Stone-Weierstrass Theorem

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Ex. 1. Let F be a finite dimensional vector subspace of a normed linear space X. Let $x \in X$. Show that there exists an element $v \in F$ such that $d(x, F) = d(x, v)$. *Hint:* F is locally compact. Consider $f(z) := \|z - x\|$ on F. Let $v_0 \in F$ be arbitrary and $r := d(x, v_0)$. Then f attains a minimum on $B[x, r] \cap F$.

Ex. 2. Consider $X := C[a, b]$ with sup norm and F_n the subspace of polynomials of degree less than or equal to n. Show that given $f \in X$, there exists a $p \in F_n$ such that $\|f - p\| \leq \|f - q\|$ for all $q \in F_n$.

Theorem 3 (Dini's Theorem)**.** *Let X be a compact metric space. Let $f_n, f \in C(X, \mathbb{R})$. Assume that (f_n) is a monotone sequence converging to f pointwise. Then $f_n \to f$ uniformly, i.e., in sup norm metric.*

Proof. Without loss of generality assume that f_n decrease to 0. Given $\varepsilon > 0$, let

$$U_n := \{x \in X : |f_m(x)| < \varepsilon, \forall m \geq n\} = \{x \in X : |f_n(x)| < \varepsilon\}.$$

Then U_n increases to X and by compactness $X = U_N$ for some N. $\square$

Proposition 4. *There is a sequence (p_n) of real valued polynomials which converge uniformly to $f(x) = \sqrt{x}$ on $[0, 1]$.*

Proof. Define (p_n) recursively as follows: $p_1 = 0$,

$$p_{n+1}(t) := p_n(t) + \frac{1}{2}\left[t - p_n^2(t)\right],$$

for $n \geq 1$. By induction we show that $0 \leq p_n(t) \leq \sqrt{t}$ for $t \in [0,]$:

$$\begin{aligned} 0 \leq p_{n+1}(t) &= p_n(t) + \frac{1}{2}\left[t - p_n^2(t)\right] \\ &= p_n(t) + \frac{1}{2}\left[t - p_n(t)\right]\left[t + p_n(t)\right] \\ &\leq p_n(t) + \sqrt{t} - p_n(t). \end{aligned}$$

Now an application of Dini's theorem to (f_n) yields the result. $\square$

Definition 5. A vector subspace $\mathcal{A}$ (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) of $C(X, \mathbb{K})$ is called a *subalgebra* of $C(X, \mathbb{K})$ if whenever f, $g \in \mathcal{A}$ so does fg.

A vector subspace $\mathcal{A}$ of $C(X, \mathbb{K})$ is called a *lattice* of $C(X, \mathbb{K})$ if whenever f, $g \in \mathcal{A}$ so do $\max\{f, g\}$ and $\min\{f, g\}$.

We say a collection $\mathcal{A}$ of functions in $C(X)$ separates points of X if given x, $y \in X$, $x \neq y$, there is an $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Lemma 6. *If X is a compact space and $\mathcal{A}$ is a closed subalgebra of $C(X, \mathbb{R})$ which contains constant functions then $\mathcal{A}$ is a lattice.*

Proof. We claim that if $f \in \mathcal{A}$ then $|f| \in \mathcal{A}$. Let $f \neq 0$ and $M := \|f\|$. By the lemma there exists a sequence (p_n) of polynomials such that $p_n \circ (\frac{f^2}{M^2}) \to |f|/M$ uniformly. Thus $|f| \in \mathcal{A}$. Now recall that $\min\{f, g\} = \frac{1}{2}(f + g - |f - g|)$ etc. $\square$

Lemma 7. *Let $\mathcal{A}$ be a subalgebra of $C(X, \mathbb{R})$ which contains constant functions and separates points of X. Then for each pair of distinct points x, y of X and pair of real numbers α and β there is a function $f \in \mathcal{A}$ such that $f(x) = \alpha$ and $f(y) = \beta$*

Proof. Choose $g \in \mathcal{A}$ such that $g(x) \neq g(y)$. Let

$$f(z) := \alpha + (\beta - \alpha)\frac{g(z) - g(x)}{g(y) - g(x)}.$$

$\square$

Theorem 8. *Let X be a compact metric space. If $\mathcal{A}$ is a subalgebra of $C(X, \mathbb{R})$ which contains constant functions and separates points of X then $\mathcal{A}$ is dense in $C(X, \mathbb{R})$ in sup norm topology.*

Proof. Let $\in C(X, \mathbb{R})$. Given any two distinct points x and y let $h_{xy} \in \mathcal{A}$ be such that $h_{xy}(x) = f(x)$ and $h_{xy}(y) = f(y)$. Given $\varepsilon > 0$, fix x. Let $U_y := \{z \in X : h_{xy}(z) < f(z) + \varepsilon\}$. By compactness $X = \cup_{i=1}^{n} U_{y_i}$. Let $h := \min\{h_{xy_1}, \ldots, h_{xy_n}\}$. Note that $h_x(x) = f(x)$. Now vary x and let $V_x := \{z \in X : h_x(z) > f(z) - \varepsilon\}$. Again, by compactness, $X = \cup_{j=1}^{m} V_{x_j}$. Let $h := \max\{h_{x_1}, \ldots, h_{x_m}\}$. Then $h \in \overline{\mathcal{A}}$ and $f(z) - \varepsilon < h(z) < f(z) + \varepsilon$ for all $z \in X$. $\square$

