Signed Measures
Theorems of Hahn, Jordan and Radon-Nikodym

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Definition 1. Let $(X, \mathcal{B})$ be a measurable space. Let $\mathbb{R}_e$ stand for the extended real number system. A map $\nu\colon (X, \mathcal{B}) \to \mathbb{R}_e$ is said to be a *signed measure* if
(i) ν takes at most one of the values ∞, $-\infty$.
(ii) $\nu(\emptyset) = 0$.
(iii) If $\{E_n : n \in \mathbb{N}\}$ is a countable family of pair-wise disjoint elements of $\mathcal{B}$, then $\nu(\cup_n E_n) = \sum_n \nu(E_n)$.

Example 2.
1. Any (positive) measure is a signed measure.
2. If μ_i, $i = 1, 2$ are (positive) measures on $(X, \mathcal{B})$ and one of them is finite, then $\nu(E) := \mu_1(E) - \mu_2(E)$ is a signed measure. Hahn's theorem says that any signed measure arises this way.
3. Let $(X, \mathcal{B}, \mu)$ be a measure space. Let f be real valued and $f \in L^1(X, \mathcal{B}, \mu)$. Define $\nu(E) := \int_E f\, d\mu$. Then ν is a signed measure. (Verify the details.)
4. If ν is a signed measure, so is $\alpha\nu$ for $0 \neq \alpha \in \mathbb{R}$. In particular, if ν is a signed measure, so is $-\nu$ where $(-\nu)(E) := -\nu(E)$.

We shall assume that our signed measures take at most ∞ and not $-\infty$. In the sequel, whenever we say measure we mean a positive measure.

Remark 3. Note that the condition (iii) in the definition of a signed measure ensures the convergence of $\sum_n |\nu(E_n)|$.

Definition 4. Let ν be a signed measure on $(X, \mathcal{B})$. A subset $P \subset X$ is called a *positive set* (with respect to ν or a ν-positive set) if $\nu(E) \geq 0$ for any measurable subset $E \subset P$.

A negative set is defined in a similar way.

A subset $E \in \mathcal{B}$ is said to be (ν-)-null if $\nu(F) = 0$ for any measurable $F \subseteq E$. It is same as saying that E is a positive set as well as a negative set. Note that if E is null then $\nu(E) = 0$ but not conversely.

Ex. 5. Let ν be as in Example 3. Can you find positive sets in this case?

Ex. 6. A subset E is ν-negative iff it is $-\nu$-positive.

Ex. 7. If ν is a signed measure and $\nu(E) = 0$ does it mean $\nu(F) = 0$ for any $F \subset E$?

Ex. 8. Give an example of a signed measure ν and a set E such that $\nu(E) = 0$ *but* E is not a null set.

Ex. 9. If P is a positive set, then $\mu_P(E) := \nu(P \cap E)$ is a (positive) measure on $(X, \mathcal{B})$. If N is a negative set, what can you say about $\nu_N(E) := -\nu(N \cap E)$?

Ex. 10. Let (E_n) be a sequence of positive sets in a signed measure space. Show that $E := \cup_n E_n$ is a positive set. Is the analogue true for negative sets?

Theorem 11 (Hahn Decomposition). *Let ν be a signed measure on $(X, \mathcal{B})$. Then there exist a positive set P and a negative set N such that $X = P \cup N$ and $P \cap N = \emptyset$.* □

The pair (P, N) is called a Hahn decomposition of ν. It is obvious that it cannot be unique, as we can remove a null set from N and add it to P. Note also that this proves our claim made in Example 3, as $\nu = \nu_P - \nu_N$ with the notation of Ex. 9.

The key step is to construct a largest negative set N and show that its complement is positive.

Proposition 12. *Let ν be a signed measure on $(X, \mathcal{B})$. If $\nu(E) < 0$, then E contains a negative set which is not null.*

Proof. The idea is to find a 'largest positive subset' of E and take its complement.

Either E contains subsets of positive measure or it does not. If it does not, then any subset of E has non-positive measure and hence E itself is a negative set. Then we are done.

So, the nontrivial case is when there exist subsets $F \subset E$ such that $\nu(F) > 0$. Let $p_1 := \text{l.u.b. } \{\nu(F) : F \subset E\}$. Then $p_1 > 0$. (It may happen that $p_1 = \infty$.) By Archimedean property, we know that there exists n such that $np_1 > 1$. We now use well-ordering property of $\mathbb{N}$ to find the least such n. Let $n_1 \in \mathbb{N}$ be the smallest such that $n_1 p_1 > 1$, that is, $1/n_1 < p_1$. By definition of LUB and p_1, there exists $F_1 \subset E$ such that $1/n_1 < \nu(F_1) \leq p_1$.

