Stokes Theorem–An Outline of a Proof

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The following is an outline of the steps leading to a proof of Stokes' theorem for parametrized surfaces. Most of them are definitions, concepts and notation. The proof itself is a computation using the terminology.

1. Jacobian matrix, chain rule, explicit notation such as $\frac{\partial(y_1,\dots,y_n)}{\partial(x_1,\dots,x_n)}$.

2. Line integral of a vector field V along a path $\gamma\colon [a,b] \to \mathbb{R}^n$:

$$\int_\gamma V := \int_a^b V(\gamma(t)) \cdot \gamma'(t)\, dt = \sum_{j=1}^n \int_a^b V_j(\gamma(t))\gamma_j'(t)\, dt = \sum_{j=1}^n \int_a^b V_j dx_j.$$

3. Green's theorem. A rigorous definition of the orientation of the boundary.

4. Parametrized surface, geometric meaning of g_u, g_v and $g_u \times g_v$, where

$$g_u := \left(\frac{\partial g_1}{\partial u}, \frac{\partial g_2}{\partial u}, \frac{\partial g_3}{\partial u}\right) \text{ and } g_v := \left(\frac{\partial g_1}{\partial v}, \frac{\partial g_2}{\partial v}, \frac{\partial g_3}{\partial v}\right).$$

5. Area element on a parametrized surface: $dS := \| g_u \times g_v \|\, dudv$

6. Integral of a vector field $V = (V_1, V_2, V_3)$ on a surface S parametrized by $g\colon D \to \mathbb{R}^3$:

$$\int_S V \cdot dS = \int_D V \cdot (g_u \times g_v)\, dudv = \int_D V_1 \frac{\partial(g_2, g_3)}{\partial(u,v)} + V_2 \frac{\partial(g_3, g_1)}{\partial(u,v)} + V_3 \frac{\partial(g_1, g_2)}{\partial(u,v)}.$$

   Using a shorthand notation such as $dx_1 \wedge dx_2 := \frac{\partial(g_1,g_2)}{\partial(u,v)}$, this can be rewritten as

$$\int_S V \cdot dS = \int_D V_1 dx_2 \wedge dx_3 + V_2 dx_3 \wedge dx_1 + V_3 dx_1 \wedge dx_2.$$

   Note that the order matters in $dx_1 \wedge dx_2$.

7. Curl of a vector field $F = (F_1, F_2, F_3)$ is defined as $\nabla \times F$, that is,

$$\operatorname{curl} F := \left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right).$$

8. Stokes theorem. Enough to establish, for $1 \leq i \leq 3$,

$$\int_{\gamma} F_i \, dx_i = \int_S \operatorname{curl} \mathbf{F}_i$$

where $\mathbf{F}_1 = (F_1, 0, 0)$ etc. The key idea is to expand the left side using the definition of the line integral and chain rule. This prepares the ground for the employment of Green's theorem.