Cardinality and Countability

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The primitive idea of "counting" a set is to set up a bijection with a known set. The words 'calculus' and 'calculation' have their origin with such a correspondence with a pile of stones!

Definition 1. We say that two sets A and B have the *same cardinality* if there is a bijection from one onto the other. (Intuitively, this means that A and B "have the same number of elements." Because of this we may even say that A and B are *equinumerous*.) Note that "having the same cardinality" is "an equivalence relation."

Example 2. (i) $\mathbb{N}$ and $2\mathbb{N}$, the set of even positive integers have the same cardinality.
(ii) Any two closed intervals $[a, b]$ and $[c, d]$ have the same cardinality.
(iii) Any two open intervals (a, b) and (c, d) have the same cardinality.
(iv) $(-1, 1)$ and $\mathbb{R}$ have the same cardinality. *Hint:* Consider the map is the map $f: (-1, 1) \to \mathbb{R}$ given by $f(x) := \frac{x}{1-|x|}$. Its inverse is given by $g(t) := \frac{t}{1+|t|}$. Or, observe that $\tan: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a bijection.
(v) $\mathbb{Z}$ and $\mathbb{N}$ have the same cardinality.
(vi) $\mathbb{R}$ and $(0, \infty)$ have the same cardinality.
(vii) (0,1) and $(1, \infty)$ have the same cardinality.

Lemma 3 (Knaster)**.** *Let $F: P(X) \to P(X)$ be a map. Assume that it is increasing in the sense that if $A \subseteq B$, then $F(A) \subseteq F(B)$. Then F has a fixed point, that is, there exists $S \subset X$ such that $F(S) = S$.*

Hint: Consider the set $\mathcal{C} := \{C \subseteq X : C \subseteq F(C)\}$. Let S be the union of all members of $\mathcal{C}$. Then $F(S) = S$. □

The next theorem is very useful. See Example 5.

Theorem 4 (Schroeder-Bernstein)**.** *Let A and B be sets. Assume that $f: A \to B$ and $g: B \to A$ be one-one. Then there exists a bijection $h: A \to B$.*

Hint: Consider $F: P(A) \to P(A)$ given by $F(C) := A \setminus g(B \setminus f(C))$. Apply the last lemma. □

Example 5. (i) $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ have the same cardinality. *Hint:* The map $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $f(m, n) := 2^m 3^n$ is one-one. For an explicit bijection, see Example 10.

(ii) The set $\mathbb{Q}$ of rational numbers and $\mathbb{N}$ have the same cardinality. Look at $\mathbb{Q} \to \mathbb{Z} \times \mathbb{N} \to \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.
(iii) There exists a bijection between the intervals $[a, b]$ and (c, d). *Hint:* The interval $[a, b]$ and a closed subinterval of (c, d) have the same cardinality by Example 2.
(iv) The sets $A := (0, 1)$ and $B := A \times A$ have the same cardinality. *Hint:* Use non-recurring decimal expansion to get a one-map of B into A. For example, consider $g(0.x_1x_2 \dots, 0.y_1y_2 \dots) := 0.x_1y_1x_2y_2 \dots$.

For any $n \in \mathbb{N}$, let I_n denote the subset $\{k : 1 \leq k \leq n\}$ of $\mathbb{N}$.

Definition 6. A set A is said to be *finite* if either $A = \emptyset$ or there is a bijection $f : A \to I_n$ for some $n \in \mathbb{N}$.

A set which is not finite is said to be *infinite.*

Theorem 7. *Let A be a finite set. Let $f : A \to I_m$ and $g : A \to I_n$ be bijections. Then $m = n$.* □

Definition 8. If A is finite with $f : A \to I_n$ is a bijection, then n is unique by the last theorem. (Note that f need not be unique.) We say that A has n elements. If A is empty, we say that A has zero elements.

Definition 9. A set A is said to be countable if either A is finite or if there exists a bijection $f : A \to N$. A set of the latter type is said to be *countably infinite.*

A set which is not countable is said to be *uncountable.*

Example 10. (i) $\mathbb{Z}_+$, $\mathbb{Z}$ are countably infinite.
(ii) Any infinite subset of $\mathbb{N}$ is countably infinite.
(iii) $\mathbb{N}\times\mathbb{N}$ is countably infinite. *Hint:* Consider the map $f(m, n) := \dfrac{(m+n-1)(m+n-2)}{2} + n$. How did one arrive at this map? What is the inverse of this map?

