Finite Dimensional Spectral Theory

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1 Spectral Theorem

We assume that the reader knows the basics about real and complex (finite dimensional) inner product spaces: definition, Cauchy-Schwartz inequality, Riesz representation theorem, existence of an orthogonal complement of a (closed) subspace and that of an orthonormal basis, adjoint A^* of an operator A etc. In the sequel we let $\mathbf{E}$ denote a real (resp. complex) (finite dimensional) vector space equipped with a real (resp. complex) inner product $\langle , \rangle$. We let the *norm* of an element $x \in \mathbf{E}$ be defined by $\| x \| := \langle x, x \rangle^{1/2}$.

A linear map (or an operator) $A : \mathbf{E} \to \mathbf{E}$ is automatically continuous. (See Ex. 11.) An operator A on $\mathbf{E}$ is said to be *self adjoint* or *hermitian* if the following identity holds:

$$\langle Ax, y \rangle = \langle x, Ay \rangle \qquad \text{for all } x, y \in \mathbf{E}.$$

In case $\mathbf{E}$ is real, then such an A is usually called *symmetric*. It is easy to see that A is self-adjoint iff $\langle Ae_i, e_j \rangle = \langle e_i, Ae_j \rangle$ for any orthonormal basis $\{e_i\}$ of $\mathbf{E}$. Since the (ij)-th entry of the matrix representation of A with respect to an orthonormal basis $\{e_i\}$ is given by $a_{ji} := \langle Ae_i, e_j \rangle$, A is hermitian iff the matrix (a_{ji}) is so. It should, however, be noted that to define hermitian-ness or symmetry of operators we need an inner product. (See Ex. 14.)

We have a natural norm on the space of operators on $\mathbf{E}$:

$$\| T \| := \sup_{\{x \in \mathbf{E} : \| x \| = 1\}} \| Tx \| .$$

It is easy to see that $T \mapsto \| T \|$ is a norm on $BL(\mathbf{E})$, the vector space of (bounded) linear operators on $\mathbf{E}$ and that $\| ST \| \leq \| S \| \, \| T \|$ for all $S, T \in BL(\mathbf{E})$.

To understand the basic ideas of the proof of results concerning self-adjoint (hermitian, symmetric) operators we suggest that the reader assumes that the space under question is real and the operator is symmetric.

We say a complex number λ is in the *spectrum* of an operator $A : \mathbf{E} \to \mathbf{E}$ iff $(A - \lambda I)^{-1}$ does not exist. Since our spaces are finite dimensional this is equivalent to saying that there exists a non-zero vector $v \in \mathbf{E}$ such that $Av = \lambda v$. In this case we say that λ is an *eigen value* and v an *eigen vector* of A.

Lemma 1. *The eigen values of a self-adjoint operator are real.*

Proof. Let v be a non-zero eigen vector of A with eigen value λ. We may assume that $\| v \| = 1$. Then $\lambda = \lambda \langle v, v \rangle = \langle Av, v \rangle = \langle v, Av \rangle = \langle v, \lambda v \rangle = \overline{\lambda} \langle v, v \rangle = \overline{\lambda}$. □

Lemma 2. *For a self-adjoint operator A the eigen vectors v_i with eigen values λ_i for $\lambda_1 \neq \lambda_2$ are orthogonal to each other.*

Proof. $\lambda_1 \langle v_1, v_2 \rangle = \langle Av_1, v_2 \rangle = \langle v_1, Av_2 \rangle = \lambda_2 \langle v_1, v_2 \rangle$. □

If $\mathbf{E}$ is finite dimensional vector space over $\mathbb{C}$, the existence of eigen-values follows from the fundamental theorem of algebra: For, the polynomial function $t \mapsto \det(tI - A)$ on $\mathbb{C}$ has a zero, say, λ in $\mathbb{C}$. But then it follows from elementary linear algebra that there exists a nontrivial linear relation among the rows (or columns) of the determinant $\det(\lambda I - A)$. The coefficients of the linear relation gives us the required eigen vector. We however prefer to give the following analytical proof of the first version of the spectral theorem for self-adjoint operators on $\mathbf{E}$.

