Riesz-Markov Representation Theorem

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Abstract

The aim of this article is to rewrite the proof of the theorem of the title (found in Rudin's book) taking into account that the target audience has already undergone a course in Lebesgue Measure and Integral and hence is not averse to the concept of outer measure and the σ-algebra of measurable sets, using Cartheodary's definition. Footnotes are points to reflect upon.

Let X denote a locally compact Hausdorff space. Let f be a real valued continuous function on X. The *support* of f is the subset

$$\operatorname{Supp}(f) := \text{Closure of } \{x \in X : f(x) \neq 0\}.$$

We say that f has compact support if the support of f is a compact subset set. Note that this is same as saying that f is zero outside a compact set. We let $C_c(X)$ denote the vector space[1] of all compactly supported real valued continuous functions on X.

A linear functional $\Lambda\colon C_c(X) \to \mathbb{R}$ is said to be *positive* if $\Lambda(f) \geq 0$ for any $f \in C_c(X)$ with $f \geq 0$. A trivial example: Fix $p \in X$ and define $\Lambda(f) := f(p)$ for $f \in C_c(X)$.

Let $\mathcal{B}$ denote the sigma algebra of Borel subsets of X. (Recall that $\mathcal{B}$ is the smallest σ-algebra containing the open subset of X.) A measure μ on $(X, \mathcal{B})$ is said to be a Borel measure if $\mu(K) < \infty$ for any compact subset $K \subset X$.

If μ is a Borel measure, then we have an associated positive linear functional Λ on $C_c(X)$ defined as follows:[2]

$$\Lambda(f) := \int_X f\, d\mu \text{ for } f \in C_c(X).$$

We then say μ *represents* the functional Λ. (The trivial example above is a special case where μ is the Dirac measure based at $p \in X$.)

We say that a Borel measure[3] μ on X is *regular* if it satisfies the following two conditions:

(i) $\mu(B) = \inf\{\mu(V) : B \subset V, V \text{ is open}\}$ for each Borel set $B \in \mathcal{B}$.
(ii) $\mu(V) = \sup\{\mu(K) : K \subset V, K \text{ is compact}\}$ for each open set V.

The theorem of the title can now be stated.

[1] Why is $C_c(X)$ a vector space?
[2] Why does $\Lambda(g)$ make sense?
[3] Our definition is slightly different from Rudin's.

Theorem 1 (Riesz-Markov)**.** *Let X be a localy compact Hausdorff space and $\Lambda\colon C_c(X) \to \mathbb{R}$ be positive linear functional. Then there exists a unique regular Borel measure μ on X such that*

$$\Lambda(f) = \int_X f\, d\mu \text{ for each } f \in C_c(X).$$

We introduce some notation which has become standard now. If V is open and $f \in C_c(X)$ is such that $0 \le f \le 1$ *on* X and Supp (f) $\subset$ V, we denote this by the symbol $f \prec V$. [4]If K is compact and $f \in C_c(X)$ is such that $0 \le f \le 1$ on X and if $f = 1$ on K, then we denote it by $K \prec f$. The symbol $K \prec f \prec V$ stands for $K \prec f$ and $f \prec V$.

We need the following two results from topology. Proofs can be found in Rudin's *Real and Complex Analysis*, Chapter 2.

Lemma 2. *Let X be a locally compact Haudorff space. Let K be a compact subset and U an open set U such that $K \subset U$. Then there exists an open set V such that (i) $\overline{V}$ is compact and $K \subset V \subset \overline{V} \subset U$.* □

Theorem 3 (Finite Partition of Unity)**.** *Let X be a locally compact Hausdorff space, $K \subset X$ a compact set. Let $\{V_i : 1 \le i \le n\}$ be a finite open cover of K. Then there exist $f_i \in C_c(X)$ such that $f_i \prec V_i$ for each i and $\sum_i f_i = 1$ on K.* □

Given a positive linear functional Λ on $C_c(X)$, the strategy is first to define an outer measure on $P(X)$. In analogy with Lebesgue measure, we define the outer measure of open sets and use it to define the outer measure on $P(X)$.

Let Λ be a positive linear functional on $C_c(X)$. Given an open set $V \subset X$, we define[5]

$$\mu(V) := \sup\{\Lambda(f) : f \prec V\}.$$

Clearly, $0 \le \mu(V) \le \infty$. Also, if V, W are open sets with $V \subset W$, then $\mu(V) \le \mu(W)$. This allows us to define $\mu(A)$ for any $A \subset X$ as follows: [6]

$$\mu(A) := \inf\{\mu(V) : A \subset V \text{ and } V \text{ is open in } X\}. \tag{1}$$

Note that if A is an open set, the apparently two definitions coincide.

