Singular Values and Spectral Theorem

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Ex. 1. Let us consider the vector space $M(n,\mathbb{R})$, the set of all real $n \times n$ matrices as $\mathbb{R}^{n^2}$ via the map

$$A = (a_{ij}) \mapsto (a_{11}, a_{12}, \dots, a_{n1}, \dots a_{nn}).$$

Let $\|A\|^2 = \sum_{ij} a_{ij}^2$. Show that the following maps are continuous:

(a) The map $(A, x) \mapsto Ax$ from $M(n, R) \times \mathbb{R}^n \to \mathbb{R}^n$.(Here $x \in \mathbb{R}^n$ is thought of as a column vector and Ax is then the matrix product.)

(b) The map $(X, Y) \mapsto XY$ from $M(n,\mathbb{R}) \times M(n,\mathbb{R}) \to M(n,\mathbb{R})$.

(c) The map $A \mapsto A^*$ from $M(n,\mathbb{R})$ to $M(n,\mathbb{R})$. (Here A^* stands for the adjoint/transpose of A.)

Proposition 2. *Let $O(k)$ denote the set of all orthogonal matrices of type $k \times k$. If we consider $O(k) \subset R^{n^2}$ via the map then $O(k)$ is closed and bounded and hence is compact.*

Proof. Note that $A^*A = I$ implies that $\sum_{ij} a_{ij}^2 = n$ and hence $\|A\|$ as a vector in $\mathbb{R}^{n^2}$ is $\sqrt{n}$. If $f\colon M(n,\mathbb{R}) \to M(n,\mathbb{R})$ is given by $f(A) = AA^*$, then f is continuous. Now, $O(n)$ is closed since $O(n) := f^{-1}(I)$ where $f(A) = AA^*$. □

Ex. 3. Let $0 \neq v \in \mathbb{R}^k$. Then there exists an $k \times k$ orthogonal matrix A such that whose first row is $v/\|v\|$ so that $Av = (\|v\|, 0, \dots, 0)$.

Theorem 4. *Let A be a nonzero $m \times n$ matrix. Then there exist orthogonal matrices M and N such that*

$$MAN = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \qquad \textit{where} \qquad D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_r \end{pmatrix} \qquad \textit{with } d_i \geq d_{i+1} > 0.$$

Proof. For any $k \times k$ matrix B, we let $a_{ij}(B)$ stand for the ij-th entry of B. Note that $a_{ij} = \langle Be_j, e_i \rangle$, where $\{e_k\}$ is the standard orthonormal basis of $\mathbb{R}^k$.

Let $O(k)$ be the set of all $k \times k$ orthogonal matrices. For $M \in O(m)$ and $N \in O(n)$, let $f(M, N) := a_11(MAN)$. Then $f\colon O(m) \times O(n) \to \mathbb{R}$ is a continuous on the compact set $O(m) \times O(n)$ and hence attains its maximum value, say, d_1. Let $f(M_1, N_1) = d_1$. Let

$$M_1AN_1 = \begin{pmatrix} d_1 & X \\ Y & A' \end{pmatrix}.$$

We claim that X and Y are zero as row and column vectors. For example, if $X \neq 0$, then the length d of the first row R_1 in $M_1 A N_1$ will be greater than d_1. If we multiply $M_1 A N_1$ on the right by an orthogonal $m \times m$ matrix, say M_1' which takes R_1 to $(d, 0, \ldots, 0)$, then $a_{11}(M_1' M_1 A N_1) = d > d_1$, a contradiction. We can argue analogously, using column and left multiplication by an appropriate orthogonal matrix that $Y = 0$.

Note that none of entries $a_{ij}(A')$ can exceed d. For otherwise, we could permute the bases and bring that entry at the $(1,1)$-th place. Now we can apply induction to A'. □

Definition 5. The positive numbers $d_1 \geq \cdots \geq d_r$ are known as the *singular values* of A.

Ex. 6. Show that $A^* A = D^2$. In fact, show that

$$A^* A v_k = \begin{cases} d_k^2 v_k & \text{for } 1 \leq k \leq r \\ 0 & \text{for } r < k \leq n \end{cases}$$

where v_k are the columns of N. *Hint:* Observe that $(MAN)^*(MAN) = N^*(A^*A)N$.

Ex. 7. Prove that the singular values of A are unique. *Hint:* Enough to show that d_k^2 are unique. Observe that if $(A^*A - \lambda I)(\sum_k a_k v_k) = 0$, then $(A^*A - \lambda I)v = \sum a_k(\lambda - d_k^2)v_k = 0$.

Ex. 8 (Polar Decomposition)**.** Let $m = n$. Then $A = PO$, where P is positive semi-definite and $O = M^* N^*$ is orthogonal.

Ex. 9 (Spectral Theorem for Symmetric Matrices)**.** If A is symmetric, then A has an eigen vector. *Hint:* If v_k is one of the columns of N, then

$$0 = (A^*A - d_k^2 I)v_k = (A + \lambda I)(A - \lambda I)v,$$

where $\lambda^2 = d_k^2$.

Ex. 10. Extend the above results to the complex case.