Matrices over a Euclidean Domain
Generators and Relations of a F.G. Abelain Group

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The aim of this article is to show that any $n \times m$ matrix A over a Euclidean domain R can be put in a particularly simple form, called the Smith Normal Form of A. This is achieved in the first section. This canonical form has important applications to the structure of finitely generated modules over a Euclidean domain, especially to the structure of finitely generated Abelian groups. In the second section we apply the result to explicate the structure of a finitely generated abelian group given by a finite set of generators and relations.

1 Smith Normal Form

To arrive at the canonical form, we employ row-operations, column operations which can be "reversed" or "inverted." To be precise, the following operations are called elementary row operations:

(1) Add any multiple of a row to another. We denote this by $R_i + cR_j$ which means to the i-th row add c-times the j-th row, $c \in R$.

(2) Multiply a row by a unit in R. We denote this by uR_j to mean that j-th row is multiplied by a unit $u \in D$.

(3) Interchange any two rows. We denote this by $R_i \leftrightarrow R_j$.

Elementary column operations are defined similarly R's replaced by C's.

Ex. 1. Let d be the greatest common divisor (gcd) of all the entries of A. Let A' be the matrix obtained from A by an elementary (row/column) operation. Show that d is the gcd of all entries of A'.

The main result of the article can now be stated.

Theorem 2 (Smith Normal Form)**.** *Let R be a Euclidean domain. Let A be an $n \times m$ matrix with entries in R. Then by employing elementary row and column operations, we can transform A to a matrix of the form $\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$ where D is a diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_r)$ with*

(i) $d_i \neq 0$ *for* $1 \leq i \leq r$,

(ii) $d_i | d_{i+1}$ *for* $1 \leq i \leq r-1$ *and 0's are zero matrices of appropriate sizes.*

Uniqueness: r is unique and d_i are unique up to associates (that is, multiplication by units in R).

Proof. Note that in view of Ex. 1, the d_1, the first entry of the canonical form must be the gcd of the entries of A.

We shall prove that we obtain B from A by elementary operations such that b_{11} divides all entries of B.

Assume that $a := a_{11}$ does not divide a_{1j} for some $j > 1$. The we write $a_{1j} = qa + r$ where $0 < d(r) < d(a)$. Hence if we employ $C_j - qC_1$, we get r as the $(1j)$-th entry. If we effect $C_1 \leftrightarrow C_j$, we obtain a new matrix whose (11)-th entry, say, b is such that $d(b) < d(a)$.

Assume that a does not divide a_{j1} for some $j > 1$. Now using row operations, we obtain similarly a new matrix whose (11)-th entry, say, b is such that $0 < d(b) < d(a)$.

Thus using these two steps, we arrive at matrix A' such that every element of its first row and first column are divisible by its (11)-th entry, say, a.

We now want to arrive at a matrix such that its (11)-th entry divides all the entries. Assume that a_{ij} is not divisible by a_{11}. Let $a_{1j} = sa_{11}$. By $C_j - (s-1)C_1$, we get the new (ij)-th entry as $a'_{ij} := a_{1j} - (x-1)a_{i1}$. Since a_{ij} is not divisible by a_{11} where a_{1j} is, we conclude that a'_{ij} is not divisible by a_{11}. Hence $a'_{ij} = qa_{11} + r$ with $0 < d(r) < d(a_{11})$. By $C_j \leftrightarrow C_1$ followed by $R_i \leftrightarrow R_1$, we get a new matrix whose 11-th entry has d-values less than $d(a)$.

Observe that in each of the above, after the indicated elementary operations, we end of up with a new matrix whose 11-th element has d-value strictly less than the d-value o the 11-th element of the original matrix. Since d takes values in $\mathbb{N}$, this process cannot go on indefinitely. After a finite number of steps, we end up with a matrix B with a_{11} dividing all other entries.

We now show that from B we obtain matrix such that $b_{1j} = 0 = b_{j1}$ for $j > 1$. If $a_{1j} = sa_{11}$, then $C_j - sa_{11}$ yields a new matrix whose $1j$-th entry is 0. Similarly, we can make any $j1$-th entry also zero,

This we have arrived at a matrix, say, B of the form $B = \begin{pmatrix} d_1 & 0 \\ 0 & C \end{pmatrix}$ where C is matrix of size $(n-1) \times (m-1)$. All entries of C are divisible by d_1.