An obvious guess and a hope is that μ is an outer measure on $P(X)$.

Lemma 4. *μ defined as in* (1) *is an outer measure on $P(X)$.*

Proof. [7] It is clear that if $A \subset B \subset X$, then $\mu(A) \le \mu(B)$. We need only establish the countable subadditivity. Let (A_n) be a countable family subsets of X. If $\sum_n \mu(A_n) = \infty$, the subadditivity is trivial. So we assume that the infinite series is convergent. In particular, $\mu(A_n) < \infty$ for each n. Let $\varepsilon > 0$ be given. For each n, there exists an open set V_n such that $\mu(V_n) < \mu(A_n) + 2^{-n}\varepsilon$. Let $V := \cup_n V_n$. Then V is an open set with $A := \cup_n A_n \subset V$. Choose an $f \in C_c(X)$ such that $f \prec V$. Let $K :=$ Supp (f). Then the compact set $K \subset V = \cup_n V_n$.

[4]In particular, $f \le \chi_V$.
[5]Draw pictures to see the motivation for this definition.
[6]Why is this natural?
[7]This proof is analogous to the one seen in the theory of Lebesgue outer measure.

Hence there exists N such that $K \subset \cup_{k=1}^N V_k$. By Theorem 3, there exist $f_k \in C_c(X)$ such that (i) $f_k \prec V_k$ for $1 \le k \le N$ and (ii) $\sum_{k=1}^N f_k = 1$ on K. Clearly, $f \le \sum_{k=1}^N f_k$. Using the monotonicity of Λ we obtain

$$\Lambda(f) \le \Lambda\left(\sum_{k=1}^N f_k\right) = \sum_{k=1}^N \Lambda(f_k) \le \sum_{k=1}^N \mu(V_k) \le \sum_{k=1}^\infty \mu(V_k) \le \sum_{k=1}^\infty \mu(A_k) + \varepsilon.$$

This is true for each $f \prec V$ so that we get $\mu(V) \le \sum_{k=1}^\infty \mu(A_k) + \varepsilon$[8]. It follows by taking infimum (over the open sets $V \supset A$) that $\mu(A) \le \sum_{k=1}^\infty \mu(A_k) + \varepsilon$ for each $\varepsilon > 0$. ☐

Let $(X, P(X), \mu)$ be the outer measure constructed above. We say that a subset $E \subset X$ is measurable if for *any* set $A \subset X$, we have $\mu(A) = \mu(A \cap E) + \mu(A \setminus E)$. In view of subadditivity, to establish the measurability of E, it is enought to show that $\mu(A) \ge \mu(A \cap E) + \mu(A \setminus E)$. It is well-known that the class $\mathcal{A}$ of measurable subsets is a σ-algebra. (The proof is verbatim the same as the one used to show that the class of Lebesgue measurable subsets of $\mathbb{R}$ is a σ-algebra.)

Theorem 5. *The σ-algebra of μ-measurable subsets contains $\mathcal{B}$. Moreover, μ restricted to $\mathcal{B}$ is a regular Borel measure.*

Proof. The proof is broken into seven claims each of which is relatively easy.

Step 1. For each compact set $K \subset X$, we have $\mu(K) < \infty$.

Proof. Let $K \subset X$ be compact. Let V be an open set such that $K \subset V$ and $\overline{V}$ is compact. Then there exists $g \in C_c(X)$ such that $\overline{V} \prec g$. If $f \in C_c(X)$ is such that $f \prec V$, then clearly $f \le g$. Hence $\Lambda(f) \le \Lambda(g)$. It follows that

$$\mu(K) \le \mu(V) \le \sup\{\Lambda(f) : f \prec V\} \le \Lambda(g) < \infty.$$

☐

Step 2. For any $E \subset X$, $\mu(E) := \inf\{\mu(V) : E \subset V, \text{ V is open.}\}$.

This is just the definition of μ.

Step 3. For any open set V, we have

$$\mu(V) := \sup\{\mu(K) : K \text{ is compact and } K \subset V\}.$$

Proof. Let $r < \mu(V)$. By the definition of supremum, there exists $f \in C_c(X)$ such that $f \prec V$ and $\Lambda(f) > r$. Let $K := \text{Supp}\,(\text{f})$. Then $K \subset V$ is compact. Let W be any open set with $K \subset W$. Clearly, $f \prec W$ and hence $\mu(W) \ge \Lambda(f) > r$. We now get

$$\mu(V) \ge \mu(K) = \inf\{\mu(W) : W \text{ open and } K \subset W\} \ge \Lambda(f) > r.$$

Since this holds true for each $r < \mu(V)$, the result follows. ☐

[8] V is special here. What happens when W is any open set containing A?

