Problems - Set Theory and Foundations

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Introduction

Students usually are exposed to only finite sets such as $\{1, 2, 3\}$ or $\{a, b, c\}$ and explicit maps between such sets shown pictorially by arrows connecting x and $f(x)$. For example, students, invariably, have not learnt that the subsets are defined using the so-called "axiom of separation", that is the elements share some common properties. Sets, functions, relations which arise more naturally in mathematics are, as a rule, never introduced. We use this set of exercises in MTTS camps for a variety of purposes: (i) as a remedial measure, (ii) to train the student in writing rigorous mathematics, (iii) to train them in problem solving. Though most often the problems are stated as complete sentences, in the actual class-room, we make the students guess the conclusion and try proving their guess. Also, if we do some problem, students are asked to suggest analogous problems and solve them. This practice gives them a sense of discovery. The target audience consists of students who have finished their first year or second year of B.Sc. The students are required to write complete solutions of these problems, as most of the solutions are a few lines length only. This helps the students acquire the art of writing rigorous mathematics in a gradual manner. We also insist on students writing the negation of most of the definitions in words as well as in symbols using the quantifiers $\exists$ and $\forall$. Even if a half of these exercises are seriously attempted by students in the first week of the programme, we find that the students understand the need for rigour and that there is a remarkable improvement in the writing skills among most of the students at the end of the programme.

The compilation is yet to be polished. The exercises are roughly grouped under these headings: (i) Basic logic, (ii) Sets, Operations on sets, Subsets, (iii) Relations, functions (injective, surjective and bijective), (iv) Equivalence relations and (v) Inverse images.

Very basic ideas about Finite and Infinite sets are also included in our Courses. These are not covered in this set. The rigorous introduction to 'Number of elements in a finite' set is a challenging but quite fulfilling task, as the students do not see the need for such rigour. We reap our reward when we see the students begin to appreciate the subtlety and are prepared for much higher abstractions and more demanding rigour in higher mathematics.

Some of the exercises are difficult for the students at this stage. In such cases what we do is to explain the solutions in complete details and ask the students to write them down on their own.

Repetitions of exercises are deliberately allowed to remain.

1 Logic

The exercises below should be taken only as indicative of what we have in mind. Some preparatory work on simple sentences and their (mathematical) negations etc are needed. Examples are 'There is a broken chair in this classroom', 'Every student here is bright' etc. At no time, the course should stoop down to what the students have gone through in their 12th standard!

Ex. 1. Write down the sentence "All the students in this room are bright" using the quantifier $\forall$. You may use R for the room and s for a student.

What is the negation of the given sentence? Which of the *quantifiers* $\forall$, $\exists$ is required to formulate the negation in mathematical language?

Ex. 2. "There is a broken chair in this room." Negate this sentence and write the sentence and its negation using the quantifiers.

Ex. 3. "There is a genius in this room." Negate this sentence and write the sentence and its negation using the quantifiers.

Ex. 4. "There is a person in this room who is both intelligent and beautiful." What doe sit mean to say that this is false? Formulate the negation in plain English and write it 'mathematically'.

Ex. 5. "There is a mango tree in this campus." Negate this sentence using quantifiers.

Ex. 6. "There is an alphonse mango tree in this campus." Negate this sentence using the quantifiers.

Ex. 7. "There is a tree in this campus all of whose leaves are green." Express this sentence using the quantifiers. Negate the sentence in plain English and write the negation using the quantifiers.

Ex. 8. Write the sentence "A chair in this room has a broken leg' using the quantifiers as required. Negate the sentence and express it in mathematical language.

Ex. 9. Write the sentence using the quantifiers: "There is a tree in this campus with at least one brown leaf."

Negate this sentence in plain English and express it using the quantifiers.

Ex. 10. "Every person in this room is either brilliant or beautiful." Express this using the quantifiers and negate it in plain English and then write it using the quantifiers.

Ex. 11. "There is a person in this room who is both bright and beautiful." Express this using the quantifiers and negate it in plain English and then write it using the quantifiers.

Ex. 12. "For any real number x, either $x > 0$ or $x < 0$." Express this using the quantifiers and negate it in plain English and then write it using the quantifiers.

