Schröeder Bernstein Theorem

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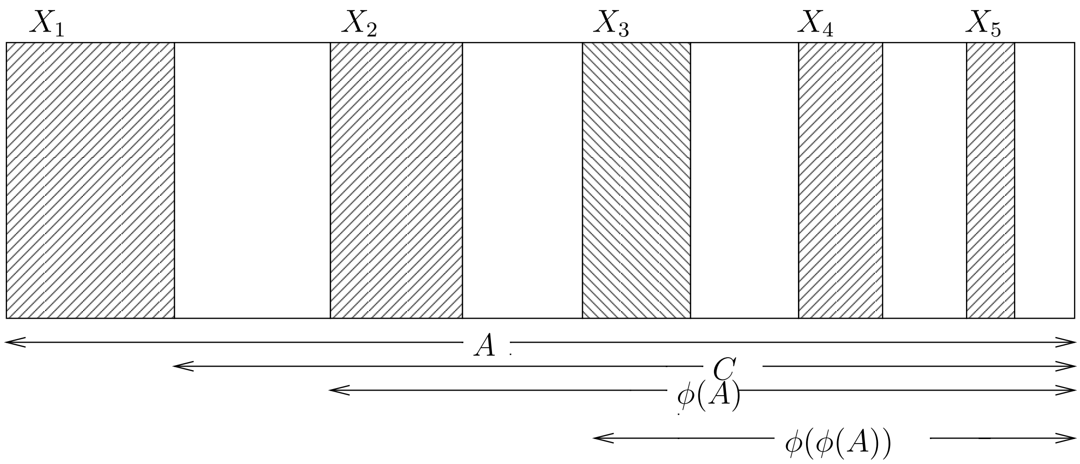
Lemma 1. *Let φ be a 1-1 map of A into itself. If $\varphi(A) \subset C \subset A$, then A is bijective with C.*



Idea behind the proof. Look at the picture. Since φ is 1-1, it carries $\varphi(A) \subset C \subset A$ into a smaller version of the same thing. If we iterate φ then successive images of A and C alternate:

$$A \supset C \supset \varphi(A) \supset \varphi(C) \supset \varphi(\varphi(A)) \supset \varphi(\varphi(C)) \supset \dots$$

Thus φ maps each of the vertical strips at the left bijectively onto its right-hand neighbour once removed. Hence a bijection is obtained if we allow φ to act on the shaded strips and leave everything else fixed. The proof below formalizes this idea.

Proof. Let $X_1 := A \setminus C$. Define inductively $X_{n+1} := \varphi(X_n)$. Let $X := \cup_{n\in\mathbb{N}} X_n$. Define $\psi\colon A \to A$ by setting

$$\psi(a) = \begin{cases} \varphi(a) & \text{if } a \in X \\ a & \text{if } a \notin X. \end{cases}$$

We claim that ψ is a bijection from A to C.

Note that ψ maps X to X and $A \setminus X$ to $A \setminus X$. It follows that ψ is 1-1.

If $a \in A$, either $\psi(a) = a \in A \setminus X \subset C$ or $\psi(a) = \varphi(a) \in \varphi(A) \subset C$. Thus ψ maps A into C.

Let $b \in C$. If $b \in A \setminus X$, then $\psi(b) = b$. If $b \in X$, then choose n so that $b \in X_n$. This integer $n \neq 1$. For, $X_1 \cap C = \emptyset$. Hence $b = \varphi(x)$ for some $x \in X_{n-1}$. Now, $x \in X$ so that $\psi(x) = b$. Thus the range of ψ is C. □

Theorem 2 (Schröder-Bernstein)**.** *Let A and B be sets. Assume that $f\colon A \to B$ and $g\colon B \to A$ are 1-1 maps. Then there exists a bijection from A onto B.*

Proof. Let $\varphi := g \circ f$. Then φ is 1-1 and $\varphi(A) \subset C := g(B) \subset A$. By the lemma there is a bijection ψ from A to $g(B)$. Then $g^{-1} \circ \psi$ is a bijection from A to B. □

Theorem 3 (KNASTER FIXED POINT THEOREM)**.** *If $F\colon P(A) \to P(A)$ is order-preserving, meaning that $F(X)$ is contained in $F(Y)$ whenever $X \subset Y$, then F has a fixed point.*

Proof. Let A_1 be the union of all sets $X \in P(A)$ such that $X \subseteq F(X)$. It is easy to see that $F(A_1) = A_1$. (In fact, this A_1 is the greatest fixed point; there is also a least one.) □

The following is a useful improvement of the usual Schroeder-Bernstein statement:

Theorem 4 (BANACH MAPPING THEOREM)**.** *Given any mappings (not necessarily injections) $f\colon A \to B$ and $g\colon B \to A$, we can partition A into disjoint sets A_1 and A_2, and B into disjoint sets B_1 and B_2, so that $f[A_1] = B_1$ and $g[B_2] = A_2$.*

Proof. Define $F\colon P(A) \to P(B)$ by setting $F(X) = A \setminus g[B \setminus f[X]]$. Clearly, F is order-preserving; by Knaster's theorem, then, F has a fixed point. Let A_1 be a fixed point of F, and let $B_1 = f[A_1]$, $B_2 = B \setminus B_1$, and $A_2 = g[B_2] = A \setminus A_1$. □