Step 4. If A and B are disjoint compact subsets, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

Proof. Let $K = A \cup B$. In view of countable subadditivity, it suffices to show that $\mu(K) \geq \mu(A) + \mu(B)$.

Let $\varepsilon > 0$ be given. Let W be an open set such that $K \subset W$ and $\mu(W) < \mu(K) + \varepsilon$. To get a handle on $\mu(A)$ and $\mu(B)$, we need to 'create' open sets containing the compact sets A and B to exploit the definition of $\mu(A)$ and $\mu(B)$. Since A is compact, it is closed and its complement $V = W \setminus A$ is an open set containing B. Hence, there exists an open set V_2 such that $B \subset V_2 \subset \overline{V_2} \subset V$ with $\overline{V_2}$ compact. Now if we set $V_1 := W \setminus \overline{V_2}$, then V_1 is an open set which is disjoint[9] from V_2. Observe that $A \subset V_1$ and $B \subset V_2$.

Consider two functions $f_1, f_2 \in C_c(X)$ such that $f_i \prec V \cap V_i$, $i = 1, 2$ with the property that $\mu(V_1) < \Lambda(f_1) + \varepsilon$ and $\mu(V_2) < \Lambda(f_2) + \varepsilon$. Since $V_1 \cap V_2 = \emptyset$, we see that $f_1 + f_2 \prec W$. We now have

$$\begin{aligned}
\mu(A) + \mu(B) &\leq \mu(V_1) + \mu(V_2) \\
&\leq \Lambda(f_1) + \varepsilon + \Lambda(f_2) + \varepsilon \\
&= \Lambda(f_1 + f_2) + 2\varepsilon \\
&\leq \mu(W) + 2\varepsilon \\
&\leq (\mu(K) + \varepsilon) + 2\varepsilon, \qquad \text{by our choice of } W.
\end{aligned}$$

Thus, we obtain $\mu(A) + \mu(B) \leq \mu(A \cup B) + 3\varepsilon$ for all $\varepsilon > 0$. The result follows. □

Step 5. If $V$is an open and K is compact such that $V \cap K = \emptyset$, then $\mu(V \cup K) = \mu(V) + \mu(K)$.

Proof. Easy consequence of Step 3 and Step 4: If $L \subset V$ is compact, then $K \cap L = \emptyset$ so that $\mu(K \cup L) = \mu(K) + \mu(L)$ by Step 4. Hence $\sup\{\mu(K) + \mu(L)\} = \mu(K) + \mu(V)$ by Step 3, where the sup is takne over all compact subsetsof V. □

Step 6. Any open subset V is meaurable.

Proof. Let V be open. We shall establish $\mu(A) \geq \mu(A \cap V) + \mu(A \setminus V)$ for the case when A is open. Since $A \cap V$ is open, we want to make use of Step 5. Let $K \subset A \cap V$ be compact. We now want to exploit Step 5. An obvious choice is $W := A \setminus K$. Note that[10] $A \setminus V \subset A \setminus K = W$. Hence we observe that

$$\mu(K) + \mu(A \setminus V) \leq \mu(K) + \mu(A \setminus K) = \mu(A), \text{ by Step 5.}$$

Taking the supremum of the left side on all compact subsets $K \subset A \cap V$, and using Step 3, we get the desired inequality.

Now let A be arbitrary. If W is an open set with $A \subset W$, then by the last observation, we have

$$\mu(A \cap V) + \mu(A \setminus V) \leq \mu(W \cap V) + \mu(W \setminus V) \leq \mu(W).$$

Taking the infimum over open sets $W \supset A$, and using Step 2, we get the result. □

[9]Draw pictures.

[10]Draw pictures.

Step 7. Every Borel set $B \in \mathcal{B}$ is μ-measurable.

Proof. Since the class $\mathcal{A}$ of μ-measurable sets is a σ-algebra, and all open sets lie $\mathcal{A}$, it follows that $\mathcal{B}$, the smallest σ-algebra generated by the class open sets, is contained $\mathcal{A}$. □

Step 7 shows that μ is a masure on $\mathcal{B}$. Step 1 shows it is a Borel measure. Step 2 (applied to Borel sets) and Step 3 show that μ is regular. Hence we conlcude that μ is a regular Borel measure on $\mathcal{B}$. □

Proof. We now begin the proof of the existence part of the Riesz-Markov Theorem. Given a positive linear functional Λ on $C_c(X)$, let μ be the regular Borel measure associated with Λ as in Theorem 5. We wish to prove that $\Lambda(f) = \int_X f\,d\mu$, for $f \in C_c(X)$.