Ex. 13. Negate the following sentences.

(i) There is a city in this country in which every person can speak at least three languages.

(ii) In every city in this country there exists at least one person who can speak minimum three languages and who can read and write two languages.

Ex. 14. Write $A \subset B$ in symbolic notation using the quantifiers $\exists$ and $\forall$. Negate this symbolically and write it in plain English.

Ex. 15. For all real numbers x, we have $x^2 \geq 0$. Write this in symbols, negate it and then write the negation in words.

Ex. 16. There exists an integer $x \in \mathbb{Z}$ such that for any $y \in \mathbb{Z}$ we have $x + y = y$. Write this in symbols using the quantifiers $\exists$ and $\forall$, negate it and write the negation in words.

Ex. 17. $\mathbb{R}$ has Archimedean property: for each $x > 0$ and $y \in \mathbb{R}$, we can find $n \in \mathbb{N}$ such that $nx > y$. Write this in symbols, negate it and write the negation in words.

Ex. 18. A set $A \subset \mathbb{R}$ is said to be *bounded above* if we can find a real number M such that $x \leq M$ whenever $x \in A$. Formulate this definitions in terms of $\exists$ and $\forall$. Negate this. Write the negation in plain English.

Such an M is said to be an upper bound for A. Complete the following sentence using the quantifiers: A real number C is an upper bound for $A \subset \mathbb{R}$ if ...

When is $\alpha \in \mathbb{R}$ not an upper bound for A?

Formulate analogous questions for a set being bounded below and being bounded.

Ex. 19. Let $f\colon X \to Y$ be map and $y, z \in Y$.

(a) Write in symbols y is in the image of f using $\exists$.

(b) Write in symbols z is not in the image of f using $\forall$.

Ex. 20. Let $f\colon X \to Y$ be a map. Then f is said to be surjective if $(\forall y \in Y)\,[\ldots]$. Complete the sentence in symbols, negate it.

Explain when you say f is not surjective in plain English.

Carry our an analogous exercise for injective maps.

Ex. 21. As you learn analysis, translate the definitions in symbolic notation, negate them and write them down in simple language.

Ex. 22. What is the negation of the statement, "She is bright and beautiful"?

What is the negation of the statement, "He is either dumb or deaf"?

Ex. 23. We say that a nonempty subset $A \subset \mathbb{R}$ is bounded if it is bounded above and bounded below. When is A not bounded? Express this using the quantifiers $\exists$ and $\forall$.

Ex. 24. Negate the sentence and express it using the quantifiers $\exists$ and $\forall$:
"For any man, there is a time when he is both angry and stupid"

2 Sets and Operations on Sets

Ex. 25. Let A, B be sets. The following are equivalent:
(a) $A \subset B$.
(b) $A \cup B = B$.
(c) $A \cap B = A$.

Ex. 26. Express the null set as a subset of $\mathbb{Q}$ in (at least) two different ways.

Ex. 27. Identify the following sets:
(a) $\{x \in \mathbb{R} : \exists y \in \mathbb{R} \text{ such that } y^2 = x\}$. (Express the meaning of this set in words.)
(b) $\{x \in \mathbb{R} : \exists y \in \mathbb{R} \text{ such that } y^{2n} = x \text{ where } n \text{ is a fixed natural number}\}$.
(c) $\{x \in \mathbb{R} : \exists y \in \mathbb{R} \text{ such that } y^{2n+1} = x \text{ where } n > 1 \text{ is a fixed natural number}\}$.
(d) $\{\frac{m}{n} \in \mathbb{Q} : m, n \text{ have the same sign and } n \text{ is a divisor of } m\}$.
(e) $S \subseteq \mathbb{N}$ such that $1 \in S$ and if $k \in S$ then $k + 1 \in S$
(f) $\{x \in \mathbb{Q} : \exists a, b \in \mathbb{Z} \text{ such that } a > 0 \text{ and } ax - b = 0\}$.
(g) $\{x \in \mathbb{Q} : \exists a, b \in \mathbb{Z} \text{ such that } a > 0 \text{ and } x^2 + ax + b = 0\}$.
(h) $\{x \in \mathbb{R} : e^x = 0\}$.