Theorem 3 (First Version)**.** *Let $\mathbf{E}$ be a finite dimensional complex vector space with a complex inner product. Let A be a self-adjoint operator on $\mathbf{E}$. Then A has an eigen-value.*

Proof. We first prove the result in the case when $\mathbf{E}$ is real and A is symmetric. The reason for this is that in this case our proof has a simple geometric interpretation. What we are going to do is to look for the minor axis of the "ellipse" $\{\langle Ax, x \rangle = 1\}$. (See Ex. 36.) Later we indicate how the same proof yields the general case.

Let $S := \{x \in \mathbf{E} : \| x \| = 1\}$ be the unit sphere in $\mathbf{E}$. Since $\mathbf{E}$ is finite dimensional, S is compact as S is closed and bounded. (See Ex. 10.)

We consider the function $f(x) := \langle Ax, x \rangle$ on S. Then since A is continuous and the inner product is continuous, f is a real valued continuous function on the compact set S. Hence it assumes a minimum, say, λ on S, at $x_0 \in S$.

Claim 1: λ is an eigen value and x_0 is an eigen vector.

This follows from

Claim 2: $\langle Ax_0, y \rangle = 0$ for all $y \in \mathbf{E}$ with $y \perp x_0$.

Claim 2 $\Rightarrow$ *Claim 1*: *Claim 2* means that Ax_0 must lie in the one dimensional space spanned by x_0, i.e., $Ax_0 = \mu x_0$ for some scalar μ. But this scalar μ must be λ: $\mu = \langle Ax_0, x_0 \rangle = \lambda$. Hence $Ax_0 = \lambda x_0$. Thus *Claim 1* and hence the theorem is proved.

We now prove *Claim 2*: The idea of the proof is simple. We consider a curve $g : \mathbb{R} \to S$ such that $g(0) = x_0$ and consider the one variable function $t \mapsto f(g(t))$. Since this function attains a minimum at $t = 0$, it derivative must be 0 at that point. Computing the derivative gives the result.

Now to get to work, let $y \in \mathbf{E}$ be such that $\langle x_0, y \rangle = 0$. Let $x(t) := x_0 + ty$. Then $\| x(t) \|^2 = 1 + t^2 \| y \|^2$. Let $u(t) := (1 + t^2 \| y \|^2)^{-1/2}(x_0 + ty)$. Then clearly $u(t) \in S$ for all

$t \in \mathbb{R}$. Consider the function $h : t \mapsto \langle Au(t), u(t)\rangle$. By our assumption on x_0, this function attains a minimum at $t = 0$ and hence $h'(0) = 0$. We compute the derivative of h:

$$\begin{aligned}
h'(t)\,|_{t=0} &= \frac{d}{dt}\langle Au(t), u(t)\rangle\,|_{t=0} \\
&= \frac{d}{dt}((1+t^2\,\|y\|^2)^{-1}\langle A(x_0+ty), x_0+ty\rangle)\,|_{t=0} \\
&= \frac{d}{dt}((1+t^2\,\|y\|^2)^{-1})\,|_{t=0}\,(\langle A(x_0+ty), x_0+ty\rangle)\,|_{t=0} \\
&\quad +(1+t^2\,\|y\|^2)^{-1}\,|_{t=0}\,\frac{d}{dt}(\langle A(x_0+ty), x_0+ty\rangle)\,|_{t=0} \\
&= (-(1+t^2\,\|y\|^2)^{-2}2t\,\|y\|^2)\,|_{t=0}\,\lambda \\
&\quad +\frac{d}{dt}(\langle Ax_0, x_0\rangle + t\langle Ax_0, y\rangle + t\langle Ay, x_0\rangle + t^2\langle Ay, y\rangle)\,|_{t=0} \\
&= 0 + \langle Ax_0, y\rangle + \langle Ay, x_0\rangle\,.
\end{aligned}$$

Since A is symmetric, the last term on the right side is $2\langle Ax_0, y\rangle$. Hence $h'(0) = 0$ iff $2\langle Ax_0, y\rangle = 0$. This completes the proof of *Claim 2*.