The inverse is given by $m \mapsto (\frac{n(n-1)}{2} - m + 1, m - \frac{(n-1)(n-2)}{2})$ where $\frac{(n-2)(n-1)}{2} < m \leq \frac{n(n-1)}{2}$. Choose ℓ so that $\frac{\ell(\ell+1)}{2} < k \leq \frac{(\ell+1)(\ell+2)}{2}$. Then $f(\ell + 1, 1) = \frac{\ell(\ell+1)}{2}$. Choose n such that $\frac{\ell(\ell+1)}{2} + n = k$. Choose m so that $m + n = \ell + 2$. Then $f(m, n) = k$.

(iv) The set of rational numbers is countably infinite.

Proposition 11. *Let A be a set. The following are equivalent.*
(i) *A is countable.*
(ii) *There is a one-one map of A into $\mathbb{N}$.*
(iii) *There is an onto map from $\mathbb{N}$ onto A.* □

Corollary 12. (i) *A subset of a countable set is countable.*
(ii) *Let I be a countable set and let A_i be countable for each $i \in I$. Then $A := \cup_{i\in I} A_i$ is countable, that is, a countable union of countable sets is countable.*
(iii) *A finite product of countable sets is countable.* □

Ex. 13. Show that $\mathbb{Q}$ is countable. *Hint:* Let $q \in \mathbb{N}$. Let A_q be the set of rational numbers whose denominator is q. Then A_q is countable and $\mathbb{Q}$ is the union of A_q's.

Ex. 14. Show that the set $F(\mathbb{N})$ consisting of finite subsets of $\mathbb{N}$ is countable.

Ex. 15. Let $f\colon X \to Y$ be onto. Prove that if X is countable so is Y.

Example 16. A complex number is said to be an *algebraic number* if it is a root of a polynomial with integer coefficients. The set of algebraic numbers is countable. *Hint:* Show that the set of polynomials with integer coefficient is countable.

Theorem 17 (Cantor)**.** *Let X be any set and $P(X)$, the power set of X. There is no onto function from X onto $P(X)$.* □

Ex. 18. Show that $P(\mathbb{N})$ is not countable.

Ex. 19. Let X be any set. Show that there exists no one-one function from $P(X)$ to X.

Ex. 20. The set of functions from $\mathbb{N}$ to $\{0, 1\}$ is not countable. *Hint:* The set under question is bijective with $P(\mathbb{N})$.

Example 21. $\mathbb{R}$ is uncountable. *Hint:* Enough to show that $[0, 1]$ is uncountable. Use Nested interval theorem. Also, diagonal trick can be used. □

Ex. 22. The set of irrational numbers is uncountable.

Ex. 23. The set $\mathbb{C}$ of complex numbers and $\mathbb{R}$ have the same cardinaliy.

A complex number is *transcendental* if it is not algebraic.

Corollary 24. *The set of transcendental numbers is uncountable.* □

Remark 25. Why is the last result historically important?

Ex. 26. Show that the set $F(\mathbb{N}, \mathbb{N}) := \{f\colon \mathbb{N} \to \mathbb{N}\}$ is uncountable. *Hint:* Diagonal trick.

Theorem 27. *The following are equivalent for a set X.*
(i) *The set X is infinite.*
(ii) *There exists a countably infinite subset S of X.*
(iii) *There exists a proper subset Y of X such that X and Y have the same cardinality.* □

Ex. 28. Let X be uncountable. Show that there exists a countably infinite subset $A \subset X$ such that X and $X \setminus A$ have the same cardinality.

Ex. 29. Let X be uncountable and $A \subset X$ a countable subset. Show that $X \setminus A$ is uncountable. More precisely, show that X and $X \setminus A$ have the same cardinality.

Reference:

J.R. Munkres, *Topology*, especially Sections 1.6, 1.7 and 1.9