Ex. 28. Let S be a subset of $\mathbb{N}$ with the property that if a positive integer $n \in S$, then $n + 1 \in S$. Describe all such subsets S.

Ex. 29. Prove that $\mathbb{Z}$ is a proper subset of $\mathbb{Q}$.

Ex. 30. Express $\mathbb{Z}$ as a subset of $\mathbb{R}$ using some trigonometric functions.

Ex. 31. Identify the set $\{(x, y) \in \mathbb{R}^2 : xy \neq 0 \text{ and } \frac{x}{y} + \frac{y}{x} \geq 2\}$.

Ex. 32. Let $S := \{(x, y) \in \mathbb{R}^2 : (1 - x)(1 - y) \geq 1 - x - y\}$. Give a simple description of S which involves signs of x and y.

Ex. 33. Describe the following subsets of $\mathbb{R}^2$:
(a) $\{(x, y) \in \mathbb{R}^2 : |x| + x = |y| + y\}$.
(b) $\{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$.
(c) $\{(x, y) \in \mathbb{R}^2 : \max\{|x|, |y|\} \leq 1\}$.

Ex. 34. Describe the sets explicitly:
(a) $A_0 := \{x \in \mathbb{R} : x < 0\}$.
(b) $A_1 := \{x \in \mathbb{R} : x(x - 1) < 0\}$.
(c) $A_2 := \{x \in \mathbb{R} : x(x - 1)(x - 2) < 0\}$.
(d) $A_3 := \{x \in \mathbb{R} : x(x - 1)(x - 2)(x - 3) < 0\}$.

Ex. 35. Let $a_1 < a_2 < \cdots < a_n$ be real numbers. Describe the set

$$\{x \in \mathbb{R} : (x - a_1) \cdots (x - a_n) < 0\}.$$

Ex. 36. Let $A := \{x \in \mathbb{R} : x^2 > x + 6\}$ and $B := \{x \in \mathbb{R} : x > 3\}$. Which of the following is true? (i) $A \subseteq B$, (ii) $B \subseteq A$?

Ex. 37. Express $\mathbb{N}$ as the union of an infinite number of finite sets I_n indexed by $n \in \mathbb{N}$.

Ex. 38. Express $\mathbb{R}$ as the union of an infinite number of intervals J_n of finite length, indexed by $n \in \mathbb{N}$.

Ex. 39. Express $\mathbb{R}$ as the union of an infinite number of intervals J_n of infinite length, indexed by $n \in \mathbb{N}$.

Ex. 40. Express $\mathbb{R}$ as the union of an infinite number of intervals J_x of finite length, indexed by $x \in \mathbb{R}_+$, the set of positive reals.

Ex. 41. Express $\mathbb{R}$ as the union of an infinite number of intervals J_x of infinite length, indexed by $x \in \mathbb{R}_+$, the set of positive reals.

Ex. 42. Express the singleton set $\{0\} = \cap_{n\in\mathbb{N}} J_n$ where J_n is an open (respectively closed) interval.

Ex. 43. Express $[0,1] = \cap_{n\in\mathbb{N}} J_n$ where J_n is an open interval. Can one replace $\cap$ by $\cup$ in this?

Ex. 44. Express $\mathbb{R}^2$, the xy-plane, as the disjoint union of a family of lines indexed by $\mathbb{R}$.

Students have difficulty in the Cartesian product of two sets. Please have the concept reviewed.

Ex. 45. Let $X = \mathbb{R} = Y$,$A = (0,\infty)$ and $B = (-\pi,\pi)$. Draw a picture of $A \times B$ as a subset of $\mathbb{R} \times \mathbb{R}$.

Ex. 46. Let X, Y be as in the last exercise. Let $A = \mathbb{Z}$ and $B = \mathbb{R}$. Describe $A \times B$ in geometric terms.

Describe in geometric terms $\mathbb{R} \times \mathbb{Z}$.

Ex. 47. Let $X = \mathbb{R} = Y$. Draw the pictures of the following subsets $A \times B$.
(a) $A = [-1,1] \times [2,3]$.
(b) $A = (-1,1) \times (2,3)$.
(c) $A = [-1,1) \times (2,3]$.