We may also consider another curve (in place of $x(t)$ above) which arises more geometrically as follows: Let $x_0 \in S$ be as above. Let $y \in S$ with $x \perp y$. Then, x_0 and y span a 2-dimensional vector subspace (a plane through the origin) which intersects the sphere S along a great circle. This curve on S is nothing other than the unit circle on the plane $\mathbb{R}x_0 + \mathbb{R}y$. Since $\|x_0\| = 1 = \|y\|$ and $\langle x_0, y\rangle = 0$, this curve is given by

$$c(t) = \cos t x_0 + \sin t y.$$

(We invite the reader to check that $c(t) \in S$.) Proceeding as earlier, we again get the result $\langle Ax_0, y\rangle = 0$.

Now how far does the above proof go in the case of a complex inner product space? The first thing to notice is that the function h is real valued, since A is self-adjoint:

$$\langle Ax, x\rangle = \langle x, Ax\rangle = \overline{\langle Ax, x\rangle}.$$

If you now go through the proof carefully, you will see up to the above computation of $h'(0)$ we are fine. Only finishing touches need to be done with a little finesse:

$$\begin{aligned}
h'(0) &= \langle Ax_0, y\rangle + \langle Ay, x_0\rangle \\
&= \langle Ax_0, y\rangle + \overline{\langle Ax_0, y\rangle}
\end{aligned}$$

so that $h'(0) = 0$ iff $Re\,\langle Ax_0, y\rangle = 0$. To show that $Im\,\langle Ax_0, y\rangle = 0$, we replace y with iy and proceed. We get

$$\begin{aligned}
h'(0) &= \langle Ax_0, iy\rangle + \langle A(iy), x_0\rangle \\
&= -i(\langle Ax_0, y\rangle - \overline{\langle Ax_0, y\rangle}) \\
&= -2iIm\,\langle Ax_0, y\rangle\,.
\end{aligned}$$

Hence we get $\langle Ax_0, y\rangle = 0$ for all $y \perp x_0$.

I could have spared you the agony of these computations as below: We keep the above notation, but now allow $t \in \mathbb{C}$. We have

$$\langle A(x_0 + ty), x_0 + ty \rangle \geq \lambda \langle x_0 + ty, x_0 + ty \rangle .$$

This implies

$$\langle Ax_0, x_0 \rangle + \bar{t} \langle Ax_0, y \rangle + t \langle Ay, x_0 \rangle + |t|^2 \langle Ay, y \rangle \geq \lambda(1 + |t|^2 \| y \|^2).$$

Hence we get

$$\bar{t} \langle Ax_0, y \rangle + \overline{\bar{t} \langle Ax_0, y \rangle} \geq |t|^2 (\lambda \langle y, y \rangle - \langle Ay, y \rangle)$$

whence

$$2Re(\bar{t} \langle Ax_0, y \rangle) \geq |t|^2 (\eta)$$

where $\eta \leq 0$ for all $t \in \mathbb{C}$. The above inequality can hold for all $t \in \mathbb{C}$ iff $\langle Ax_0, y \rangle = 0$. (Justify this!) Hence *Claim 2* is proved. □

We indicate another proof of the above theorem.

Proof. We now use the operator norm introduced above. Let $\lambda := \| A \| := \sup_{x \in S} \| Ax \|$. Let $x_k \in S$ be such that $\| Ax_k \| \to \lambda$. Since S is compact there exists a subsequence of x_k which we again denote by x_k, converging to some point $x_0 \in S$. Since the norm function is continuous (see Ex. 9), $\| Ax_0 \| = \lambda$. We claim that $\| A^2 \| \geq \| A \|^2 = \lambda^2$:

$$\lambda^2 = \| Ax_0 \|^2 = \langle Ax_0, Ax_0 \rangle = \langle A^2 x_0, x_0 \rangle \leq \| A^2 x_0 \| \| x_0 \| \leq \| A^2 \| \| x_0 \| \| x_0 \| = \| A^2 \| .$$

Since $\| AB \| \leq \| A \| \| B \|$ always, the equality holds everywhere. In particular, $\langle A^2 x_0, x_0 \rangle = \| A^2 x_0 \| \| x_0 \|$. But the equality part of the Cauchy-Schwartz inequality says that this can happen iff $A^2 x_0 = \mu x_0$ for some $\mu \in \mathbb{C}$. Clearly, $\mu = \lambda^2$:

$$\lambda^2 = \langle A^2 x_0, x_0 \rangle = \langle \mu x_0, x_0 \rangle = \mu.$$

