Riesz-Fredholm-Schauder Theory

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Lemma 1. *Let X be a Banach space. Assume that $T \in BL(X)$ is compact and that $\lambda \in \mathbb{C}^*$ is not an eigenvalue of T. Then there exists $C > 0$ such that for all $x \in X$ we have*

$$\|(T - \lambda I)x\| \geq C\|x\| \text{ for } x \in X. \tag{1}$$

Proof. If no such C exists, then there exists a sequence (x_n) of unit vectors such that $Tx_n - \lambda x_n \to 0$. Since T is compact there exists a subsequence of (Tx_n) which is convergent. After re-indexing, let us assume that $Tx_n \to y$. Let $y_n := Tx_n$ so that $y_n \to y$. Observe that

$$(T - \lambda I)Tx_n = T(T - \lambda I)x_n \to 0.$$

That is, $(T - \lambda I)y_n \to 0$. Since $y_n \to y$, it follows that $Ty_n - \lambda y_n \to 0$ or $Ty - \lambda y = 0$. Hence λ has an eigenvector, provided y is nonzero. Can y be zero? If it is, the facts that $Tx_n \to 0$ and that $Tx_n - \lambda x_n \to 0$ imply $\lambda x_n \to 0$. That is, $|\lambda| = |\lambda| \|x_n\| \to 0$. This contradicts our assumption that $\lambda \neq 0$. □

Corollary 2. *Keep the notation of the last lemma. Then the range $R(T - \lambda I)$ is closed.*

Proof. Let $(T - \lambda I)x_n \to y$. From (1), it follows that (x_n) is Cauchy. Since X is complete, (x_n) converges to some x. Clearly, $(T - \lambda I)x = y$. □

Theorem 3 (Fredholm Alternative). *Let X be a Banach space and $T \in BL(X)$ compact. Let $\lambda \in \mathbb{C}^*$. Then exactly one of the following is true.*
(i) *λ is an eigenvalue of T so that* $\ker(T - \lambda I)$ *is nonzero.*
(ii) *$T - \lambda I$ is a topological isomorphism, that is, $(T - \lambda I)^{-1} \in BL(X)$.*

Proof. Assume that λ is not an eigenvalue of T. In view of BIT, it suffices to show that $(T - \lambda I)$ is bijective. It is one-one thanks to (1). So, we need only establish that $(T - \lambda I)$ is onto. Assume that this does not happen.

Let $X_1 := (T - \lambda I)X$. Observe that (1) shows that $(T - \lambda I)$ is one-one. Since by the last corollary, X_1 is closed and hence complete. Thus, $T - \lambda I$ is a continuous linear bijection of the Banach space X onto the Banach space X_1. Hence by BIT, $T - \lambda I$ is a topological linear isomorphism of X onto X_1. Let $X_{n+1} := (T - \lambda I)X_n$, $n \geq 1$. Then by the argument above, we see that (i) each X_{n+1} is a proper closed linear subspace of X_n and that $T - \lambda I$ is a topological linear isomorphism of X_n onto its proper closed subspace X_{n+1}. Hence by Riesz lemma, there exist a unit vector $x_n \in X_n$ such that $d(x_n, X_{n+1}) \equiv \inf\{\|x_n + y\| : y \in X_{n+1}\} \geq 1/2$.

We now claim that the sequence (Tx_n) cannot a convergent subsequence. For, since $(T - \lambda I)x_n \in X_{m+1}$ and $(T - \lambda I)x_m \in X_{m+1}$, we obtain

$Tx_n - \lambda x_n - (Tx_m - \lambda x_m) \in X_{m+1}$ so that $Tx_n - Tx_m \in -\lambda x_m + (\lambda x_n + X_{m+1}) = -\lambda x_m + X_{m+1}$.

Since $d(x_m, X_{m+1}) = \|x_m + X_{m+1}\| \geq 1/2$, it follows that $\|-\lambda x_m + X_{m+1}\| \geq \frac{|\lambda|}{2}$. Therefore, $\|Tx_n - Tx_m\| \geq \frac{|\lambda|}{2}$. The claim is proved. This contradicts out assumption that T is compact.

□