Ex. 48. Given a map $f\colon X \to Y$, think of a "natural" subset of $X \times Y$ associated with f. *Hint:* Start with the case when $X = \mathbb{R} = Y$.

Ex. 49. Let $A, C \subset X$ and $B, D \subset Y$.
(a) True or false? $A \times B \subset C \times D$ iff $A \subset C$ and $B \subset D$.
(b) True or false? $(A \cap C) \times (B \cap D) = (A \times B) \cap (C \times D)$.

While the construction of subsets of the form $A \times B$ is important, the students should be aware of the fact that there are subsets of $X \times Y$ that are not of the form $A \times B$.

Ex. 50. Show that the circle $\{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$ is not of the form $A \times B$ for any subsets $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$.

3 Functions

Ex. 51. For the function $f\colon X \to Y$ plot the set G of points $(x, f(x))$ on a graph with X along the horizontal axis.
(a) If each horizontal line contains at most one point of G, what type of function is f?
(b) If each vertical line contains at least one point of G, what type of function is f?
(c) Can any vertical line meet G at two points?

Ex. 52. Let $f\colon \mathbb{R} \to \mathbb{R}$ be strictly increasing. Then show that f is injective.

If you remember your Calculus, then conclude e^x, $\log x$ are injective.

How about strictly decreasing functions?

Ex. 53. Find a one-one map from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ and a one-one map from $\mathbb{N}$ to $\mathbb{N} \times \mathbb{N}$.

Ex. 54. Which of the following functions are surjective?
(a) $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = e^x$.
(b) $f\colon (0, \infty) \to \mathbb{R}$ where $f(x) = \log x$.
(c) $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = \sin x$.

Ex. 55. Exhibit a 1-1 correspondence between the set E of even natural numbers and $\mathbb{N}$.

Ex. 56. Exhibit a 1-1 correspondence between the set O of odd natural numbers and $\mathbb{N}$.

Ex. 57. Exhibit a 1-1 correspondence between $\mathbb{N}$ and $\mathbb{Z}$.

Ex. 58. Fid a bijection between $\mathbb{Q}$ and $\mathbb{Z} \times \mathbb{N}$.

Ex. 59. Let A be set of all odd natural numbers and B be set of all even natural numbers. Give examples of functions from A to B which are
(a) 1-1, not onto,
(b) onto, not 1-1,
(c) neither 1-1 nor onto,
(d) 1-1 and onto.

Ex. 60. Find out whether the following functions are 1-1 or onto.
(a) $f : \mathbb{N} \to \mathbb{N}$, $f(n) = n$ if n is odd and $f(n) = 2n$ if n is even.
(b) $f : [0,\ 1] \to [0,\ 1]$, $f(x) = (1 - x)/(1 + x)$.
(c) $f : [0,\ 1] \to [a,\ b]$, $f(x) = bx + (1 - x)a$.
(d) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}(x + |x|)$.
(e) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + x + 1$.
(f) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + x + 1$ if $x \geq 0$ and $f(x) = x + 1$ for $x < 0$.
(g) $f : [0,\ 2\pi) \to D = \{(x,\ y)\ :\ x^2 + y^2 = 1\}$, $f(x) = (\cos x,\ \sin x)$.
(h) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.
(i) $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^3$.
(j) $h\colon \mathbb{R} \to \mathbb{R}$ given by $h(x) = ax^2 + bx + c$, $a \neq 0$.
(k) Let $a, b, c, d \in \mathbb{R}$ be such that $ad - bc \neq 0$. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

Ex. 61. Construct a function $f\colon (0,1) \to \mathbb{R}$ which is 1-1 and onto.

Ex. 62. If f is a function and A and B are subsets of its domain of definition, then prove that $f(A \cap B) \subseteq f(A) \cap f(B)$ and show that proper inclusion can occur.

Ex. 63. Prove that a function $f\colon X \to Y$ is one-to-one iff $f(A \cap B) = f(A) \cap f(B)$ holds for all subsets A and B of X.

Ex. 64. Prove that a function $f\colon X \to Y$ is onto Y iff $f(X \setminus A) \supseteq Y \setminus f(A)$ for all sets $A \subset X$.

