Tangents

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Abstract

All you wanted to know about tangents but could not find a convenient reference! To be more precise, this article gives you a gentle introduction to the notion of tangents so as to make your journey through Differential Geometry enjoyable.

1. A line in a real vector space V: If $u, v \in V$, then the line joining u, v is defined by

$$\ell(u, v) := \{x \in V : x = u + t(v - u), \text{ for some } t \in \mathbb{R}\}.$$

It is seen that this gives rise to the equation of a line joining two points in $\mathbb{R}^2$ or $\mathbb{R}^3$.

2. Equivalently, a line through $p \in V$ and in the direction of $0 \neq v \in V$ is given by

$$\ell(p; v) := \{x \in \mathbb{R}^n : x = p + tv \text{ for some } t \in \mathbb{R}\}.$$

v is called the direction vector of $\ell(p, v)$ and corresponds to the 'slope' of ℓ.

3. Let $c\colon (a, b) \to U \subset \mathbb{R}^n$ be a (continuously differentiable) curve. If we write $c(t) = (x_1(t), \dots, x_n(t))$, then $c'(t) = (x_1'(t), \dots, x_n'(t))$ is called the tangent (or the velocity) vector at t.

4. Why is $c'(t)$ called the tangent vector? If c is a parametrization of a standard conic section in $\mathbb{R}^2$, then the tangent line at $c(t)$ is the line through $c(t)$ in the direction of $c'(t)$. Details for circles, ellipses, parabolas and hyperbolas should be worked out to explain this.

5. If $S \subset \mathbb{R}^n$ and $p \in S$, we denote by T_pS the tangent space at p to S and define it as the collection of all tangent vectors $c'(0)$ where $c\colon (-\varepsilon, \varepsilon) \to S$ is a smooth curve with $c(0) = p$:

$$T_pS := \{v \in \mathbb{R}^n : \exists c\colon (-\varepsilon, \varepsilon) \to S \text{ with } c(0) = p \text{ and } v = c'(0)\}$$

6. As $c(t) = p$ for all t has 0 as the tangent vector, $T_pS \neq \emptyset$. Also, if $v \in T_p$, then $\lambda v \in T_pS$ for any $\lambda \in \mathbb{R}$. Thus T_pS is a subset of $\mathbb{R}^n$ which is nonempty and closed under scalar multiplication.

7. Does T_pS contain nonero vectors? Not necessarily. We have $T_x\mathbb{Q} = \{0\}$ for any $x \in \mathbb{Q}$ and $T_xC = \{0\}$ for any x in the Cantor set C.

8. If $v_1, v_2 \in T_pS$, can we conclude $v_1 + v_2 \in T_pS$? No, we cannot. If $S = \{xy = 0\}$, the union of the axes in $\mathbb{R}^2$, then $e_1, e_2 \in T_{(0,0)}S$, but $e_1 + e_2 \notin T_{(0,0)}S$.

9. Whether T_pS contains nonzero vectors or whether it is closed under (vector) addition (so that it becomes a vector space) depends on some geometric properties of S. We look at some special cases below which are very important for differential geometry and for which the tangent spaces are vector spaces.

10. If $S = U$ is an open subset of $\mathbb{R}^n$, then $T_p(S) = \mathbb{R}^n$.

11. If S is a the vector subspace $H := \{x \in \mathbb{R}^n : x \cdot a = 0\}$ for some nonzero $a \in \mathbb{R}^n$, then $T_pS = H$. If $v \in T_pS$, then $v \in H$ with a correpsonding curve c, then $c(t) \cdot a = 0$ for all t. Differentiating this equation we get $v \in H$.

    More generally, if $W := w + H$ is a plane, then $T_pW = H$.

12. Exercise: Let $W \leq \mathbb{R}^n$ be a vector subspace. Identify T_pW. The same question if S is a coset of W in $\mathbb{R}^n$.

13. Consider $S = S^{n-1} := \{x \in \mathbb{R}^n : x \cdot x = 1\}$. Then $v \in T_pS$ iff $v \perp p$. Enough to show that $v \perp p$ is of unit norm, then $v \in T_pS$. Consider the curve which is the intersection of the sphere S with the two dimensional subspace span$\{p, v\}$. It is parametrized as $t \mapsto \cos tp + \sin tv$.

    This is a special case of a more general phenomenon. See Item 15.

14. Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}$ be smooth. Let S be the "surface" in $\mathbb{R}^{n+1}$ defined as the graph of f: $S := \{(x, f(x)) : x \in U\}$. If γ is a curve in S, and if we write $\gamma(t) = (x(t), f(x(t)))$, then $c(t) := x(t)$ is a curve in U. This sets up a 1-1 correspondence between curves in S and curves in U. If $n = 2$, so that

    $$\gamma(t) = (x(t), y(t), z(t)) = (x(t), y(t), f(x(t), y(t))),$$

    then

    $$\begin{aligned}\gamma'(t) &= \left(x'(t), y'(t), \frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t)\right)\\ &= x'(t)\left(1, 0, \frac{\partial f}{\partial x}\right) + y'(t)\left(0, 1, \frac{\partial f}{\partial y}\right)\\ &= x'(t)\partial_x + y'(t)\partial_y, \text{ say.}\end{aligned}$$

    ∂_x and ∂_y are the tangent vectors to S that correspond to the curves $(x, b, f(x, b))$ and $(a, y, f(a, y))$. Also, note that

    $$\partial_x \times \partial_y = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

    the 'normal' to the tangent plane.