We now apply induction (on what?) to transform C its canonical form with $d_2 | d_3 | \cdots | d_r$. Note that the elementary operations do not alter the fact that d_1 divides the elements of matrix obtained from C. Hence we get $d_1 | d_2$ also.

Now we wish to prove the uniqueness part. We observe that d_1 must be the gcd of all entries of the original matrix A, as observed earlier. Now the crucial observation is the following: If we consider all the $j \times j$ principal submatrices, then the gcd of their determinants remains the same under elementary operations. Hence at the end we obtain 'the' canonical form in which the gcd of all its $j \times j$ submatrices is obviously $d_1 \cdots d_j$ for $1 \le j \le r$. Since none of $d_1, \ldots, d_{j-1}$ are zero, we conclude that d_j's are unique.

We shall now establish the crucial observation. Fix $1 \le j \le r$. Let d denote the gcd of the determinants of all $j \times j$ submatrices of A. Let A' be the matrix obtained row A by means an elementary operation. and d' denote the gcd of the determinants of all $j \times j$ submatrices of A'. It suffices to show that d and d' are associates. To prove this, it is enough to prove the result for the first the row operations in view of the fact that $\det(X) = \det(X')$ for any square matrix X.

Among the row operations, the interchange of rows can be obtained from the other two row operations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{R_1+R_2} \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} \xrightarrow{R_2-R_1} \begin{pmatrix} a+c & b+d \\ -a & -b \end{pmatrix} \xrightarrow{R_1+R_2} \begin{pmatrix} c & d \\ -a & -b \end{pmatrix} \xrightarrow{-R2} \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

Hence it is enough to prove this for the operations of the addition of rows and multiplication of a row by a unit of R.

If we multiply a row of A by a unit u, then an $i \times i$ submatrix either remains unchanged or one of its rows is multiplied by u. In the first case, the minor, namely, its determinant, is unchanged; in the second case, the minor is changed by a unit. Therefore, every $i \times i$ minor of A is an associate of the corresponding $i \times i$ minor of , and so d and d' are associates, as claimed.

If we employ $R_j + rR_k$, an $i \times i$ submatrix of A either does not involve R_i this row or it does. In the first case, the corresponding minor is unchanged. In the second case, it has the form $M + rM'$ where M and M' are $i \times i$ minors of A. For, det is a multilinear function of the rows of a matrix:

$$\det(S_1, \dots, R_j + rR_k, \dots, S_i) = \det(S_1, \dots, R_j, \dots, S_i) + r \det(S_1, \dots, R_k, \dots, S_i),$$

where S are the rows of the $i \times i$-submatrix.

It follows that $d|M$ and $d|M'$. Hence $d|d'$. Since A' can be obtained from A by reversing the elementary transformation, it follows that $d'|d$. Since R is a domain, we have d and d' are associates. □

Remark 3. The 'diagonal' form of the matrix stated in the theorem is called the Smith normal form (SNF) of A.

The entries d_j of the SNF of A are called the *invariant factors* of A. They are unique up to associates by the uniqueness part of the theorem.

If one goes through the uniqueness part of the proof, one can extract the following lemma. First let us establish a convention and notation. Let R be a commutative ring with 1. Let $A \in M_{m\times n}(R)$. Let $1 \leq i \leq \min\{m, n\}$. By an i-minor, we mean the determinant of an $i \times i$-submatrix of A. Let $J_i(A)$ denote the ideal generated by all the i-minors of A. We say that an elementary row/column operation is admissible, if it is invertible.

Lemma 4. *Let $A \in M_{m\times n}(R)$. Let B be obtained from an elementary row or column operation. Then $J_i(B) \subseteq J_i(A)$. If the operation is admissible, then $J_i(A) = J_i(B)$.*

In particular, if R is a PID, and the operation is admissible, then the generators of the ideals $J_i(A)$ and $J_i(B)$ are associates. □

Remark 5. The proof of the uniqueness part tells us how to "compute" the invariant factors. Go through the example below.