Ex. 65. Prove that a function $f\colon X \to Y$ is bijection iff $f(X \setminus A) = Y \setminus f(A)$ for all sets $A \subset X$.

Ex. 66. Let $f\colon X \to Y$ be a map. Then the following are equivalent:
(a) f is injective.
(b) $f(A \cap B) = f(A) \cap f(B)$ holds for all pairs A, B of subsets of X.
(c) For all pairs A, B of subsets of X such that $A \cap B = \emptyset$, we have $f(A) \cap f(B) = \emptyset$.

Ex. 67. Let $f\colon A \to B$ and $g\colon B \to C$ be functions. Then
(a) If $g \circ f$ is one-one (injective), then f is injective.
(b) If $g \circ f$ is onto (surjective), then g is surjective.

Ex. 68. Let $f\colon X \to X$ be any map. If $f \circ f$ is injective, then f is injective.

Ex. 69. Define functions $f, g\colon \mathbb{Z} \to \mathbb{Z}$ such that f is not surjective but $g \circ f$ is.

Ex. 70. Is $f\colon (0,1) \to (0,1)$ given by $f(x) = x^2$ a bijection? How about $g(x) = x^n$ for a fixed $n \in \mathbb{N}$? *Hint:* Analysis is needed!

Ex. 71. The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is neither an injection nor a bijection.

Ex. 72. Show that $f(x) := x^{2k} + x^{2k-1} + \cdots + x + 1$ for $x \in \mathbb{R}$ is not surjective by showing that 0 is not in the image of f. More generally, show that the only $(x, y) \in \mathbb{R}^2$ such that

$$x^{2k} + x^{2k-1}y + \cdots + xy^{2k-1} + y^{2k} = 0$$

is (0,0). *Hint:* What is $(x^n - 1)/(x - 1)$ and hence $(a^n - b^n)/(a - b)$?

Ex. 73. Give an example of a cubic polynomial function on $\mathbb{R}$ which is not injective.

Ex. 74. Show that any cubic polynomial function on $\mathbb{R}$ is surjective. *Hint:* Requires analysis!

Ex. 75. Find a bijection $f\colon [a,b] \to [0,1]$.

Ex. 76. Let $f\colon \mathbb{R} \to (-1,1)$ be given by $f(x) = \frac{x}{\sqrt{1+x^2}}$. Show that f is a bijection.

Ex. 77. Discuss one-one, onto, bijective nature of the following functions. Also, indicate what modifications either on the domain, or on the range or on both is needed to make the function one-one, or onto or bijective. The problem is open ended so that the students can investigate as thoroughly as possible.
(a) $f\colon \mathbb{Q} \to \mathbb{Q}$ given by $f(x) = x^2$.
(b) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.
(c) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$.

(d) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$.

(e) $f\colon \mathbb{C} \to \mathbb{C}$ given by $f(z) = P(z)$ where $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ is a nonconstant polynomial, that is, $a_n \neq 0$ and $n \geq 1$.

(f) $f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$, $a \neq 0$ and $x \in \mathbb{R}$.

(g) Let X be the set of all people on the earth. Let Y be the subset of all men. Let $f\colon X \to Y$ be defined by $f(x)$ to be the father of x for $x \in X$.

Ex. 78. Find a "natural" bijection between the two sets X and Y given below.

(a) X is the set of all lines in $\mathbb{R}^2$ parallel to the x-axis and $Y = \mathbb{R}$.

(b) X is the set of all maps from $\{1, 2\}$ to $\mathbb{R}$ and Y is $\mathbb{R}^2$.

(c) X is the set of all natural numbers that leave 2 as a remainder when divided by 3 and $Y = \mathbb{N}$.

Ex. 79. Let $f, g, h\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \frac{x}{1+x^2}, \quad g(x) := \frac{x^2}{1+x^2}, \quad h(x) := \frac{x^3}{1+x^2}.$$

(a) Determine which of them are injective.

(b) Show that f and g are not surjective. *Hint:* Recall the notion of $\lim_{x\to\pm\infty}$ of a rational function. What is the sign of $g(x)$ for $x \in \mathbb{R}$?