Hence we see that $(A^2 - \lambda^2)x_0 = 0$. Hence either $(A - \lambda)x_0 = 0$ in which case λ is an eigen value with x_0 as an eigen vector or $(A + \lambda)v = 0$ where $v := (A - \lambda)x_0 \neq 0$ in which case $-\lambda$ is an eigen value. □

Theorem 4 (Second Version)**.** *Let the notation be as in Thm. 3. Then there exist eigen values $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ of A and eigen subspaces $V_j := \{x \in \mathbf{E} : Ax = \lambda_j x\}$ such that $\mathbf{E} = \oplus_{j=1}^n V_j$ is a direct sum of eigen subspaces of A. (Here $n = \dim \mathbf{E}$.)*

Proof. Since A is self-adjoint, by the previous theorem, there exists an eigen value λ_1 and an eigen vector v_1. We let $V_1 := \mathbb{C}v_1$. We let $W := V_1^{\perp}$, the orthogonal complement of V_1 in $\mathbf{E}$. Then A maps W to itself, i.e., A leaves W invariant, since A is self-adjoint:

$$\langle Aw, v \rangle = \langle w, Av \rangle = \langle w, \lambda_1 v \rangle = 0, \quad \text{for } v \in V_1, w \in W.$$

Hence $\mathbf{E} = V_1 \oplus W$. It is easy to see that $A : W \to W$ is self-adjoint so that induction establishes the result. □

Theorem 5 (Third Version)**.** *Let $A : \mathbf{E} \to \mathbf{E}$ be self-adjoint. Then there exists an orthonormal basis consisting of eigen vectors of A.* $\square$

Let T be the transition matrix which takes the given orthonormal basis e_i of $\mathbf{E}$ to v_i: $Te_i = v_i$. Then T is orthogonal:

$$\langle T^*Te_i, e_j\rangle = \langle Te_i, Te_j\rangle = \langle v_i, v_j\rangle = \delta_{ij}.$$

We then have

$$T^{-1}AT(e_i) = T^{-1}Av_i = T^{-1}\lambda_i v_i = \lambda_i T^{-1}v_i = \lambda_i e_i.$$

We have thus proved the fourth version of the spectral theorem:

Theorem 6 (Fourth Version)**.** *Given a self-adjoint matrix A on $\mathbf{E}$ with respect to an orthogonal basis, there exists an orthogonal matrix T such that $T^{-1}AT$ is diagonal.* $\square$

What we now do is to group all equal eigen values together and prove a "natural" spectral decomposition of the operator A.

Theorem 7 (Fifth Version)**.** *Let A be a self-adjoint operator on $\mathbf{E}$. Then there exist distinct eigen values $a_1 < a_2 \cdots < a_k$ of A and eigen subspaces $W_j := \{x \in \mathbf{E} : Ax = a_j x\}$. Let E_j denote the (orthogonal) projection of $\mathbf{E}$ onto the subspace W_j. Then the following hold:*
1) $\mathbf{E}$ is an orthogonal direct sum of W_j: $\mathbf{E} := \oplus_{j=1}^k W_j$ and $W_j \perp W_k$ for $j \neq k$.
2) $E_j E_l = \delta_{jl} E_j$ and $I = \sum_{j=1}^k E_j$.
3) $A = \sum_{j=1}^k a_j E_j$.

Proof. Since $\mathbf{E} = \sum W_j$ is an orthogonal direct sum, it is clear that $E_j E_l = \delta_{jl} E_j$. The rest of the claims are obvious. $\square$

Question: In what sense the above decomposition is more natural than the one given in Thm. 4? Hint: If $\dim W_j > 1$, then there is no natural way of writing it as an orthogonal direct sum of one dimensional eigen subspaces.

The above data $\{(a_j, W_j, E_j)\}$ satisfying the conditions 1)–3) of the theorem is called the *spectral decomposition* of A and $\sum_j E_j = I$ is called the *resolution of the identity*.

An important question arises now. Is the spectral decomposition unique? It is unique but for a re-indexing. This can be directly and easily seen and we leave it as an exercise. However we address ourselves with a different question, viz., whether it is possible to get hold of the projections from the knowledge of the spectrum.