By our choice of n_1, we note that $1/n_1 < p_1 \leq 1/(n_1 - 1)$.

If $E \setminus F_1$ has no subsets of positive measure, then $E \setminus F_1$ is a negative set and we have

$$\nu(E \setminus F_1) = \nu(E) - \nu(F_1) < 0.$$

In particular, $\nu(E \setminus F_1)$ is not null and we are done. If it is not a negative set, we let

$$p_2 := \text{l.u.b. } \{\nu(F) : F \subset E \setminus F_1\}.$$

As earlier, $0 < p_2 \leq \infty$. Let n_2 be the smallest integer such that $1/n_2 < p_2$. Select $F_2 \subset E \setminus F_1$ such that

$$1/n_2 < \nu(F_2) \leq p_2.$$

We continue this process, Either it stops at a finite stage and so we end up with a non-null negative subset of form $E \setminus (F_1 \cup \ldots \cup F_n)$. In such a situation, the theorem is proved.

Or, we find a disjoint sequence (F_k) such that for all k we have

$$\frac{1}{n_k} < \nu(F_k) \leq p_k \leq \frac{1}{n_k - 1}. \tag{1}$$

Using the countable additivity of ν, we see that

$$\nu(E \setminus \cup_k F_k) + \sum_k \nu(F_k) = \nu(E).$$

Since ν does not take the $-\infty$, the first term on the left, namely, $\nu(E \setminus \cup_k F_k) \neq -\infty$. The term on the right is a finite negative. Hence we conclude that $\sum_k \nu(F_k)$ is convergent. Consequently, the series $\sum_k (1/n_k)$ is convergent and hence $n_k \to \infty$. In view of the last inequality in (1), we deduce that $p_k \to 0$.

Let $F := \cup_k F_k$. We claim that $E \setminus F$ is a negative set. Suppose not. Then there exists $A \subset E \setminus F$ such that $\nu(A) > 0$. Since $p_k \to 0$, there exists $N \in \mathbb{N}$ such that $\nu(A) > p_N$. Since $A \subset E \setminus F \subset E \setminus (F_1 \cup \ldots \cup F_{N-1})$, this violates our choice of p_N. Hence our claim is established.

We need to show that $E \setminus F$ is not null. Observe that

$$\nu(E \setminus F) + \sum_k \nu(F_k) = \nu(E) < 0.$$

Since the second term, the sum of the series is positive, it follows that $\nu(E \setminus F) < 0$. □

Proof. We now prove Hahn's theorem. If $\mathcal{N} := \{E \in \mathcal{B} : \nu(B) < 0\}$ is empty, then X itself is a positive set, that is, ν is a (positive) measure. So we take $P = X$ and $N = \emptyset$.

If $\mathcal{N} \neq \emptyset$, then there exists E with $\nu(E) < 0$ and hence by the last proposition there exist negative subsets.

Let $q := \inf\{\nu(E) : E \text{ is a negative set.}\}$. Note that $q < 0$. Let E_n be such that each E_n is a negative set and $\nu(E_n) \to q$. Let $B_1 := E_1$ and $B_2 = B_1 \cup (E_2 \setminus B_1)$. Then B_2 is negative. Inductively, define $B_n := B_{n-1} \cup (E_n \setminus B_{n-1})$. Then (B_n) is an increasing sequence of negative sets such that $\nu(B_n) \to q$. Let $N := \cup_n B_n$. Then N is negative and $\nu(N) = \lim_n \nu(B_n) = q$.

We claim that $P := X \setminus N$ is a positive set. If not, then we can find a subset of P which is of negative measure and hence a subset A of P which is negative. Note that $N \cup A$ is negative. Since N and A are disjoint, we then conclude that $\nu(N \cup A) = \nu(N) + \nu(A)$. Since ν does not assume the value $-\infty$, in this equation all are finite negative reals and hence we conclude $\nu(N \cup A) < q$, contradicting our definition of q. □

The "uniqueness" part of Hahn decomposition is as follows: if (P_1, N_1) and (P_2, N_2) are two Hahn decompositions, then $P_1 \Delta P_2$ and $N_1 \Delta N_2$ are null sets.

For, $P_1 \setminus P_2 = P_1 \cap N_2$ so that it is both positive and negative. Similarly, $P_2 \setminus P_1$ is a null set. The symmetric difference is the union of these two sets.

We now arrive at the Jordan decomposition of ν. Fix a Hahn decomposition (P, N) of ν. Define

$$\nu^+(E) := \nu(E \cap P) \text{ and } \nu^-(E) := -\nu(N \cap E).$$

Then ν^+ and ν^- are (positive) measures and we have $\nu = \nu^+ - \nu^-$. This is called the Jordan decomposition associated with the Hahn decomposition.

How do Jordan decompositions associated with two Hahn decompositions differ? They do not!