Ex. 80. Let $a, b \in \mathbb{R}$. Consider $f(x) := ax + b$ for $x \in \mathbb{R}$. Under what conditions on a, b

(i) $f\colon \mathbb{R} \to \mathbb{R}$ is an injective map?

(ii) $f\colon \mathbb{Z} \to \mathbb{Z}$ is an injective map?

(iii) $f\colon \mathbb{N} \to \mathbb{N}$ is an injective map?

Ex. 81. If $f\colon \mathbb{N} \to A$ and $g\colon \mathbb{N} \to B$ are surjective, then there exists a surjective map $h\colon \mathbb{N} \to A \cup B$.

Ex. 82. Let $f\colon \mathbb{N} \to X$ be onto. Find a $g\colon 2\mathbb{N} \to X$ which is onto. (Here $2\mathbb{N}$ stands for the set of even integers in $\mathbb{N}$.) Can you replace the set of even integers by the set of odd integers? Any further generalisation?

Ex. 83. Let X be a finite set. Let $\mathcal{A}$ denote the set of all subsets A of X such that $|A|$ is even and $\mathcal{B}$ the set of subsets B of X such that $|B|$ is odd. Find a bijection $\varphi\colon \mathcal{A} \to \mathcal{B}$.

Note that $|\mathcal{A}| = |\mathcal{B}|$ is obvious in view of the identity: $\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$.

Ex. 84. Let X be any nonempty set and let $P(X)$ be its power set, that is, the set of all subsets of X. Give an injective map $f\colon X \to P(X)$.

Ex. 85 (Cantor's Theorem)**.** There exists no surjective map $f\colon X \to P(X)$.

Ex. 86. Let 2^X denote the set of all functions on X into a set of two elements which we will ordinarily take to be $\{0,\ 1\}$. If X is any set then exhibit a one-to-one correspondence between $\mathcal{P}(X)$ and 2^X.

A General Remark: The set of all functions on X into Y, symbolized Y^X, is a subset of $\mathcal{P}(X \times Y)$. Also $Y^{\emptyset} = \{\emptyset\}$ and $\emptyset^X = \emptyset$ if $X \neq \emptyset$.

Ex. 87. Let X, Y be sets. Consider the map $f\colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y)$ given by $f(A, B) = A \times B$. Is this map onto? one-one?

4 Equivalence Classes

Ex. 88. We say that two real numbers $x \sim y$ if $x - y$ is an integer. Show that this is a equivalence relation.

Ex. 89. Define $(x_1, y_1) \sim (x_2, y_2)$ for points of $\mathbb{R}^2$ if $x_1 = x_2$. Show that this is an equivalence relation and give geometric descriptions of the equivalence classes.

Ex. 90. Let $(x_j, y_j) \in \mathbb{R}^2$, $j = 1, 2$. We say $(x_1, y_1) \sim (x_2, y_2)$ if $x_1^2 + y_1^2 = x_2^2 + y_2^2$. Show that this defines an equivalence relation on $\mathbb{R}^2$. Identify the equivalence classes.

Ex. 91. Let $(x_j, y_j) \in \mathbb{R}^2$, $j = 1, 2$. We say $(x_1, y_1) \sim (x_2, y_2)$ if there exists $\alpha \in \mathbb{R}$, $\alpha \neq 0$, such that $x_1 = \alpha x_2$ and $y_1 = \alpha y_2$. Show that this defines an equivalence relation on $\mathbb{R}^2$. Identify the equivalence classes.

Ex. 92. Fix $k \in \mathbb{N}$. Let us say that $m \sim n$ iff $m - n$ is a multiple of k. Show that this defines an equivalence relation. How many equivalence classes are there? Can you find a natural representative in each class?

Ex. 93. For $x, y \in \mathbb{R}$, define $x \sim y$ iff $x - y \in \mathbb{Z}$. Show that this is an equivalence relation.

Ex. 94. Define a relation in $\mathbb{R}^6$ by setting $x \sim y$ iff the coordinates of y obtained from a permuation of coordinates of x. Find the equivalence classes.

