Completeness of \mathbb{R}

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Theorem. Any Cauchy sequence in \mathbb{R} converges.

Proof. Let (x_n) be a Cauchy sequence in \mathbb{R} . Given $\varepsilon > 0$, there exists a positive integer N such that for all $m \ge N$ and $n \ge N$, we have $|x_n - x_m| < \varepsilon/2$. In particular we have $|x_n - x_N| < \varepsilon/2$ Or, equivalently,

$$x_n \in (x_N - \varepsilon/2, x_N + \varepsilon/2)$$
 for all $n \ge N$.

From this we make the following observations:

i) For all $n \ge N$, we have $x_n > x_N - \varepsilon/2$.

ii) If $x_n \ge x_N + \varepsilon/2$, then $n \in \{1, 2, ..., N-1\}$. Thus the set of n such that $x_n \ge x_N + \varepsilon/2$ is finite.

Let $S := \{x \in \mathbb{R} : \text{ there exists infinitely many } n \text{ such that } x_n \ge x\}$. We claim that S is nonempty, bounded above and that $\sup S$ is the limit of the given sequence.

From i) we see that $x_N - \varepsilon/2 \in S$. Hence S is nonempty.

From ii) it follows that $x_N + \varepsilon/2$ is an upper bound for S. That is, we claim that $y \leq x_N + \varepsilon/2$ for all $y \in S$. If this were not true, then there exists a $y \in S$ such that $x_n \geq y$ for infinitely many n. This implies that $x_n > x_N + \varepsilon/2$ for infinitely many n. This contradicts ii). Hence we conclude that $x_N + \varepsilon/2$ is an upper bound for S.

By the LUB axiom, there exists $\ell \in \mathbb{R}$ which is $\sup S$. As ℓ is an upper bound for S and $x_N - \varepsilon/2 \in S$ we infer that $x_N - \varepsilon/2 \leq \ell$. Since ℓ is the least upper bound for S and $x_N + \varepsilon/2$ is an upper bound for S we see that $\ell \leq x_N + \varepsilon/2$. Thus we have $x_N - \varepsilon/2 \leq \ell \leq x_N + \varepsilon/2$ or

$$|x_N - \ell| \le \varepsilon/2.$$

For $n \geq N$ we have

$$|x_n - \ell| \leq |x_n - x_N| + |x_N - \ell|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

We have thus shown that $\lim_{n\to\infty} x_n = \ell$.