T_pS for Two Special Classes of Surfaces

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1. T_pS where S is a graph of a smooth function

Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}$ be a smooth (at least continuously differentiable) function. Let S be the graph surface $S := \{(x, f(x) : x \in U\}$. Let $p := (a, f(a)) \in S$. We wish to identify T_pS as a subset of $\mathbb{R}^{n+1}$. There are 3 steps in this.

1. First, we show that any tangent vector in T_pS is a linear combination of vectors ∂_i where
$$\partial_i := \left(0, \ldots, 0, 1, 0, \ldots, 0, \frac{\partial f}{\partial x_i}(p)\right),$$
where 1 is at the i-th place.

2. Each of the vectors ∂_i is a tangent vector.

3. Adapting the proof of the last item, any linear combination of ∂_i's is a tangent vector to S at p so that $T_pS = \text{span}\{\partial_i : 1 \leq i \leq n\}$.

Step 1: Let $c\colon (-\varepsilon, \varepsilon) \to S$ be a differentiable curve such that $c(0) = p$. Then $c(t) \in S$ is of the form $(x_1(t), \ldots, x_n(t), f(x_1(t), \ldots, x_n(t)))$. We let $x(t) := (x_1(t), \ldots, x_n(t))$. (Note that $c \mapsto x$ sets up a bijection between curves c passing through p and curves passing through a.) We compute $c'(t)$ using the chain rule:

$$\begin{aligned} c'(t) &= \left(x_1'(t), \ldots, x_n'(t), \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x(t))x_i'(t)\right) \\ &= \sum_{i=1}^n x_i'(t)\frac{\partial f}{\partial x_i}(x(t))x_i'(t). \end{aligned}$$

Hence $c'(0) = \sum_{i=1}^n x_i'(0)\partial_i$. This completes Step 1.

Step 2: Consider the i-th 'coordinate curve' passing through a in U:

$$\gamma(t) = a + te_i = (a_1, \ldots, a_{i-1}, a_i + t, a_{i+1}, \ldots, a_n).$$

Then $c(t) := (\gamma(t), f(\gamma(t))$ is a curve through p. Note that, for $1 \leq j \leq n$, $x_j(t) = a_j + \delta_{ij} t$ so that $x_j'(t) = \delta_{ij}$. We therefore have $c_i'(0) = \partial_i$. Thus, $\partial_i \in T_pS$.

Step 3: More generally, given scalars $v_i \in \mathbb{R}$ for $1 \leq i \leq n$, let v be the vector $v := (v_1, \ldots, v_n) \in \mathbb{R}^n$. Consider the curve $\gamma(t) := a + tv$ and the corresponding curve $c(t) := (\gamma(t), f(\gamma(t)))$ in S. Then c is a differentiable curve through p with $x_j(t) = a_j + tv_j$ so that $x_j'(t) = v_j$. It follows that $c'(0) = \sum_{i=1}^n v_i \partial_i$, as seen in Step 1. Hence any linear combination of vectors ∂_i is again a tangent vector to S at p.

In view of Steps 1 and 3, we conclude that $T_pS = \text{span}\{\partial_i : 1 \leq i \leq n\}$.

The tangent plane to S at p is by definition the plane through p parallel to the tangent space T_pS and hence it is $p + T_pS$. We derive the equation of the tangent plane when $n = 2$.

In the case when $n = 2$, we let $\partial_1 = \partial_x$ and $\partial_2 = \partial_y$. Then the vector which is normal to the space spanned by ∂_x, ∂_y is given by

$$\partial_x \times \partial_y = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right).$$

Hence the tangent space T_pS is given by

$$\left\{ (x, y, z) \in \mathbb{R}^3 : (x, y, z) \cdot \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right) = 0 \right\}.$$

Recall that if W is a vector subspace given by $W := \{x \in \mathbb{R}^{n+1} : x \cdot v = 0\}$, the plane passing through p parallel to W, (that is , $p + W$) is given by $\{x \in \mathbb{R}^{n+1} : x \cdot v = p \cdot v\}$. Hence the tangent plane to S at $(a, b, f(a, b))$ is given by

$$(x, y, z) \cdot \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right) = (a, b, f(a, b)) \cdot \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right).$$

Thus the equation of the tangent plane is

$$z = (x - a)\frac{\partial f}{\partial x} + (y - b)\frac{\partial f}{\partial y} + f(a, b).$$

The geometric meaning of the existence of the derivative of f at (a, b) is the existence of a tangent plane which 'approximates' the graph of f at $(a, b, f(a, b))$. The rigorous defintion quantifies what is meant by approximation.

2. T_pS where $S = f^{-1}(0)$ is a level surface

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function. Assume that 0 is a regular value of f, that is, $\text{grad}\, f(a) \neq 0$ for all $a \in f^{-1}(0)$. Let $S := f^{-1}(0)$, called the level set of f at the level 0. We wish to show that $T_aS = (\text{grad}\, f(a))^\perp$. This is done in three steps:

1. $T_aS \subset \text{grad}\, f(a)^\perp$.
2. Any level surface S is locally a graph G of a smooth function.

3. The observation that any tangent at a to G is also a tangent to S at a so that T_aS is an n-dimensonal vector subspace of $\operatorname{grad} f(a)^\perp$.

Step 1. Let $v \in T_aS$. If c is a curve in S with $c(0) = a$ and $c'(0) = v$, then $f \circ c(t)$ is the constant function 0. Hence its derivative is zero. By chain rule, $(f \circ c)'(t) = \operatorname{grad} f(c(t)) \cdot c'(t) = 0$. In particular, $\operatorname{grad} f(a) \cdot c'(0) = 0$. Hence $v \perp \operatorname{grad} f(a)$.

Step 2. Since $\operatorname{grad} f(a) \neq 0$, we may assume, without loss of generality, that $\frac{\partial f}{\partial_{x_{n+1}}}(a) \neq 0$. By the implicit function theorem, there exists an open neighbouthood of a in $\mathbb{R}^{n+1}$, an open set $V \ni (a_1, \dots, a_n)$ in $\mathbb{R}^n$ and a smooth function $g \colon V \to \mathbb{R}$ such that $g(a_1, \dots, a_n) = a_{n+1}$ and $U \cap S = \{(x, g(x)) : x \in V\}$. We let $a' = (a_1, \dots, a_n)$. Thus, S is 'locally a graph'.

Step 3. If $c \colon (-\varepsilon, \varepsilon) \to G$ is any (smooth) curve with $c(0) = a$, then c may also be considered as a smooth curve in S passing through a, since $G = S \cap U$. In particular, by the earlier result, $c'(0) \in \mathbb{R}^{n+1}$ is a tangent vector to S. In other words, we have shown that the n-dimensional vector subspace T_aG is a vector subspace of T_aS. Since by Step 1, $T_aS \subset \operatorname{grad} f(a)^\perp$, we conclude that T_aS is an n-dimensional vector subspace of $\operatorname{grad} f(a)^\perp$ and hence both are equal. Thus, we have shown that any vector perpendicular to $\operatorname{grad} f(a)$ is a tangent vector to S at p.

A different proof which, for a given $v \perp \operatorname{grad} f(a)$, exhibits a curve c in S with the initial data $c(0) = p$ and $c'(0) = v$ is given in my article "Lagrange Multipliers—A Geometric Approach". This artciel is available in MTTS notes.