First of all, we note that we have $A^n = \sum_j a_j^n E_j$ for any $n \geq 1$. Thus given any polynomial $p(z) := \sum_{i=0}^m c_i z^i$ of 1-variable we can associate an operator $p(A) := \sum_{i=0}^m c_i \sum_j a_j^i E_j = \sum_j p(a_j)E_j$. There are some very important polynomials that allow us to recover the projections E_j. Recall that if $\{\lambda_j : 1 \leq j \leq k\}$ is set of k distinct complex numbers then the i-th Lagrange polynomial p_i is given by

$$p_i(z) := \frac{\prod_{j=1\&j\neq i}^k (z - \lambda_j)}{\prod_{j\neq i}(\lambda_i - \lambda_j)}.$$

p_i have the property that $p_i(\lambda_j) = \delta_{ij}$. Let p_j be the j-th Lagrange polynomial associated with the distinct eigen values a_j of A. We then have $p_j(A) = E_j$ for $1 \leq j \leq k$. We shall use this observation below.

We say $T \in BL(\mathbf{E})$ is *normal* if $TT^* = T^*T$. If $T \in BL(\mathbf{E})$ is any operator which admits a spectral resolution then T must be normal. This can be seen as follows: By Ex. 19, λ is an eigen value of T iff $\overline{\lambda}$ is an eigen value of T^*. Hence, if $T = \sum_j \alpha_j E_j$ is a spectral decomposition of T then $T^* = \sum_j \overline{\alpha_j} E_j$, since E_j are self-adjoint. Hence we have

$$TT^* = \sum_{j,l} \alpha_j \overline{\alpha_l} E_j E_l = \sum_j |\alpha_j|^2 E_j = T^*T,$$

since E_j are "orthogonal" to each other. Our final version of the spectral theorem says that for any normal operator T we have a spectral decomposition:

Theorem 8 (General Version)**.** *Let $T \in BL(\mathbf{E})$ be normal. Then there exists a spectral resolution of T.*

Proof. Let $T = A + iB$ where $A := (T + T^*)/2$ and $B := (T - T^*)/2i$. Then A and B are self-adjoint. Since T is normal, A and B commute. (Verify!) Let $A = \sum_j \alpha_j E_j$ and $B = \sum_k \beta_k P_k$ be the spectral resolutions. Here P_k denotes orthogonal projection onto eigen subspaces of B. Since A and B commute and since E_j (resp. P_k) is a polynomial in A (resp. B), we see that E_j and P_k commute for all j and k. Hence by Ex. 25, $R_{jk} := E_j P_k$ is an orthogonal projection onto the subspace $E_j(\mathbf{E}) \cap P_k(\mathbf{E})$. Thus we have the spectral decomposition:

$$\begin{aligned} T = A + iB &= \sum_j \alpha_j E_j + i \sum_k \beta_k P_k \\ &= (\sum_j \alpha_j E_j)(\sum_k P_k) + i(\sum_k \beta_k P_k)(\sum_j E_j) \\ &= \sum_{j,k} a_j R_{jk} + i \sum_{j,k} \beta_k R_{jk} \\ &= \sum_{j,k} (\alpha_j + i\beta_k) R_{jk}. \end{aligned}$$

It is easy to check that for $v \in E_j(\mathbf{E}) \cap P_k(\mathbf{E})$ we have $R_{jk} v = (A + iB)v = (\alpha_j + i\beta_k)v$. (Note however that $R_{jk} = 0$ possibly for many (j,k).) $\square$

Let $\Lambda := \{\lambda_1, \ldots, \lambda_k\}$ be the set of distinct eigen values of $T \in BL(\mathbf{E})$, $\mathbf{E}$ a complex finite dimensional inner product space. For any $f : \Lambda \to \mathbb{C}$ we define $f(T)$ be setting $f(T) := \sum_j f(\lambda_j) E_j$. Note that this coincides with our earlier definition if f is a polynomial function on $\mathbb{C}$ restricted to Λ. Let $\mathcal{F}$ denote the set of operators obtained this way and $\mathcal{P}$ obtained by means of polynomials. Then we claim that $\mathcal{F} = \mathcal{P}$. For, let f be any function on Λ. Let $a_j := f(\lambda_j)$. Consider the polynomial $p(z) := \sum_j a_j p_j$ where p_j is the Lagrange polynomial of Λ. Then we have $p(T) = \sum_j a_j p_j(T) = \sum_j f(\lambda_j) E_j = f(T)$. Question: What do we get for the function $f(z) = \bar{z}$? (See Ex. 35.)

