Stokes Theorem — A simple and intuitive proof

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The proof uses two ingredients: (i) the integral definition of the exterior derivative of a differential form and (ii) a generalized Riemann integral (called the Heinstock-Kurzweil integral).

(i) **Integral definition of $d\omega$:** Let ω be an $(n-1)$-form on $\mathbb{R}^n$. We claim that if $d\omega = \varphi\, dx_1 \wedge \cdots \wedge dx_n$, then the density φ is given as follows:

$$\varphi(x) := \lim_{\substack{x\in Q \\ \operatorname{diam}(Q)\to 0}} \frac{1}{|Q|}\int_{\partial Q} \omega,$$

where the limit is taken over all cubes Q containing x. In the sequel, we shall not distinguish between $d\omega$ and its 'density' φ.

(ii) **A special case of Henstock-Kurzweil Integral:** A function $\delta\colon [0,1]^n \to (0,\infty)$ is called a *gauge*. A *tagged partition* of $[0,1]^n$ is a finite collection $\{c_j, Q_j\}$, where Q_j's are subcubes, $c_j \in Q_j$, $[0,1]^n = \cup_{j=1}^N Q_j$ and Q_j's are disjoint except for the boundaries. The points c_j's are called tags. Given a gauge δ, a tagged partition $\{c_j, Q_j\}$ is said to be δ-*fine* if $\operatorname{diam}(Q_j) \le \delta(c_j)$, $1 \le j \le N$.

Let $\varphi = f\, dx_1 \wedge \cdots \wedge dx_n$ be an n-form on $[0,1]^n$. The generalized integral $(HK)\int_{[0,1]^n} \varphi \equiv (HK)\int_{[0,1]^n} f$ is a real number α such that given $\varepsilon > 0$, there exists a gauge δ on $[0,1]^n$ such for any δ-fine partition $\{c_j, Q_j\}$ we have

$$\left| \alpha - \sum_{j=1}^N f(c_j) \operatorname{Volume}(Q_j) \right| \le \varepsilon.$$

For this definition to make sense, we need to make two observations.

Observation 1. Given a gauge δ on $[0,1]^n$, there exists a δ-fine partition. (This is known as Cousin's lemma.) This is a typical application of Cantor intersection theorem. If a cube Q does not admit a δ-fine partition, and if we subdivide it into cubes, then one of the subcubes will not admit a δ-fine partition. (Here δ for the subcube is the restriction of δ on Q.) Thus, we construct a nested sequence (Q_k) of subcubes with $\operatorname{diam}(Q_k) \to 0$. Let $x \in \cap Q_k$. Choose k so that $\operatorname{diam}(Q_k) < \delta(x)$. Then $\{Q_k, x\}$ is a delta-fine partition of Q_k, a contradiction.

Observation 2. If such an α in the definition exists, it is unique. This is trivial to see.

We claim that if f is continuous on $[0,1]^n$ or more generally if f is Lebesgue integrable on $[0,1]^n$, then the HK-integral exists and is equal to the Riemann integral (or the Lebesgue integral) of f. (See Theorem stokes;thm2.)

With these preliminaries over, we shall give a quick proof of Stokes' theorem for cubes.

Theorem 3 (Stokes Theorem for cube)**.** *Let ω be a differential $(n-1)$-form defined on an open set $U \supseteq [0,1]^n$. If $d\omega$ exists on $[0,1]^n$, then $d\omega$ is HK-integrable on $[0,1]^n$ and we have*

$$(HK) \int_{[0,1]^n} d\omega = \int_{\partial[0,1]^n} \omega. \tag{1}$$

Proof. Given $\varepsilon > 0$, we define a gauge δ on $[0,1]^n$ is as follows. For $x \in [0,1]^n$, by the integral definition of $d\omega$, there exists a $\delta(x) > 0$ such that for any cube Q such that $x \in Q$ and $\operatorname{diam}(Q) < \delta(x)$ we have

$$\left| \int_{\partial Q} \omega - d\omega(x)\text{Volume}\,(Q) \right| < \varepsilon\text{Volume}\,(Q).$$

Let $\{c_j, Q_j\}$ be a δ-fine partition of $[0,1]^n$. We have

$$\begin{aligned} \left| \int_{\partial[0,1]^n} \omega - \sum_{j=1}^{N} d\omega(c_j)\text{Volume}\,(Q_j) \right| &= \left| \sum_{j=1}^{N} \left(\int_{\partial Q_j} \omega - \sum_{j=1}^{N} d\omega(c_j)\text{Volume}\,(Q_j) \right) \right| \\ &< \sum_{j=1}^{N} \varepsilon\text{Volume}\,(Q_j) = \varepsilon. \end{aligned}$$

