k - f\circ\varphi) j(x)\,dx - \int_{S_N} (f\circ\varphi)(x) j(x)\,dx. \tag{17}$$

Since φ is C^1, it is Lipschitz on the support of f, that is, there exists $L > 0$ such that

$$\|\varphi(x_1) - \varphi(x_2)\| \leq L\,\|x_1 - x_2\| \text{ on the support of } f. \tag{18}$$

Also, since the support of f is compact, f is uniformly continuous on U. Hence given $\varepsilon > 0$, we can choose the mesh $\delta > 0$ so small such that

$$|f\circ\varphi(x) - \alpha_k| < \varepsilon, \qquad \text{for all } x \in Q_k, 1 \leq k \leq N. \tag{19}$$

Using the Lipschitz nature of φ, by shrinking $\delta > 0$ is necessary, we may assume that

$$|f(y) - \alpha_k| < \varepsilon \qquad \text{for all } y \in \varphi(Q_k), 1 \leq k \leq N. \tag{20}$$

Hence it follows that

$$E \leq \int_{\varphi(S_N)} f(y)\,dy - \int_{S_N} (f\circ\varphi)(x) j(x)\,dx + \varepsilon\left(|\varphi(S)| + |S|\right), \tag{21}$$

where S is the support of f.

Choose a mesh so that $|S_N| < \varepsilon$. It follows that $|\varphi(S_N)| \leq L^n\varepsilon$. If M is a bound for f, then we arrive at the following inequality:

$$E \leq \varepsilon\left(ML^n + M + |\varphi(S)| + |S|\right). \tag{22}$$

Hence we conclude $D \leq 0$. Most importantly, we obtain

$$\int_V f(y)\,dy \leq \int_U f(\varphi(x)) j(x)\,dx. \tag{23}$$

Replacing φ by φ^{-1}, we deduce from (23)

$$\int_V f(y)\,dy \leq \int_U f(\varphi(x)) j(x)\,dx \leq \int_V f(\varphi(\varphi^{-1}))(y) |D\varphi^{-1}(y)||D\varphi(x)|\,dy = \int_V f(y)\,dy.$$

□

[3]Note that the union may not be pairwise disjoint. One can easily show using the Lipschitz nature of φ that the common intersections are of measure zero.