2 Spectral Decomposition for Orthogonal Operators

Let $\mathbf{E}$ be a real inner product space, $T : \mathbf{E} \to \mathbf{E}$ an orthogonal operator. The following steps will lead to the spectral theorem for T.

1) Let $A := T + T^{-1} = T + T^*$. Then A is symmetric and let $\mathbf{E} := \sum_i V_i$ be the spectral decomposition of A into distinct eigen spaces as in Thm. 8. Then T leaves each V_i invariant, as $AT = TA$. We concentrate on one such eigen space V with eigen value λ. Then we have $T^2 v - \lambda T v + I v = 0$.

2) If $\lambda = \pm 2$, then T acts as $\pm I$ on V.

3) If $\lambda \neq \pm 2$, then $W := \mathbb{R}v + \mathbb{R}(Tv)$ is a two-dimensional subspace and $V = W \oplus W^\perp$. Also, $TW \subset W$, $TW^\perp \subset W^\perp$.

4) On any two dimensional space, an orthogonal operator has a matrix representation with respect to an orthonormal basis: $k(\theta) := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in \mathbb{R}$.

5) Putting all these things together, we see that there exists an orthonormal basis with respect to which T can be represented as follows:

$$\left(\pm 1, \cdots, \pm 1, k(\theta_1), \cdots, k(\theta_r)\right).$$

3 Exercises

3.1 Problem Set I

Ex. 9. The norm function on any normed linear space is continuous.

Ex. 10. Show that in a finite dimensional normed linear space the unit ball $B := \{x \in \mathbf{E} : \|x\| \leq 1\}$ is compact.

Ex. 11. Any linear operator between finite dimensional normed linear spaces is continuous.

Ex. 12. Let $M(n, \mathbb{R})$ be the n^2-dimensional vector space over $\mathbb{R}$ consisting of all $n \times n$ matrices with real entries. Show that $\operatorname{tr}(A) := \sum_{i=1}^n \langle Ae_i, e_i\rangle$ where e_i is an orthonormal basis. Show that the trace functional $A \mapsto \operatorname{tr}(A)$ is independent of the orthonormal basis. Prove that i) $\operatorname{tr}(A) = \operatorname{tr}(A^*)$, ii) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, iii) $\operatorname{tr}(ATA^{-1}) = \operatorname{tr}(T)$. Show that $\langle A, B\rangle := \operatorname{tr}(AB^*)$ defines an inner product on $M(n, \mathbb{R})$.

Ex. 13. Let $T \in BL(\mathbf{E})$. Using Riesz, show that there exists a unique $T^* \in BL(\mathbf{E})$ such that $\langle Tx, y\rangle = \langle x, T^* y\rangle$ for all $x, y \in \mathbf{E}$. Show that i) $\|T^*\| = \|T\|$, ii) $\|TT^*\| = \|T\|^2$, iii) $(aA + bB)^* = \bar{a}A^* + \bar{b}B^*$ for $A, B \in BL(\mathbf{E})$ and scalars a, b, iv) $(AB)^* = B^*A^*$, v) $(T^*)^* = T$, vi) $R(T)^\perp = N(T^*)$ and $N(T)^\perp = \overline{R(T^*)}$, where $N(A)$ and $R(A)$ denote the null and range spaces of A. vii) If V is a vector subspace of $\mathbf{E}$ such that $TV \subset V$, then $T^*(V^\perp) \subset V^\perp$.

Ex. 14. Show that the linear map A from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $e_1 \mapsto 2e_1 + 3e_2$ and $e_2 \mapsto 3e_1 + e_2$ has matrix representation $\begin{pmatrix} 2 & 3 \\ 3 & 1 \end{pmatrix}$ with respect to the natural basis $e_1 := (1, 0)$ and $e_2 :=$

$(0,1)$. However with respect to the basis $v_1 := e_1 + e_2$ and $v_2 := 2e_1 - e_2$ it is represented by the matrix $\begin{pmatrix} 13/3 & 11/3 \\ 1/3 & -4/3 \end{pmatrix}$. *Moral: To define symmetric maps we need an extra structure on the vector space, viz., that of an inner product.*

Ex. 15. A linear map $A : \mathbf{E} \to \mathbf{E}$ is symmetric (self-adjoint) iff $\langle Ae_i, e_j \rangle = \langle e_i, Ae_j \rangle$ for any orthonormal basis $\{e_i\}$ of $\mathbf{E}$.

