Subspace Topology

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Let $Y \subset X$ of a topological space $(X, \mathcal{T})$. We say that a set $V \subset Y$ is open **in** Y if there exists an open set $U \in \mathcal{T}$ such that $V = U \cap Y$. Let $\mathcal{T}_Y$ denote the set of all subsets $V \subset Y$ which are open in Y. Then $\mathcal{T}_Y$ is a topology on Y. It is called the subspace topology. Given below are some examples-cum-exercises which will help you master this concept. We concentrate on "basic" open sets in Y, that is, those sets whose arbitrary unions will produce all elements of $\mathcal{T}_Y$. In the following any $\mathbb{R}^n$ is endowed with the standard topology coming from the Euclidean metric $(x, y) := \sqrt{\sum_{i=1}^n (x_i^2 - y_i^2)}$.

Ex. 1. Let (X, d) be a metric space. If we restrict d to $Y \times Y$ we get a metric on Y. Observe that $B_Y(y, r) := B(y, r) \cap Y$, where $B_Y(y, r) := \{z \in Y : d(z, y) < r\}$. The collection $\{B_Y(y, r) : y \in Y, r > 0\}$ is a family of basic open sets for $\mathcal{T}_Y$.

Ex. 2. Let $Y := [0, \infty) \subset \mathbb{R}$. Then the sets of the form $[0, x)$ with $x > 0$ are open in Y. In fact, the basic open sets are $[0, r)$, $r > 0$ and sets of the form (a, b), $0 < a < b$.

Ex. 3. Let $Y := \{(x, 0) : x \in \mathbb{R}\} \subset \mathbb{R}^2$. Then the sets of the form $(a, b) \times \{0\}$ are basic open sets.

Ex. 4. Let $S := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ be the unit circle in $\mathbb{R}^2$. The basic open sets in S are open arcs of the circle.

Ex. 5. Consider two circles in $\mathbb{R}^2$ which 'touch' (or, which are tangential) at the origin. Then the basic open sets around the origin are two arcs (through the origin) of the two circles.

Ex. 6. Consider $Y := \{(x, y) : xy = 0\} \subset \mathbb{R}^2$ be the two axes. Then the basic open sets near (0,0) are crosses (of two line segments along the x and y-axes.) At other points, just intervals around them.

Ex. 7. This is a generalization of Ex. 3. It requires the knowledge of product topology.

Let X and Y be topological spaces. We consider the product topology on $X \times Y$. Fix $y_0 \in Y$. Let $S := X \times \{y_0\}$. Then the basic open sets of S are of the form $U \times \{y_0\}$ where U is an arbitrary open set in X.

Ex. 8. Let $Y \subset X$ be open in X. Then $Z \subset Y$ is open in Y iff Z is open in X.

The result is not true if Y is not open in X

Ex. 9. Let $Y \subset X$ be closed in X. Then $Z \subset Y$ is closed in Y iff Z is closed in X.

The result is not true if Y is not closed in X

Ex. 10. Let $A := \{1/n : n \in \mathbb{N}\} \cup \{0\}$. Then the basic open sets are the singletons $\{1/n\}$ for $n \in \mathbb{N}$ and $\{1/n : n \geq n_0\} \cup \{0\}$. The latter are basic opens sets near 0 in A.

Ex. 11. $\mathbb{Z} \subset \mathbb{R}$ has discrete topology as the subspace topology.

Ex. 12. The basic open sets in $\mathbb{Q}$ with the subspace topology from $\mathbb{R}$ are of the form $(a, b)_{\mathbb{Q}} := \{x \in \mathbb{Q} : a < x < b\}$ for $a, b \in \mathbb{R}$.

Is the collection $\{(a, b)_{\mathbb{Q}} : a, b \in \mathbb{Q}\}$ a family of basic open sets in $\mathbb{Q}$?

Ex. 13. Let $A \subset X$. If the subspace topology on A is the discrete topology on A, then every $a \in A$ is an *isolated* point in X, that is, there exists an open set $U_a \ni a$ in X such that $U_a \cap A = \{a\}$.

Ex. 14. Let $Y := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ be the first quadrant in $\mathbb{R}^2$. Let $A := \{(x, y) \in Y : 0 \leq x < 1, 0 \leq y < 1\}$. Is A open in Y?

Ex. 15. Let $Y \subset X$. Let f be the restriction of the identity of X to Y. (Thus, f is th einclusion map of Y into X.) Then $f \colon (Y, \mathcal{T}_Y) \to (X, \mathcal{T})$ is continuous.

We can say more. If $\mathcal{T}'$ is a topology on Y such that $f \colon (Y, \mathcal{T}') \to (X, \mathcal{T})$ is continuous, then $\mathcal{T}_Y \subset \mathcal{T}'$. Thus, the subspace topology is the smallest topology on Y making the natural inclusion map f continuous.

Ex. 16. Let X and Y be topological spaces. Let $f \colon X \to Y$ be a (not necessarily continuous) map. Let $G(f) := \{(x, f(x)) : x \in X\} \subset X \times Y$ be the graph of f. Let $X \times Y$ be equipped with the product topology. Let $G(f) \subset X \times Y$ be given with the subspace topology. What are the basic open sets of $G(f)$?