Ex. 95. Let p be a prime number. In $\mathbb{R}^p$ define $x \sim y$ iff the coordinates of y are obtained from the coordinates of x by a cyclic permutation. Find the equivalence classes.

Ex. 96. Let X be a set with an equivalence relation $\sim$. Let $[x]$ be an equivalence class. If $a \in [x]$, we say that a is a representative of $[x]$. By a *transversal*, we mean a set $S \subset X$ such that S has precisely one representative from each equivalence class. In other words, for each equivalence class $[x]$, we have $S \cap [x]$ is a singleton.

Find "geometrically nice" transversals for the equivalence relations in Exercises 88– 93. (They are by no means unique!)

Ex. 97. Let $f \colon X \to Y$ be a map. We say that $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. Show that this defines an equivalence relation on X.

Ex. 98. Consider $f, g \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = x + y$ and $g(x, y) = x^2 + y^2$. Find the equivalence classes induced by these functions. (See the last exercise.)

Ex. 99. Define a relation on $\mathbb{N} \times \mathbb{N}$ by setting $(m, n) \sim (r, s)$ if $m + s = n + r$. Show that this is an equivalence relation.

There is a natural way in which each equivalence class can be associated with an integer. This process will set up a bijection between the set of equivalence classes and $\mathbb{Z}$. Can you find this?

Ex. 100. Let $X := \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : n \neq 0\}$. Say that $(m, n) \sim (a, b)$ if $mb = na$. Show that this defines an equivalence relation on X. Let Y be the set of all equivalence classes. Can you set up a bijection from Y to $\mathbb{Q}$?

Ex. 101. Find an explicit expression involving the coordinates for the equivalence relation which induces the partition of $X := \mathbb{R}^2 \setminus$ the coordinate axes.

5 Inverse Images

Most of the students are under the misconception that $f^{-1}(B)$ is the image of B under f^{-1}. That this is a mere notation should be emphasized.

Ex. 102. Let $f\colon X \to Y$ be a constant map $f(x) = y_0$ for all $x \in X$. What is $f^{-1}(B)$ for $B \subset Y$?

Ex. 103. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. What are $f^{-1}(1)$, $f^{-1}([0,1])$, $f^{-1}((0,1))$, $f^{-1}([-1,1])$, $f^{-1}([-4,4])$, $f^{-1}((-4,4))$, $f^{-1}([0,4])$ and $f^{-1}((0,4))$?

Ex. 104. Let $f\colon (0,\infty) \to (0,\infty)$ be given by $f(x) = 1/x$. What are $f^{-1}((0,1))$, and $f^{-1}((1,\infty))$?

Ex. 105. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) := \sum_{k=0}^{n} a_k x^k$. Show that there exists a natural number N such that the number of elements in $f^{-1}(c)$ for any $c \in \mathbb{R}$ is at most N.

Ex. 106. Let $f\colon [-2\pi, 2\pi] \to \mathbb{R}$ be given by $f(x) = \sin x$. Find $f^{-1}([0,1])$.

Ex. 107. Let $f\colon \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \cos x$. Find $f^{-1}(1)$.

Ex. 108. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = x$. What are $f^{-1}(r)$ for $r \in \mathbb{R}$ and $f^{-1}([a,b])$? Draw pictures of these inverse images.

Ex. 109. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = x^2 + y^2$. What are $f^{-1}(r)$ for $r \in \mathbb{R}$ and $f^{-1}([a,b])$? Draw pictures of these inverse images.

Ex. 110. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = xy$. What are $f^{-1}(r)$ for $r \in \mathbb{R}$? Draw pictures of these inverse images.

Ex. 111. Let $f\colon M(n,\mathbb{R}) \to \mathbb{R}$ be given by $f(X) = \det(X)$. Identify the sets $f^{-1}(0)$ and $f^{-1}(\mathbb{R}^*)$, where $\mathbb{R}^*$ denotes the set of nonzero real numbers.

Ex. 112. Let $f\colon M(n,\mathbb{R}) \to M(n,\mathbb{R})$ be given by $f(X) = XX^T$. Identify the sets $f^{-1}(I)$.