Ex. 16. Two self-adjoint operators A and B are such that AB is self-adjoint iff $AB = BA$.

Ex. 17. Let $\mathbf{E}$ be a complex inner product space. $A \in BL(\mathbf{E})$ is 0 iff $\langle Ax, x \rangle = 0$ for all $x \in \mathbf{E}$. The analogous result is false for real inner product spaces.

Ex. 18. Let $\mathbf{E}$ be a complex inner product space. Then $A \in BL(\mathbf{E})$ is self-adjoint iff $\langle Ax, x \rangle \in \mathbb{R}$.

Ex. 19. i) Show that the eigen values of a self-adjoint operator are real.
ii) Show that the eigen values of a skew-hermitian operator (i.e., $A^* = -A$) are purely imaginary.
iii) Let $T \in BL(\mathbf{E})$ be invertible. If λ is an eigen-value of T iff λ^{-1} is an eigen value of T^{-1}.
iv) λ is an eigen value of T iff $\overline{\lambda}$ is an eigen value of T^*.
v) $A, B : E \to E$ be linear. Show that AB and BA have the same set of eigen values.

3.2 Problem Set–II

Ex. 20. Let $\mathbf{E} := \mathbb{R}^n$ with the euclidean inner product. Find the characteristic equation of $A := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & \ddots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$. Find the eigen values and eigen vectors.

Ex. 21. Do the same for $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Ex. 22. What are the eigen values of an orthogonal projection?

Ex. 23. Let $\mathbf{E} := V \oplus W$ be a direct sum of vector subspaces. We can then define *projections* P_V and P_W in an obvious manner: $P_V(x) = y$ if $x := y + z$ with $y \in V$ and $z \in W$. Note that $P_V^2 = P_V$ etc. Show that in an inner product space $\mathbf{E}$ a projection P is self-adjoint iff P is an **orthogonal projection**. (You need to explain the meaning of the phrase in boldface.)

Ex. 24. Let P be an orthogonal projection of $\mathbf{E}$ onto a closed subspace V. Then $T \in BL(\mathbf{E})$ leaves V invariant, i.e., $TV \subset V$, iff $PTP = TP$.

Ex. 25. Let P_i be orthogonal projections onto closed linear subspaces V_i of $\mathbf{E}$. Then i) $V_1 \perp V_2 \iff P_1 P_2 = 0 \iff P_2 P_1 = 0$, ii) $P_1 P_2$ is an orthogonal projection iff $P_1 P_2 = P_2 P_1$. In this case $P := P_1 P_2$ is the orthogonal projection onto the subspace $V_1 \cap V_2$.

Ex. 26. Let E_i be nonzero orthogonal projections with $E_1E_2 = 0$. Then $\|E_1 + E_2\| < \|E_1\| + \|E_2\|$.

Ex. 27. Let $\mathbf{E}$ be an inner product space. For any two self adjoint operators A and B we write $A \leq B$ (or $B \geq A$) if $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in \mathbf{E}$. (Why does this make sense?) Prove that if $A \leq B$ and T is any self-adjoint operator, then $A + T \leq B + T$ and also that $\lambda A \leq \lambda B$ for any real number $\lambda \geq 0$. Is it true that $A \leq B$ and $B \leq A$ implies $A = B$?

Ex. 28. A self-adjoint operator A is said to be *positive* iff $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbf{E}$, i.e., iff $0 \leq A$. (See Ex. 27.) Prove the following:
i) If $\mathbf{E}$ is a complex inner product space the requirement that A is self-adjoint is superfluous (see Ex. 18) whereas in the case of real inner product spaces it is essential. (See Ex. 17.)
ii) A self-adjoint operator is positive iff all its eigen values are non-negative.
iii) A positive operator T has a unique positive square root, i.e., there exists a positive operator A such that $A^2 = T$.
iv) What can you say about an operator which is both positive and unitary?

Ex. 29. Let $A \geq 0$, $B \geq 0$ and $A + B = 0$. Then $A = 0 = B$.