Thus by the definition of HK-integral, $(HK)\int_{[0,1]^n} d\omega$ exists and is equal to $\int_{\partial[0,1]^n} \omega$. ☐

To get the standard version, we need to prove (i) that the integral definition of the exterior derivative coincides with the standard definition and (ii) that if f is Lebesgue integrable over the cube it is HK-integrable and both the integrals coincide. Of these (ii) in the case of a continuous function is immediate. If f is continuous on $[0,1]^n$, then it is uniformly continuous on $[0,1]^n$ and hence given $\varepsilon > 0$, there corresponds a constant $\delta > 0$ by the uniform continuity. As α in the HK-integral we take the Riemann integral $\int_{[0,1]^n} f$ and for a given $\varepsilon > 0$, we choose the constant function δ obtained by the uniform continuity. Then for any δ-fine partition $\{c_j, Q_j\}$, we have by the definition of Riemann integral,

$$\left| \int_{[0,1]^n} f - \sum_j f(c_j)\text{Volume}\,(Q_j) \right| < \varepsilon.$$

In Theorem 4, we establish that any Lebesgue integrable function on $[0,1]^n$ is HK-integrable and that the integrals coincide.

We now prove the claim in (ii) concerning Lebesgue integral.

Theorem 4. *Let $f\colon [0,1]^n \to \mathbb{R}$ be Lebesgue integrable. Then it is HK-integable and both the integrals are the same.*

Proof. In the proof, all integrals are Lebesgue.

Let $\varepsilon > 0$ be given. By the absolute continuity of the Lebesgue integral, there exists $\eta > 0$ such that if $\mu(A) < \eta$, then $\int_A |f|\, d\mu < \varepsilon$.

For each $k \in \mathbb{Z}$, we define

$$E_k := \{x \in [0,1]^n : k\varepsilon < f(x) \le (k+1)\varepsilon\}.$$

Then each E_k is measurable and E_k's form a measurable partition of $[0,1]^n$.

By the outer regularity, there exists an open set $G_k \supseteq E_k$ such that

$$\mu(G_k \setminus E_k) < \frac{\eta}{C_k}, \text{ for a constant } C_k \text{ to be chosen later.}$$

We are now ready to define the gauge: set $\delta(x) = d(x, [0,1]^n \setminus G_k)$ for $x \in E_k$. Let $\{c_j, Q_j\}$ be a δ-fine partition of $[0,1]^n$. Let $n_k \in \mathbb{Z}$ be such that $c_k \in E_{n_k}$. Decompose $Q_k = A_k \cup B_k$ where $A_k := Q_k \cap E_{n_k}$ and $B_k := Q_k \setminus E_{n_k}$. We have

$$\begin{aligned} &\left| \int_{[0,1]^n} f - \sum_{k=1}^{N} f(c_k) \text{Volume}\,(Q_k) \right| \\ &\qquad = \left| \sum_{k=1}^{n} \int_{Q_k} [f(x) - f(c_j)] \right| \\ &\qquad \le \sum_{k=1}^{N} \int_{A_k} |f(x) - f(c_k)| + \sum_{k=1}^{N} \int_{B_k} |f(x)| + \sum_{k=1}^{N} \int_{B_k} |f(c_k)| . \end{aligned} \tag{2}$$

We show that each of the three terms on the right is less than ε. This will complete the proof.

First term: Since $c_k, x \in A_k \subseteq E_{n_k}$, we observe that $|f(x) - f(c_k)| < \varepsilon$. Since Q_k's are essentially disjoint, we see that A_k's are almost disjoint. Hence

$$\sum_{k=1}^{N} \int_{A_k} |f(x) - f(c_k)|\, dx \le \sum_{k=1}^{N} \int_{A_k} \varepsilon\, dx \le \int_{[0,1]^n} \varepsilon = \varepsilon.$$

Second term: As in the case of A_k's, the sets B_k's are also almost disjoint. We note that $Q_k \subset G_{n_k}$. For, $\{c_j, Q_j\}$ is a δ-fine partition and hence $\text{diam}\,(Q_k) \le \delta(c_k) = d(c_k, [0,1]^n \setminus G_{n_k})$. We delete E_{n_k} from both sides of the inclusion $Q_k \subseteq G_{n_k}$ to get $B_k \subseteq G_{n_k} \setminus E_{n_k}$. It follows that the almost disjoint union of all the B_k's with the same n_k is contained in $G_{n_k} \setminus E_{n_k}$. Hence we deduce

$$\sum_{k=1}^{N} \mu(B_k) \le \sum_{n_k} \mu(G_{n_k} \setminus E_{n_k}) < \sum_{n_k} \frac{\eta}{C_{n_k}} \le \sum_{k} \frac{\eta}{C_k} < \eta,$$

if C_k are suitably chosen. Thus, we may choose $C_k = 3 \cdot 2^{|k|}$. (The presence of 3 is thanks to two infinite series indexed by $\mathbb{N}$ and the term corresponding to 0.) By the choice of η (that is, by the absolute continuity), we conclude that $\sum_k \int_{B_k} |f| < \varepsilon$.