Ex. 113. Let $f, g\colon M(n,\mathbb{R}) \to M(n,\mathbb{R})$ be given by $f(X) = X + X^T$ and $g(X) = X - X^T$. Identify the sets $f^{-1}(0)$ and $g^{-1}(0)$.

Ex. 114. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps. Let $C \subset Z$. Show that $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.

Ex. 115. The notion of inverse images is important because of the following facts which show that they behave well with respect to set theoretic operations.

Let $f\colon X \to Y$ be a map. Let B_1, B_2, B be subsets of Y. Prove the following:
(a) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
(b) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
(c) Do (a) and (b) remain true if we deal with arbitrary unions and intersections?
(d) $f^{-1}(B^c) = (f^{-1}(B))^c$, where ^{c} denotes the complement in appropriate sets. In other words, $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$.
(e) If $B_1 \subseteq B_2$, then $f^{-1}(B_1) \subseteq f^{-1}(B_2)$.

Ex. 116. Let $f\colon X \to Y$ be a map. Let $A \subset X$ and $B \subset Y$. Prove the following:
(a) $f(f^{-1}(B)) \subset B$.
(b) $A \subseteq f^{-1}(f(A))$.
(c) f is onto iff $f(f^{-1}(B)) = B$ for all $B \subset Y$.
(d) f is one-one iff $A = f^{-1}(f(A))$ for all $A \subset X$.

Ex. 117. Let $f\colon X \to Y$ be a bijection. Let $B \subseteq Y$. Show that $f^{-1}(B)$ is *the image* of B *under* the map f^{-1}.

Ex. 118. * Let $f\colon M(n,\mathbb{R}) \to M(n,\mathbb{R})$ be given by $f(X) = X^n$. Identify the sets $f^{-1}(0)$.

Ex. 119. * Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and strictly increasing. Assume that $\alpha < \beta$ are in the image of f. What is $f^{-1}([\alpha,\beta])$?

Answer the same question if f is strictly decreasing.

6 Partial Order

1. Definition of partial order, total order.
2. Typical examples.
   (a) Standard $\leq$ relation on $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$.
   (b) In $P(X)$, $A \leq B$ iff $A \subseteq B$. This is a partial order which is not a total order.
   (c) In $\mathbb{N}$, we define $a \preceq b$ iff a divides b. This is a partial order which is not a total order. Students seem to know this example better. (I did not probe much!)
   (d) In $\mathbb{R}^2$, we define $(x_1, y_1) \leq (x_2, y_2)$ iff either $x_1 < x_2$ (and no requirement on y_1 and y_2) or $x_1 = x_2$ and $y_1 \leq y_2$. This is a total order, called dictionary or lexicographic order.
   (e) $\mathbb{C}$ is a totally ordered set. Compare this with what you learnt in complex analysis. Most students have problem here.
3. Definition of an upper bound, lower bound, l.u.b., and g.l.b. Examples in $\mathbb{R}$ and $P(\mathbb{R})$.
4. Definition of minimum, maximum, minimal and maximal elements.

   Maximum and minimum are unique.

   Any maximum is a maximal element but the converse is not true. Analogue for minimum and minimal elements.

   Minimal and maximal elements in $P(\mathbb{R}), P(\mathbb{R}) \setminus \{\emptyset\}, P(\mathbb{R}) \setminus \{\mathbb{R}\}, P(\mathbb{R}) \setminus \{\emptyset, \mathbb{R}\}$ and in $\mathcal{H}_1$, the set of all nontrivial proper subgroups, $\mathcal{H}_2$, the set of all proper subgroups and $\mathcal{H}_3$, the set of all subgroups of the group $(\mathbb{Z}, +)$.

   Find the LUB and GLB of $m\mathbb{Z}$ and $n\mathbb{Z}$ in the partially ordered set $\mathcal{H}$ of all subgroups of $\mathbb{Z}$.
5. Intervals in a totally ordered set. Give a geometric and explicit description of the intervals $[(0,0),(0,1)]$ and $[(0,0),(1,0)]$ and $[(a,b),(c,d)]$ in $\mathbb{R}^2$ with dictionary order.

6. Are there partially ordered sets in which every maximal element is a minimal element and vice-versa?

For more details, consult my article “Patial order, total order and well-ordering”.