Proposition 13. *The Jordan decompositions associated with two Hahn decompositions of a signed measure ν are the same.*

Proof. Let (P_1, N_1) and (P_2, N_2) be two Hahn decompositions of ν. Observe that

$$\nu(P_1 \cup P_2) = \nu(P_1 \cap P_2) + \nu(P_1 \Delta P_2) = \nu(P_1 \cap P_2),$$

by the uniqueness of Hahn decomposition.

Note that $P_1 \cap P_2$ is ν-positive. For any E, we have

$$\nu(E \cap (P_1 \cap P_2)) \leq \nu(E \cap P_1) \leq \nu(E \cap (P_1 \cup P_2)).$$

Interchanging P_1 and P_2 in the above we get

$$\nu(E \cap (P_1 \cap P_2)) \leq \nu(E \cap P_2) \leq \nu(E \cap (P_1 \cup P_2)).$$

In each of these displayed equations, the first and the third terms are the same. Hence we conclude that $\nu(E \cap P_1) = \nu(E \cap P_2)$ for any E. Since $\nu = \nu^+ - \nu^-$ we also conclude that ν^- corresponding to these Hahn decompositions also coincide. $\square$

Two measures μ and ν on $(X, \mathcal{B})$ are said to be *mutually singular* if there exists a set A such that $\mu(A) = 0 = \nu(X \setminus A)$. One denotes this by $\nu \perp \mu$. The measures ν^+ and ν^- in the Jordan decomposition are mutually singular.

Proposition 14. *If a signed measure ν is expressed as a difference of two measures which are mutually singular, then such measures are unique.*

Proof. The basic idea is to prove that any such expressions arises out of a Hahn decomposition of ν and it is none other than the associated Jordan decomposition.

Let $\nu = \nu_1 - \nu_2$. Assume that $\nu_1 \perp \nu_2$. Assume that A is such that $\nu_1(A) = \nu_2(X \setminus A) = 0$. Let $E \subset A$. Then

$$\nu(E) = \nu_1(E) - \nu_2(E) = -\nu_2(E) < 0. \tag{2}$$

(We used the fact monotonicity of the positive measure ν_1!) Hence A is a ν-negative set . One shows in an analogous way that $B := X \setminus A$ is a ν-positive. Hence (B, A) is a Hahn decomposition of the signed measure ν. We claim that the decomposition $\nu = \nu_1 - \nu_2$ is the Jordan decomposition associated with the Hahn decomposition. For,

$$\nu(E) = \nu(E \cap A) + \nu(E \cap B) = -\nu_2(E \cap A) + \nu_1(E \cap B),$$

by (2) and its analogue. This proves our claim.

In view of Proposition 13, the theorem follows. $\square$

Definition 15. Given two measures μ, ν on a measurable space X, we say that ν is absolutely continuous with respect to μ if $\mu(E) = 0$ implies $\nu(E) = 0$. We demote this by $\nu \ll \mu$.

Example 16. Let f be non-negative and measurable on $(X, \mathcal{B}, \mu)$. Define $\nu(E) := \int_E f\, d\mu$. Then $\nu \ll \mu$. We usually write this as $\frac{d\nu}{d\mu} = f$.

Radon-Nikodym theorem says that this is the only way to get meausres absolutely continuous with respect to a (σ-finite) measure μ.

Example 17. Any measure ν on X is absolutely continuous with respect to a counting measure.

Example 18. Consider the counting measure μ on $\mathbb{N}$. Let ν be a measure which assigns the weight p_n at $n \in \mathbb{N}$, that is, $\nu(E) = \sum_{n\in E} p_n$. Then each of these two measures is absolutely continuous with respect to the other. Identify $\frac{d\nu}{d\mu}$ and $\frac{d\mu}{d\nu}$.

Proposition 19 (Radon-Nikodym for Finite Measures)**.** *Let $(X, \mathcal{B}, \mu)$ be a finite measured space and let ν be a finite measure on $(X, \mathcal{B})$ Assume further that $\nu \ll \mu$. Then there exists a non-negative measurable function f such that $\nu(E) = \int_E f\, d\mu$ for all measurable E.*

Proof. Let $\mathcal{F}(\nu;\mu)$ denote the set of all non-negative measurable functions f such that $\int_E f\, d\mu \le \nu(E)$ for all E. Note that $f = 0$ lies in the set. Let $M :=$ l.u.b. $\{\int_X f\, d\mu : f \in \mathcal{F}(\nu;\mu\}$. Then there exists a sequence (f_n) such that $\int_X f_n\, d\mu \to M$.