Example 6. Let $R = \mathbb{Z}$. Consider the matrix $A := \begin{pmatrix} 4 & 8 & 12 \\ 6 & 4 & 16 \end{pmatrix}$. The gcd of the entries of A is 2. The determinants of the 2×2 submatrices of A are (up to sign) $\{32, 80, 8\}$ and their gcd is 8. Hence $d_1 d_2 = 8$ so that d_2 must 4. Hence the SNF of A must be $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}$. This may be obtained by the following steps: $R_2 \leftrightarrow R_1$, $R_1 - R_2$, $(C_2 + 2C_1, C_3 + C_2)$, $R_2 - 2R_1$, $C_3 + 2C_2)$, $C_2 + 2C_3$, $C_2 \leftrightarrow C_3$. (This is not the only way of arriving at SNF of A!) You maybe able to arrive at it in less number of steps.

2 Finitely Generated Abelian groups

Let G be a finitely generated abelian group, say, by n-elements x_i, $1 \leq i \leq n$. Then we have an obvious homomorphism

$$f \colon \mathbb{Z}^n \to G \text{ given by } f(m_1, \dots, m_n) = m_1 x_1 + \cdots + m_n x_n.$$

This is onto. If K is the kernel of f, by the first fundamental theorem of homomorphism, we have $\mathbb{Z}^n/K \simeq G$. It is well-known that K is a also finitely generated and free over $\mathbb{Z}$. If we write the generators, say, $y_1, \dots, y_k$ of K in terms of the standard basis of $\mathbb{Z}^n$, we get an $k \times n$ matrix (called the relation matrix) over $\mathbb{Z}$. The Smith normal from of the relation matrix gives us a new basis of $\mathbb{Z}^n$ and a new set of generators of K which are aligned/compatible with each other in the sense that we can read off the quotient $\mathbb{Z}^n/K$ easily and hence identify G in its invariant factor decomposition. We look at some examples in this article.

To understand what follows, it is important to have a good idea of what effect the elementary (column/row) operations have on the basis of $\mathbb{Z}^n$ or on the set of generators of K.

Let us start with a change of basis of $\mathbb{Z}^n$. Let us start with the standard basis $\{e_1, \dots, e_n\}$ of $\mathbb{Z}^n$. Let $\{x_1, \dots, x_m\}$ be a set of generators of K. We claim that the set $\{e_1, \dots, e_i + \alpha e_j, \dots, e_n\}$ is a basis of $\mathbb{Z}^n$. Assume for definiteness sake, $i < j$. Observe

$$\begin{aligned} x_k &= x_{k1} e_1 + \cdots + x_{ki} e_i + \cdots + x_{kj} e_j + \cdots + x_{kn} e_n \\ &= x_{k1} e_1 + \cdots + x_{ki}(e_i + \alpha e_j) + \cdots + (x_{kj} - \alpha x_{ki}) e_j + \cdots + x_{kn} e_n. \end{aligned}$$

That is, the change of the basic element $e_i \mapsto e_i + \alpha e_j$ has the effect $C_j \mapsto C_j - \alpha C_i$. To put in a form which will be useful for us, the elementary column operation $C_i + \alpha C_J$ comes from the change of basis element $e_j \mapsto e_j - \alpha e_i$.

One sees in a similar way, any row operation of the form $R_i \mapsto R_i + \alpha R_j$ comes from the change of the generator $x_i \mapsto x_i + \alpha x_j$.

In the tables below, the first column is about the elementary column/row operation on the relation matrix in the 2nd column of the previous row. The third column of the table indicates the effect of the operation on the basis of $\mathbb{Z}^n$ and the fourth column expresses the set of generators in the new basis of $\mathbb{Z}^n$.

Example 7. Let G be an abelian group generated by two elements a and b satisfying the relations $2a + 4b = 0$ and $-2a + 6b = 0$. The relation matrix is $\begin{pmatrix} 2 & 4 \\ -2 & 6 \end{pmatrix}$. We now show how to reduce it to Smith normal form and find the basis for $\mathbb{Z}^3$ and a set of generators of G which are compatible so that we can identify G.

Elem.Op.	**Matrix**	**Basis**	**Generators**
	$\begin{pmatrix} 2 & 4 \\ -2 & 6 \end{pmatrix}$	a b	$2a + 4b$ $-2a + 6b$
$C_2 - 2C_1$	$\begin{pmatrix} 2 & 0 \\ -2 & 10 \end{pmatrix}$	$a + 2b$ b	$2(a + 2b)$ $-2(a + 2b) + 10b$
$R_2 + R_1$	$\begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix}$	$a + 2b$ b	$2(a + 2b)$ $10b$

Thus, if we take $u = a + 2b$ and $v = b$, they form a basis of $\mathbb{Z}^2$. Relative to this basis, the generators a and b $\mathbb{Z}$-module K satisfy the relations $2u = 0$ and $10b = 0$. Hence the the abelian group is $(\mathbb{Z}u + \mathbb{Z}v)/(\mathbb{Z}2a + \mathbb{Z}10b) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_{10}$.