Corollary 9. *The space of polynomials in $C[a, b]$ is dense $C([a, b], \| \; \|_\infty)$.* $\square$

Theorem 10. *Let X be a compact space. If $\mathcal{A}$ is a subalgebra of $C(X, \mathbb{C})$ which contains constant functions, separates points of X and contains $\overline{f}$ whenever $f \in \mathcal{A}$, then $\mathcal{A}$ is dense in $(C(X, \mathbb{C}), \| \; \|_\infty)$.* $\square$

Ex. 11. Let K be a compact subset of $\mathbb{R}^n$. Show that the set of polynomial functions (with coefficients in $\mathbb{K}$) on K are dense in $C(X, \mathbb{K})$ with sup norm.

Ex. 12. Let $f \colon [0, 1] \to \mathbb{R}$ be continuous. Assume that $\int_0^1 f(t)t^n \, dt = 0$ for all $n \in \mathbb{Z}$ with $n \geq 0$. Then $f = 0$.

Ex. 13. Let X and Y be compact spaces. Consider the set $\mathcal{A}$ of functions h of the form $h(x, y) := f(x)g(y)$ for $f \in C(X)$ and $g \in C(Y)$. Show that $\mathcal{A}$ is dense in $C(X \times Y)$ with sup norm .

Ex. 14. How can you generalize the above exercise to infinite products?

Ex. 15. Let X be a compact space. Let $\mathcal{A}$ be a subalgebra of $C(X,\mathbb{R})$. Assume that $\mathcal{A}$ separates points of X. Show that the closure of $\mathcal{A}$ in $C(X)$ with sup norm is either $C(X)$ or there exists an $x \in X$ such that $\mathcal{A} = \{f \in C(X) : f(x) = 0\}$.

Remark 16. As a application, we prove Tieze extension theorem for $\mathbb{R}^n$.

Let us prove the result when the closed set is compact. So, we assume that $f\colon K \to \mathbb{R}$ is a continuous function on a compact subset of $\mathbb{R}^n$. By Weierstrass approximation theorem, for each $k \in \mathbb{Z}_+$, there exists a polynomial p_k such that $|f(x) - p(x)| < 2^{-k-2}$ for all $x \in K$. We let $q_0 = p_0$ and $q_k := p_k - p_{k-1}$. Then $p_k = \sum_{i=1}^k q_i$ and $\sum q_k$ converges uniformly to f on K.

Let $M := \sup\{|f(x)| : x \in K\}$. Then $|p_0(x)| \leq 2^{-2} + M$ for $x \in K$. Also, $|q_k(x)| < 2^{-k}$ for $k \geq 1$ and $x \in K$. We let

$$\begin{aligned} h_0 &:= \max\{-2^{-2} - M, \min\{q_0, 2^{-2} + M\}\}, \\ h_k &:= \max\{-2^{-k}, \min\{q_k, 2^{-k}\}\}, \qquad \text{for } k \geq 1. \end{aligned}$$

Then $h_k(x) = q_k(x)$ for $x \in K$, h_k is continuous on $\mathbb{R}^n$ and $|h_k(x)| \leq 2^{-k}$ for $x \in \mathbb{R}^n$ and for all k. Hence $\sum h_k$ converges uniformly on $\mathbb{R}^n$ to a continuous function h. Then h is continuous and $h(x) = f(x)$ for $x \in K$.

We now extend to result if the subset K is any arbitrary closed subset. If K is bounded the result follows from the previous paragraph. So, we assume that K is not bounded. Let $k \in \mathbb{N}$ be such that $B[0,k] \cap K$ is nonempty. Let f_k be the restriction of f to this nonempty compact set. Then there exists a continuous function h_k on $\mathbb{R}^n$ which extends f_k. Define

$$g_k(x) := \begin{cases} h_k(x), & \text{if } x \in B[0,k] \\ f(x), & \text{if } x \in K \cap B[0,k+1]. \end{cases}$$

Then g_k is continuous on the compact set $B[0,k] \cup (K \cap B[0,k+1])$. There is an extension h_{k+1} on $\mathbb{R}^n$. Let

$$g_{k+1}(x) := \begin{cases} h_{k+1}(x), & \text{if } x \in B[0,k+1] \\ f(x), & \text{if } x \in K \cap B[0,k+2]. \end{cases}$$

Continuing in this way, we obtain a sequence (g_m) whose domains are increasing to $\mathbb{R}^n$. Define $g(x) := g_m(x)$ if $x \in B[0,m]$. Then g is an extension of f.