$\pi_1(S^n)$ is trivial for $n \geq 2$

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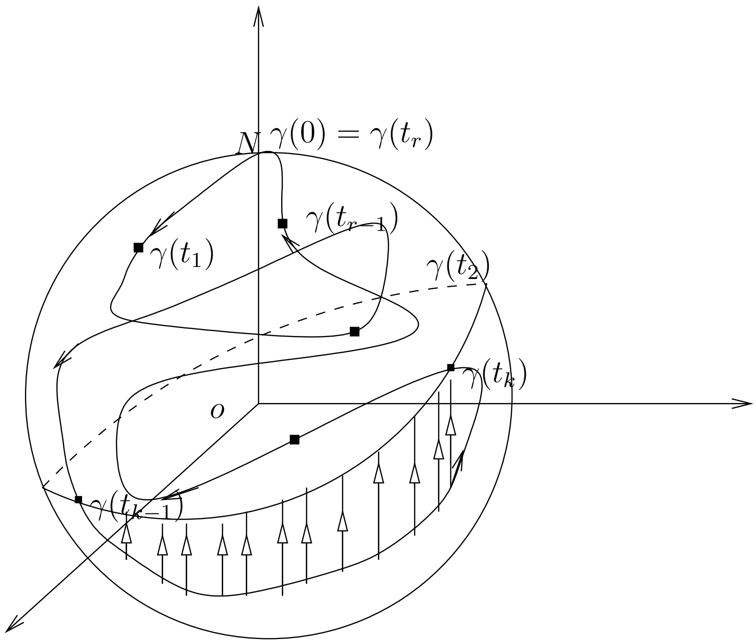
We prove the result in the title by showing that any loop $\gamma\colon I \to S^n$, for $n \geq 2$, is homotopic to the constant loop at the north pole $N := (0, 0, \dots, 1) \in S^n$.

To give an idea, you may visualize S^n as S^2 and assume that $N \notin \gamma(I)$. Then $S^n \setminus \{N\}$ is homeomorphic to $\mathbb{R}^n$. Since $\mathbb{R}^n$ is contractible, the loop γ is homotopic to a constant loop in $S^n \setminus \{N\}$ and hence in S^n. But if we recall Peano's space filling curves, we may perceive the difficulty in extending this argument for all loops.

We can surmount this problem with a little analysis and geometric thinking. The basic idea is to break up the loop into finitely many pieces (using the uniform continuity) in such a way that each of these pieces lies in an open hemisphere. These pieces can be made homotopic to shorter arcs (of great circles) which connect the end points of these arcs. Thus the given curve is homotopic to a curve made up of finitely many pieces, each of which is an arc of a great circle lying entirely in an open hemisphere. The image of the latter curve is not all of S^n and hence the result will follow.

For each $x \in S^n$, let $H_x := \{y \in S^n : \|y - x\| < \sqrt{2}\}$ be the open hemisphere. The family $\mathcal{H} := \{H_x : x \in S^n\}$ is an open cover of S^n and hence of $\gamma(I)$. The set $\gamma^{-1}(H_{\gamma(t)})$ is an open neighbourhood of t in I. Let J_t be the connected component of this set containing t. Then $\{J_t : t \in I\}$ is an open cover of I and hence we have a finite subcover. Let $\{J_i := J_{t_i} : 1 \leq i \leq N\}$ be such a finite subcover. Assume, without loss of generality, that $0 \in J_1$. We choose J_k inductively as follows: Assuming that we have chosen J_k, we choose J_{k+1} so that $J_{k+1} \cap J_k \neq \emptyset$ and $\sup J_{k+1} > \sup J_k$. This process stops when $1 \in J_r$ and $\{J_k : 1 \leq k \leq r\}$ is a finite subcover of I.

Thus we can find a finite number of points $t_k \in I$, $1 \leq k \leq r$ such that $0 = t_1 < t_2 < \cdots < t_k = 1$ with $t_k \in J_k \cap J_{k+1}$ for $1 \leq k < r$. Note that the interval $[t_k, t_{k+1}] \subset J_k$ for $1 \leq k < r$. This completes the first step of dividing the domain of γ into finite number of pieces each of which is mapped into a hemisphere.



Let $1 \leq k \leq r$. We wish to show that γ is homotopic to a loop σ such that $\sigma = \gamma$ on $I \setminus [t_{k-1}, t_k]$ and $\sigma([t_{k-1}, t_k])$ is a great circle arc from between the end points $\sigma(t_{k-1})$ and $\sigma(t_k)$. If $\gamma(t_{k-1}) = \gamma(t_k)$, then we can invoke the contractibility of $H_{\gamma(t_k)}$ to dispose of this case. So, we assume that such is not the case. Then $\{\gamma(t_{k-1}), \gamma(t_k)\}$ is a linearly independent set and hence they span a two dimensional vector subspace. On this two dimensional vector subspace, these two vectors lie on an open semicircle of the unit circle. There is a shorter arc of that great circle from $\gamma(t_{k-1})$ to $\gamma(t_k)$. We can find a fixed end point homotopy from the restriction of γ to $[t_{k-1}, t_k]$ to the arc of the great circle which connects the endpoints.

By induction, we proceed to get a loop σ consisting of finite number of arcs of great circles. Obviously, the image of σ is not all of S^n. (For, its measure is zero, as we can find open subsets of arbitrarily small measure of S^n which contain the image.) Now we an argue as in the beginning to show that σ is homotopic to a constant loop. By transitivity, it follows that γ is homotopic to a trivial loop. Hence we get $\pi_1(S^n) = \{e\}$ for $n \geq 2$.