We shall run a check. Do the elements $2u$ and $10b$ lie in K? Note that $2u = 2a + 4b = 0$ and $10v = 10b = (2a + 4b) + (-2a + 6b) \in K$. Can we write the original set of generators as a $\mathbb{Z}$-linear combination of these new 'generators'? Observe that

$$2a + 4b = 2u \text{ and } -2a + 6b = 10v - u.$$

Also, we can find P and Q such that $PAQ = \begin{pmatrix} 1 & 0 \\ 0 & 17 \end{pmatrix}$. The matrix Q is got from the identity matrix by applying the column operations whereas P got from the identity matrix by applying the row-operations. We find:

$$Q : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{C_2 - 2C_1}{\longrightarrow} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

Similarly,

$$P : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{R_2 + R_1}{\longrightarrow} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

One checks that PAQ is the Smith normal form of $A := \begin{pmatrix} 2 & 4 \\ -2 & 6 \end{pmatrix}$.

Example 8.

Elem.Op.	**Matrix**	**Generators**	**Relations**
	$\begin{pmatrix}2 & 3\\ 1 & -7\end{pmatrix}$	a b	$2a+3b$ $a-7b$
$R_1 \leftrightarrow R_2$	$\begin{pmatrix}1 & -7\\ 2 & \end{pmatrix}$	a b	$a-7b$ $2a+3b$
$C_2 + 7C_1$	$\begin{pmatrix}1 & 0\\ 2 & 17\end{pmatrix}$	$a-7b$ b	$a-7b$ $2a+3b = 2(a-7b)+17b$
$R_2 - 2R_1$	$\begin{pmatrix}1 & 0\\ 0 & 17\end{pmatrix}$	$a-7b$ b	$a-7b$ $17b$

Note that $2a+3b = 2(a-7b)+17b$ and $(2a+3b)-2(a-7b) = 17b$. Thus clearly, the $\mathbb{Z}$-submodule generated by $\{2a+3b, a-7b\}$ is the same as $\{a-7b, 17b\}$.

Also, we can find P and Q such that $PAQ = \begin{pmatrix}1 & 0\\ 0 & 17\end{pmatrix}$. The matrix Q is got from the identity matrix by applying the column operations whereas P got from the identity matrix by applying the row-operations.

$$Q: \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \xrightarrow{C_2+7C_1} \begin{pmatrix}1 & 7\\ 0 & 1\end{pmatrix}$$

Similarly,

$$P: \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \xrightarrow{R_2-2R_1} \begin{pmatrix}0 & 1\\ 1 & -2\end{pmatrix}.$$

One checks that PAQ is the Smith normal from of A, as claimed

Example 9.

Elem.Op.	**Matrix**	**Generators**	**Relations**
	$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$	a b	$2a$ $3b$
$C_1 - C_2$	$\begin{pmatrix} 2 & 0 \\ -3 & 3 \end{pmatrix}$	a $a+b$	$2a$ $3b = 3(a+b) - 3a$
$R_1 + R_2$	$\begin{pmatrix} -1 & 3 \\ -3 & 3 \end{pmatrix}$	$-2a-3b$ $a+b$	$2a+3b$ $3b$
$C_2 + 3C_1$	$\begin{pmatrix} -1 & 0 \\ 3 & -6 \end{pmatrix}$	$-2a-3b$ $a+b$	$2a+3b$ $-6(a+b)$
$R_2 - 3R_1$	$\begin{pmatrix} -1 & 0 \\ 0 & -6 \end{pmatrix}$	$-2a-3b$ $a+b$	$-2a-3b$ $-6(a+b)$
$(-1)R_1$	$\begin{pmatrix} 1 & 0 \\ 0 & -6 \end{pmatrix}$	$-2a-3b$ $a+b$	$-2a-3b$ $6(a+b)$
$(-1)R_2$	$\begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}$	$-2a-3b$ $a+b$	$-2a-3b$ $6(a+b)$

Thus, if we take $u = -2a - 3b$ and $v = a + b$ as a basis of $\mathbb{Z}^2$, the kernel subgroup is generated by u and $6v$. They satisfy the relations $u = 0$ and $6v = 0$ and hence G is $\mathbb{Z}_6$.