We claim that we may assume that the sequence (f_n) is increasing. It suffices to show that if $f, g \in \mathcal{F}(\nu;\mu)$, then $h := \max\{f, g\} \in \mathcal{F}(\nu;\mu)$. Look at the following.

$$\begin{aligned}\int h\, d\mu &= \int_{E\cap\{f\ge g\}} f\, d\mu + \int_{E\cap\{f<g\}} g\, d\mu \\ &\le \nu(E\cap\{f \ge g\}) + \nu(E\cap\{f < g\}) \\ &= \nu(E).\end{aligned}$$

Let $f = \lim f_n$. By MCT, $f \in \mathcal{F}(\nu;\mu)$ and $\int_X f\, d\mu = M$.

We shall show that $\nu(E) = \int_E f\, d\mu$ for all E. Consider $\nu_0(E) = \nu(E) - \int_E f\, d\mu$. Clearly, ν_0 is a measure, $\nu_0 \ll \mu$.

We claim that the class $\mathcal{F}(\nu_0;\mu)$ has only the zero function. For, if $g \in \mathcal{F}(\nu_0;\mu)$, then

$$\int_E g d\mu \le \nu(E) - \int_E f\, d\mu.$$

In particular, $\int_E (f+g)\, d\mu \le \nu(E)$. Hence $f + g \in \mathcal{F}(\nu;\mu)$. If $g > 0$ on a set whose μ-measure is positive, then $\int_X (f+g)\, d\mu > M$, a contradiction.

> **Reason:** Let $A := \{g > 0\}$. Then A is the limit of increasing sequence (A_n) where $A_n := \{g > 1/n\}$. Thus if $\mu(A) > 0$ then there exists $N \in \mathbb{N}$ such that for $n \ge N$, $\mu(A_n) > 0$. Since $g \ge 0$, we obtain
>
> $$\int_X g\, d\mu \ge \int_{A_n} g\, d\mu \ge \frac{1}{n}\mu(A_n) > 0.$$

This implies

$$\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu \geq M + \frac{1}{n}\mu(A_n) > M.$$

Hence $\mathcal{F}(\nu_0;\mu) = \{0\}$.

To complete the proof, we need to show that $\nu_0 = 0$. If not, let $n \in \mathbb{N}$ be such that $\nu_0(X) - \frac{1}{n}\mu(X) > 0$. (This is possible, since μ is a finite measure. Is there any other place where we invoked the finiteness of the measures?) Let $\sigma := \nu_0 - \frac{1}{n}\mu$. Let (P, N) be a Hahn decomposition of σ. Let $g = \frac{1}{n}\chi_P$. We claim that $g \in \mathcal{F}(\nu_0;\mu)$.We have

$$\int_E \frac{1}{n}\chi_P\,d\mu = \frac{1}{n}\mu(P\cap E) = \nu_0(P\cap E) - \sigma(P\cap E) \leq \nu_0(P\cap E) \leq \nu_0(E).$$

The claim is proved. By the Claim in the last para, we conclude that $g = 0$ a.e.$[\mu]$.

Hence $\chi_P = ng = 0$ or $\mu(P) = 0$. Since $\nu_0 \ll \mu$, we see that $\nu_0(P) = 0$. Hence $\sigma(P) = 0$. That is, $\sigma \leq 0$. We conclude that $\nu_0(X) - \frac{1}{n}\mu(X) \leq 0$. This is a contradiction to our assumption on n. □

We now extend the result to σ-finite measures. Let $\{A_n\}$ and $\{B_n\}$ be countable partitions of X such that $\mu(A_n) < \infty$, $\nu(B_n) < \infty$ for $n \in \mathbb{N}$. Then the family $\{A_m \cap B_n : m, n \in \mathbb{N}\}$ is countable partition of X each of whose members is μ as well as ν finite. So, we may assume that we have a common σ-finite partition, say, $\{E_n\}$ for both the measures. Let $\mathcal{B}_n := \{E \cap E_n : E \in \mathcal{B}\}$. It is a σ-algebra on E_n. Let $\mu_n(E) := \mu(E_n \cap E)$, $E \in \mathcal{B}$ defines a finite measure on $\mathcal{B}_n$. ν_n is defined similarly. Then $\nu_n \ll \mu_n$.

By the special case of Radon-Nikodym theorem, there exist functions f_n such that $\nu_n(E) = \int_E f_n\,d\mu_n$ for all E. Note that $\nu_n(X \setminus E_n) = 0$. So, we may assume that $f_n(x) = 0$ for $x \notin E_n$. We define f on X by setting $f(x) = f_n(x)$ if $x \in E_n$. Then f is a nonnegative measurable function on X. Thus we obtain

$$\nu(E\cap E_n) = \int_{E\cap E_n} f\,d\mu, \text{ for all } n \text{ and } E \in \mathcal{B}.$$

Summing over n, we get $\nu(E) = \int_E f\,d\mu$. □