Third term: We have

$$\begin{aligned}\sum_k \int_{B_k} |f(c_k)|\ dx &\leq \sum_k |f(c_k)|\, \mu(B_k) \\ &\leq \sum_k (1+|n_k|) \frac{\eta}{C_{n_k}} \\ &< \varepsilon,\end{aligned}$$

provided that $0 < \eta < \varepsilon$ and we choose $C_k := 3 \cdot 2^{|k|}(1+|k|)$. $\square$

We now attend to item (i).

Theorem 5. *Let $\omega = \sum_{i=1}^n f_i(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ be an $(n-1)$-form, where the hat over dx_i indicates that dx_i is omitted. If the f_i are differentiable at 0, then $d\omega(0)$ exists, that is, the limit in the integral definition of $d\omega(0)$ exists and is given by the usual algebraic formula:*

$$d\omega(0) = \sum_{j=1}^n (-1)^{j-1} \partial_j f_j(0) = \sum_{j=1}^n (-1)^{j-1} \frac{\partial}{\partial x_j} f_j(0).$$

Proof. We must show that

$$\lim_{\substack{x \in Q \\ \operatorname{diam}(Q) \to 0}} \frac{1}{\operatorname{Volume}(Q)} \int_{\partial Q} \sum_{j=1}^n f_j(x)\ dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n = \sum_{j=1}^n (-1)^{j-1} \partial_j f_j(0). \tag{3}$$

We prove (3) for cubes with sides parallel to the axes. For such cubes, it is enough to show that for an arbitrary k and differentiable function f,

$$\lim_{\substack{x \in Q \\ \operatorname{diam}(Q) \to 0}} \frac{1}{\operatorname{Volume}(Q)} \int_{\partial Q} f(x)\ dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n = (-1)^{k-1} \partial_k f_(0). \tag{4}$$

Let Q have width ε and sides $s_j^{\pm}$, on which x_j is a constant. Recall that the orientation on $s_k^{\pm}$ in Q is $\pm(-1)^{k-1}$ times the oreintation of $s_k^{\pm}$ in $\mathbb{R}^n$, that is, the boundary orientation is $\pm(-1)^{k-1}(x_1, \ldots, \widehat{x_k}, \ldots, x_n)$. The only sides in ∂Q contributing to the integral in (4) are $s_k^{\pm}$. Thus, we can rewrite (4) as follows:

$$\lim_{\varepsilon \to 0} \frac{(-1)^{k-1}}{\varepsilon^n} \left(\int_{s_k^+} f(x) - \int_{s_k^-} f(x) \right) = (-1)^{k-1} \partial_k f(0). \tag{5}$$

Since f is differentiable at 0, we have

$$f(x) = f(0) + \sum_{k=1}^n \partial_k f(0) + E(x), \tag{6}$$

where the remainder $E(x)$ is such that $E(x)/\|x\| \to 0$ as $x \to 0$.

We now prove (4) by substituting the three terms on the right side of (6) in place of f in the integral terms of (5).

First term: If we substitute $f(0)$ in place of $f(x)$, then $\int_{s_k^+} f(0) = \int_{s_k^-} f(0)$ and hence the contribution is zero.

Second term: If $x \in s_k^+$, the corresponding point y on the opposite side s_k^- is given by $y_j = x_j$ for $j \neq k$ and $y_k = x_k - \varepsilon$. We now substitute $\partial_k f(0)$ for $f(x)$ in the integrals of (5). Without the limit, we are looking at the following expression:

$$\frac{(-1)^{k-1}\partial_j f(0)}{\varepsilon^n} \int_{s_j^+} (x_j - y_j).$$

The integrand is zero if $j \neq k$ and is ε if $j = k$. In the latter case, the integral is $\varepsilon^{n-1}\varepsilon$ so that the expression is $(-1)^{k-1}\partial_k f(0)$.