Observe that $-2a - 3b$ and $6(ab + b)$ lie in K and we have

$$2(-2a - 3b) + (6a + 6b) = 2a \text{ and } -3(-2a - 3b) - (6a + 6b) = 3b.$$

Thus $-2a - 3b$ and $6(a + b)$ generate K.

The matrices P and Q such that PAQ is the Smith normal from of the given matrix are given by

$$P = \begin{pmatrix} -1 & -1 \\ 3 & 2 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}$$

Example 10.

Elem.Op.	Matrix	Basis	Generators
	$\begin{pmatrix} 2 & 4 & 3 \\ -1 & 2 & 2 \end{pmatrix}$	a b c	$2a+4b+3c$ $-a+2b+2c$
R_1+R_2	$\begin{pmatrix} 1 & 6 & 5 \\ -1 & 2 & 2 \end{pmatrix}$	a b c	$a+6b+5c$ $-a+2b+2c$
R_2+R_1	$\begin{pmatrix} 1 & 6 & 5 \\ 0 & 8 & 7 \end{pmatrix}$	a b c	$a+6b+5c$ $8b+7c$
C_2-C_3	$\begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 7 \end{pmatrix}$	a b $b+c$	$a+b+5(b+c)$ $b+7(b+c)$
C_2-C_1	$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 7 \end{pmatrix}$	$a+b$ b $b+c$	$a+b+5(b+c)$ $b+7(b+c)$
C_3-5C_1	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 7 \end{pmatrix}$	$a+b+5(b+c)$ b $b+c$	$(a+b+5(b+c))$ $b+7(b+c)$
C_3-7C_2	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$a+b+5(b+c)$ $b+7(b+c)$ $b+c$	$a+b+5(b+c)$ $b+7(b+c)$.

The reader may run the checks and find the matrices P and Q are as given below.

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -7 \\ 0 & -1 & 8 \end{pmatrix}$$

Example 11.

Elem.Op.	Matrix	Basis	Generators
	$\begin{pmatrix} 5 & 9 & 5 \\ 2 & 4 & 2 \\ 1 & 1 & -3 \end{pmatrix}$	a b c	$5a+9b+5c$ $2a+4b+2c$ $a+b-3c$
$C_2 - C_1$	$\begin{pmatrix} 5 & 5 & 5 \\ 2 & 2 & 2 \\ 1 & 0 & -3 \end{pmatrix}$	$a+b$ b c	$5(a+b)+4b+5c$ $2(a+b)+2b+2c$ $(a+b)-3c$
$R_1 - 2R_2$	$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & -3 \end{pmatrix}$	$a+b$ b c	$(a+b)+c$ $2(a+b)+2b+2c$ $(a+b)-3c$
$C_3 - C_1$	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 0 & -4 \end{pmatrix}$	$a+b+c$ b c	$a+b+c$ $2(a+b+c)+2b$ $(a+b+c)-4c$
$C_2 - C_1$	$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & -4 \end{pmatrix}$	$a+2b+c$ b c	$(a+2b+c)-b$ $2(a+2b+c)$ $(a+2b+c)-b-4c$
$R_3 - R_1$	$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix}$	$a+2b+c$ b c	$(a+2b+c)-b$ $2(a+2b+c)$ $-4c$
$(-1)R_3$	$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$	$a+2b+c$ b c	$(a+2b+c)-b$ $2(a+2b+c)$ $4c$
$(-1)C_2$	$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$	$a+2b+c$ $-b$ c	$(a+2b+c)+(-b)$ $2(a+2b+c)$ $4c$
$C_1 \leftrightarrow C_1$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$	$-b$ $a+2b+c$ c	$(a+2b+c)+(-b)$ $2(a+2b+c)$ $4c$
$C_2 - C_1$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$	$a+b+c$ $a+2b+c$ c	$a+b+c$ $2(a+2b+c)$ $4c$