Let $f \in C_c(X)$ be given. Let $M > 0$ such that $|f| \le M$ on X. Let $K := \mathrm{Supp}\,(\mathrm{f})$. Let V be an open set such that $K \subset V$ and $\mu(V) < \infty$. [11]

Let $1 > \varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that $(2M/N) < \varepsilon$. Let us partition the interval $[-M, M]$ into N subintervals of equal length. Let the vertices be given by $y_k := -M + k\frac{2M}{N}$. Let, for $1 \le k \le N$,

$$\begin{array}{rclcl} A_k & := & K \cap f^{-1}((y_k, y_{k+1}]) & = & \{x \in K : y_k < f(x) \le y_{k+1}\} \\ U_k & := & V \cap f^{-1}(y_k - \varepsilon, y_k + \varepsilon) & = & \{x \in V : y_k - \varepsilon < f(x) < y_{k+1} + \varepsilon\} \end{array}$$

Then $A_k \in \mathcal{B}$ (Why?) and U_k is an open set containing A_k, $1 \le k \le N$. Note also that A_k's are pairwise disjoint and $\cup_k A_k = K$.

Using the regularity of the measure μ, we can find open sets V_k such that $A_k \subset V_k \subset U_k$ with $\mu(V_k) - \mu(A_k) < \varepsilon/N$ for $1 \le k \le N$. Note that $K \subset \cup_k V_k \subset V$. By Theorem 3, there exist $g_k \in C_c(X)$ such that $g_k \prec V_k$ and such that $\sum_k g_k = 1$ on K. We rewrite f using the partiton of unity g_k as $f = \sum_k f g_k$. We split the integral over K as a sum of integrals over A_k's and use the obvious estimates $f \le y_k + \varepsilon$ on V_k for $1 \le k \le N$ to show that $\Lambda(f) - \int f\,d\mu < C\varepsilon$ for some constant $C = C(K, M)$. The details follow. [12]

$$\begin{aligned} \Lambda(f) - \int_X f\,d\mu &= \sum_k \Lambda(f g_k) - \sum_k \int_{A_k} f\,d\mu \\ &\le \sum_k (y_k + \varepsilon)\Lambda(g_k) - \sum_k (y_k - \varepsilon)\mu(A_k) \\ &\le \sum_k (y_k + \varepsilon)\mu(V_k) - \sum_k (y_k - \varepsilon)\mu(A_k) \\ &= \sum_k (y_k + \varepsilon)\left(\mu(V_k) - \mu(A_k)\right) + 2\varepsilon \sum_k \mu(A_k) \\ &\le \sum_k (M + \varepsilon)\frac{\varepsilon}{N} + 2\varepsilon\mu(K) \\ &\le \varepsilon\left(M + 1 + 2\mu(K)\right). \end{aligned}$$

[11] Why is this possible?

[12] You may now go ahead on your own and fix the details.

Since $\varepsilon > 0$ is arbitrary, it follows that $\Lambda(f) - \int f\, d\mu \leq 0$. Same argument with $-f$ in place of f yields $\Lambda(f) - \int f\, d\mu \geq 0$. Hence we obtain $\Lambda(f) = \int f\, d\mu$.

We now attend to the uniqueness part of the theorem. Let μ and ν be two regular Borel measures that represent Λ. We claim that $\mu(K) = \nu(K)$ for any compact subset $K \subset X$. Let $\varepsilon > 0$ be given. By the regularity of μ, there exists an open set $V \supset K$ such that $\mu(V) < \mu(K) + \varepsilon$. There exists $f \in C_c(X)$ such that $K \prec f \prec V$. Note that $\chi_K \leq f \leq \chi_V$. Using the monotonicty (positivity) of Λ, we obtain

$$\nu(K) = \int_X \chi_K\, d\nu \leq \int_X f\, d\nu = \int_X f\, d\mu \leq \int_X \chi_V\, d\mu = \mu(V) < \mu(K) + \varepsilon.$$

This being true for all $\varepsilon > 0$, it follows that $\nu(K) \leq \mu(K)$. Interchanging μ and ν in the argument above, we get $\mu(K) \leq \nu(K)$. Hence the claim follows. The regularity of μ and ν leads us to conclude[13] that $\mu(A) = \nu(A)$ for all $A \in \mathcal{B}$. □

Remark 6. Recall that a measure μ on a σ-algebra $(X, \mathcal{A})$ is said to be *complete* if whenever $E \in \mathcal{A}$ with $\mu(E) = 0$ and if $A \subset E$, then $A \in \mathcal{A}$. Note that $\mu(A)$ is necessarily zero.

It is trivial exercise to show that a measure arising out of an outer measure is always complete. In particular, the measure μ on X corresponding to the postive linear functional Λ on $C_c(X)$ is complete.

[13]Supply the details.