Third term: Since $E(x)/\|x\| \to 0$, and since $\|x\| \leq \sqrt{n}\varepsilon$ on Q, we have

$$\begin{aligned} \left| \frac{(-1)^{k-1}}{\varepsilon^n} \int_{s_k^\pm} E(x) \right| &= \left| \frac{(-1)^{k-1}}{\varepsilon^n} \int_{s_k^\pm} \frac{E(x)}{\|x\|} \|x\| \right| \\ &\leq \frac{1}{\varepsilon^n} \int_{s_k^\pm} \sqrt{n}\varepsilon \frac{|E(x)|}{\|x\|} \\ &\leq \sqrt{n} \sup_{\|x\| \leq \sqrt{n}\varepsilon} \frac{|E(x)|}{\|x\|} \to 0. \end{aligned} \tag{7}$$

We have thus proved (3) for cubes with sides parallel to the axes. As the limit is taken over all cubes Q with $\operatorname{diam}(Q) \to 0$, we need to show that the limit is independent of the rotation and is also 'uniform'. But this is almost clear, for the only place in the proof where we have taken limits is when dealing with the third term. Now, $E(x)$ is independent of the rotation, as f and $\operatorname{grad} f \cdot x = \nabla f(x) \cdot x$ are so. □

We can now improve our version of Stokes theorem which is much stronger than the standard versions.

Corollary 6. *Let ω be a continuous differential form of degree $n-1$ on $[0,1]^n$. Assume that $d\omega$ exists on $(0,1)^n$ and it is Lebesgue integrable there. Then*

$$\int_{[0,1]^n} d\omega = \int_{\partial[0,1]^n} \omega. \tag{8}$$

Proof. Assume $d\omega = \varphi\, dx_1 \wedge \cdots \wedge dx_n$.

Let $Q_k := [\frac{1}{k}, 1 - \frac{1}{k}]^n$. From Theorem 3 and the fact that Lebesgue integrable functions are HK-integrable, it follows that

$$\int_{Q_k} d\omega = \int_{\partial Q_k} \omega. \tag{9}$$

We let $k \to \infty$ in the above equation. The integrand on the left side is $\chi_{Q_k}\varphi$ is dominated by the integrable function $|\varphi|$ and $\chi_{Q_k}\varphi \to \varphi$. Hence by the dominated convergence theorem, the integral on the right side of (9) approaches $\int_{[0,1]^n} d\omega$.

The right side of (9) approaches that of (8) by the uniform continuity of ω on $[0,1]^n$, the boundedness of ω along with the fact that the measure of the "obvious complement" of ∂Q_k in $\partial[0,1]^n$ is very small. □

Remark 7. The standard versions require the continuity of $d\omega$. Hence, the Green's theorem which is a special case of Stokes requires the coefficients P and Q in the 1-form $Pdx + Qdy$ to have continuous partial derivatives. This prevents us from deriving the Cauchy-Goursat and Cauchy's theorem as immediate applications of Green's theorem. With our version in the corollary, Cauchy-Goursat can be derived from Green's theorem!

The version of Stokes theorem on manifolds follows by standard trick of employing a partition of unity. We shall only sketch the arguments for the sake of completeness.

Theorem 8 (Stokes Theorem for Manifolds)**.** *Let ω be a continuous differential $(n-1)$-form on a compact oriented manifold M with smooth boundary ∂M. Assume that ∂M is given the boundary orientation. Assume that ω is differentiable on $M \setminus \partial M$ and $d\omega$ is Lebesgue integrable on it. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. Choose $p \in M \setminus \partial M$. Given a coordinate chart (U, φ) around p, we may assume that $[0,1]^n \subset U$. Similarly, given $q \in \partial M$, we choose a chart $\varphi\colon U \to M$ such that $[0,1]^n \subset U$, U is open in the half-space $\{x \in \mathbb{R}^n : x_n \geq 0\}$ and $q \in V := \varphi((0,1)^{n-1} \times \{0\})$.

To start with, we assume that the support of ω is contained in a V as above. Since $\varphi^*\omega$ is differentiable, the integral definition and the standard definition of the exterior derivative of $\varphi^*\omega$ coincide. Hence it follows that $\varphi^*(d\omega) = d(\varphi^*\omega)$. Thus using the corollary, we get

$$\int_M d\omega = \int_{[0,1]^n} \varphi^*(d\omega) = \int_{[0,1]^n} d(\varphi^*\omega) = \int_{\partial[0,1]^n} \varphi^*\omega = \int_{\partial M} \omega.$$

Now the general case. Cover the compact manifold M with a finite number of open V_i of the type above. Let $\{f_i\}$ be a partition of unity subordinate to the open cover $\{V_i\}$: $\mathrm{Supp}\,(\mathrm{f_i}) \subset \mathrm{V_i}$, $f_i \geq 0$, f_i are smooth and $\sum_i f_i(x) = 1$ for all $x \in M$. The $\omega = \sum_i f_i\omega$. Since $\mathrm{Supp}\,(\mathrm{f_i}\omega) \subset \mathrm{Supp}\,(\mathrm{f_i}) \subset \mathrm{V_i}$, we can apply our earlier result to each of the $f_i\omega$:

$$\int_M d\omega = \sum_i \int_M d(f_i\omega) = \sum_i \int_{\partial M} f_i\omega = \int_{\partial M} \omega.$$

This completes the proof of the Stokes' theorem. □

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