We let $u := a+b+c$, $v := a+2b+c$ and $w := c$. Then $\{u, v, w\}$ is a basis of $\mathbb{Z}^3$ and $\{u, 2v, 4w\}$

is a basis of K. Hence we arrive at

$$G \simeq (\mathbb{Z}u \oplus \mathbb{Z}v \oplus \mathbb{Z})/(\mathbb{Z}u \oplus \mathbb{Z}2v \oplus \mathbb{Z}4w) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_4.$$

The matrices P and Q are given below:

$$P = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & -1 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Ex. 12. Let the relation matrix be $\begin{pmatrix} 3 & 5 & 3 \\ 3 & 3 & 5 \\ 7 & 3 & 7 \end{pmatrix}$. Show that the Smith diagonal form of this matrix is the diagonal matrix with entries 1,2, and 26. If the elements of the basis for $\mathbb{Z}^3$ are by a,b and c, find the compatible bases of $\mathbb{Z}^3$ and K. Ans: $3a + 5b$, $a + 2b$ and $c + a + 2b$. Find the matrices P, Q such that PAQ is the Smith normal form of the relation matrix.

A sequence of the elementary operations is $C_2 - C_1$, $C_1 - C_2$, $C_2 - 2C_1$, $C_3 - 3C_1$, $R_2 - 3R_1$, $R_3 - 11R_1$,$-R_2$, $-R_3$, $C_2 - C_3$ and $C_3 - 2C_2$.

Relative to these operations,we have

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -1 & 0 \\ 11 & 0 & -1 \end{pmatrix} \text{ and } Q = \begin{pmatrix} 2 & 1 & -8 \\ -1 & 0 & 3 \\ 0 & -1 & 3 \end{pmatrix}.$$

Ex. 13. For $n = 5$, consider the matrix

$$P_5 := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}.$$

Do you observe the pattern of Pascal's triangle? Write down P_6. For each $n \in \mathbb{N}$, we can form P_n. If P_n is th relations matrix of a finitely generated abelian group G_n, identify each G_n.

Ex. 14. Let the relations matrix be $\begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$. Let the basis of $\mathbb{Z}^3$ be a, b, c. Show that after elementary column/row operations,

$$C_2 - 2C_1,\ C_3 - 3C_1,\ R_2 - 4R_1,\ R_3 - 7R_1,\ R_3 - 2R_2, C_3 - 2C_2,\ -R_2$$

we end up with $\{a + 2b + 3c, b + 2c, c\}$ as a basis of $\mathbb{Z}^3$ and $\{2(a + 2b + 3c), 6(b + 2c)\}$ as a basis of K. The Smith normal form is the diagonal matrix with diagonal entries $2, 6, 0$. The group G is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_6$. Compute the matrices P and Q so that PAQ is the Smith normal form of the given matrix.

The details are worked out in the next example.

Example 15.

Elem.Op.	Matrix	Basis	Generators
	$\begin{pmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$	a b c	$2a+4b+6c$ $8a+10b+12c$ $14a+16b+18c$
$C_2 - 2C_1$	$\begin{pmatrix} 2 & 0 & 6 \\ 8 & -6 & 12 \\ 14 & -12 & 18 \end{pmatrix}$	$a+2b$ b c	$2a+4b+6c$ $8a+10b+12c$ $14a+16b+18c$
$C_3 - 3C_1$	$\begin{pmatrix} 2 & 0 & 0 \\ 8 & -6 & -12 \\ 14 & -12 & -24 \end{pmatrix}$	$a+2b+3c$ b c	$2a+4b+6c$ $8a+10b+12c$ $14a+16b+18c$
$R_2 - 4R_1$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -6 & -12 \\ 14 & -12 & -24 \end{pmatrix}$	$a+2b+3c$ b c	$2a+4b+6c$ $-6b-12c$ $14a+16b+18c$
$R_3 - 7R_1$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & -12 & -24 \end{pmatrix}$	$a+2b+3c$ b c	$2a+4b+6c$ $-6b-12c$ $-12b-24c$
$R_3 - 2R_2$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -6 & -12 \\ 0 & 0 & 0 \end{pmatrix}$	$a+2b+3c$ b c	$2a+4b+6c$ $-6b-12c$ 0
$C_3 - 2C_2$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$a+2b+3c$ $b+2c$ c	$2(a+2b+3c)$ $-6(b+2c)$ 0
$-R_2$	$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$a+2b+3c$ $b+2c$ c	$2(a+2b+3c)$ $6(b+2c)$ 0

Run the usual checks and write down P and Q.