Ex. 30. Let $A \geq 0$, $B \geq 0$ and $AB \geq 0$. Then $AB = BA$. (This is trivial; for its nontrivial converse, see Ex. 43.

3.3 Problem Set–III

Ex. 31. Let A be a symmetric positive operator on a real finite dimensional inner product space. If A is strictly positive, show that $\det A > 0$. Hint: Think of a curve joining A and I points on which are all positive and use the continuity of det. Or, think of some relation between the determinant and eigen values.

Ex. 32. Let $\mathbf{E}$ be a real (resp. complex) inner product space. $A \in BL(\mathbf{E})$ is said to be *orthogonal* (resp. *unitary*) iff $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in \mathbf{E}$.
i) Show that A is orthogonal (resp. unitary) iff $\|Ax\| = \|x\|$ for all $x \in \mathbf{E}$ iff $AA^* = A^*A = I$.
ii) A is orthogonal (unitary) iff A takes one orthonormal basis to another.
iii) In any matrix representation of an orthogonal operator with respect to an orthonormal basis the rows (columns) are mutually "orthonormal". Analogous result is true for unitary operators.
iv) Show that the eigen values of an orthogonal (resp. unitary) operator are of unit modulus.
v) An orthogonal projection is orthogonal iff $\cdots$. Complete it and prove it.

Ex. 33. Let $\mathbf{E}$ be a real (resp. complex) finite dimensional inner product space. Let $O(\mathbf{E})$ (resp. $U(\mathbf{E})$) be the set of all orthogonal (resp. unitary) operators on $\mathbf{E}$. Show that it is a compact subset of $BL(\mathbf{E})$ and that it is a group with respect to composition.

Ex. 34. Let A, a self-adjoint operator be such that $A^n = I$ for some $n \geq 1$. Then show that $A^2 = I$.

Ex. 35. Show that A is normal iff A^* is a polynomial in A.

Ex. 36. Let $\mathbf{E} = \mathbb{R}^2$ be with the euclidean inner product. Let $A := \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then the locus of the points $\{\langle Ax, x\rangle = 1\}$ can be considered as a conic section. It is an ellipse if $a > 0$ and

$c > 0$ and it is a hyperbola if $a > 0$ and $c < 0$, for example. In the case of an ellipse, the eigen values of A are got by the minor and major axes of the ellipse. This may help you understand the proof of Thm. 3.

Ex. 37. Diagonalize the following operators on the real euclidean plane $\mathbb{R}^2$:
i) $\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$. ii) $\begin{pmatrix} 5 & 1 \\ 1 & -1 \end{pmatrix}$ iii) $\begin{pmatrix} 1 & 4 \\ 4 & -1 \end{pmatrix}$. iv) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

Ex. 38. Let T be self-adjoint on $\mathbf{E}$. Let $v \in \mathbf{E}$ be such that for some $r \geq 1$ $T^r v = 0$. Is it true that $Tv = 0$?

Ex. 39. Let $A, B \in BL(\mathbf{E})$. Assume $AB = BA$. Show that there exists a normal operator T such that A and B are polynomials in T. Hint: Go through the proof of Th. 8.

Ex. 40. A is normal iff $\| Ax \| = \| A^* x \|$ for all $x \in \mathbf{E}$. ($\mathbf{E}$ may be real or complex.)

Ex. 41. For any $A \in BL(\mathbf{E})$, we have $\| AA^* \| = \| A \|^2$.

Ex. 42. For any $A \geq 0$. Show that there exists a unique positive square root B of A, i.e., there exists a unique $B \geq 0$ such that $B^2 = A$.

Ex. 43. If $A \geq 0$, $B \geq 0$ and $AB = BA$, then $AB \geq 0$. Hint: Use the spectral resolution of A etc. Or use the square roots of A and B. An elementary proof runs as follows: Assume $A > 0$, strictly positive. Then A defines an inner product $b(v, w) := \langle Av, w \rangle$. Use the commutativity etc. to conclude that B is a self-adjoint, positive with respect to b.

Remark. This article is a revised and expanded version of the set of notes for a course of lectures given in a Refresher Course in Topology and Functional Analysis, December 6–8, 1990 at Ramanujan Institute, Madras. I take this opportunity to thank Professor M.S. Rangachari for the invitation.