15. Look at a level surface $S = f^{-1}(0)$, $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, with 0 as a regular value. If γ is a curve in S, then $\gamma'(t) \perp \operatorname{grad} f(\gamma(t))$. Thus if $v \in T_pS$, then $v \perp \operatorname{grad} f(p)$. Converse is also true. It needs an implicit/inverse function theorem argument.

    If $v \perp \operatorname{grad} f(p)$ and if $\frac{\partial f}{\partial x_{n+1}}(p) \neq 0$, by implicit function theorem, there exists an open set $U \subset \mathbb{R}^n$, a smooth function $g\colon U \to \mathbb{R}$ and an open set $V \subset \mathbb{R}^{n+1}$ with $p \in V$ such that $V \cap S = \{(x, g(x)) : x \in U\}$. Now if c is a curve in U, we then have a corresponding γ in $U \cap S$. Now the images of all coordinate curves in U give rise to a set of n linearly independent tangent vectors at p. Hence $\ker Df(p) = T_pS$.

    My article "Lagrange Multipliers — A Geometric Treatment" explains this in more detail in a more general setting and shows its application to constrained maxima and minima.

16. The notion of tangent occurs even in differential calculus in a subtle way. To start with, if we wish to compute the directional derivative $D_v f(p)$ of a differentiable function, we employ the difference quotient: $\frac{f(p+tv)-f(p)}{t}$. The numerator is the composition of f with the straight line curve $\ell(t) := p + tv$. This curve has the property that $\ell(0) = p$ and $\ell'(0) = v$. Also, we note that the directional derivative is $\frac{d}{dt} f \circ \ell(t)\,|_{t=0}$. If γ is any curve with the same properties $\gamma(0) = p$ and $g'(0) = v$, then again we get (by an application of the chain rule)

    $$D_v f(p) = \frac{d}{dt} f \circ \gamma(t)\,|_{t=0}\,.$$

    In particular, we may consider the vectors v that are fed to the derivative map $Df(p)$ as the tangent vectors at p!

17. If $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ is a nonzero linear map, then its graph is a plane passing through 0. For, if $f(x) := x \cdot a$ for some $a \in \mathbb{R}^{n-1}$, then the graph $G(f) := \{(x, f(x)) : x \in \mathbb{R}^{n-1}$ is the plane given by

    $$\begin{aligned} G(f) = \{(x_1, \ldots, x_{n-1}, f(x)\} &= \{x \in \mathbb{R}^n : a_1x_1 + \cdots + a_{n-1}x_{n-1} - x_n = 0\} \\ &= \{x \in \mathbb{R}^n : x \cdot (a_1, \ldots, a_{n-1}, -1) = 0\} \end{aligned}$$

18. Let a plane W passing through the origin be given by $x \cdot a = 0$. Then the equation for the plane $p + W$, the translate of W by p, is given by $x \cdot a = p \cdot a$. (This reminds us of the high-school geometry formula: the equation of the line through (x_0, y_0) parallel to the line $ax + by = 0$ is $ax + by = ax_0 + by_0$.)

19. Now the tangent space to the graph of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is given by $(x, y, z) \cdot (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}, -1) = 0$. Hence the equation of the tangent plane to the surface through the point $(a, b, f(a, b))$ is given by

    $$(x, y, z) \cdot (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}, -1) = (a, b, f(a, b)) \cdot (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial x}, -1).$$

    That is,

    $$(x - a)\frac{\partial f}{\partial x} + (y - b)\frac{\partial f}{\partial y} + f(a, b) = z.$$

20. The importance of tangent space in differential geometry is as follows. Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $p \in U$. To know $Df(p)$ it is enough to know $Df(p)(v)$. The latter is the directional derivative $D_v f(p)$ and to compute it, we can use any curve γ such that $\gamma(0) = p$ and $\gamma'(0) = v$. (This follows by a trivial application of the chain rule.) Hence v, which is fed to $Df(p)$, can be thought of a tangent vector to U at p! The interpretation of the domain of the derivative in the last item leads us to the definition of the derivative of a map between two surfaces.

21. We now use our knowledge of T_pS of a level set to bring out the underlying geometry of the method of Lagrange multipliers. First of all look at the following examples in a geometric way.

    (a) Find the extrema of the function $g(x, y) = x$ subject to the constraint $f(x, y) := x^2 + y^2 - 1 = 0$.

    (b) Find the extrema of the function $g(x, y) = x^2 + y^2$ subject to the constraint $f(x, y) := ax + by + c = 0$

    (c) Find the extrema of the function $g(x, y) = x^2 + y^2$ subject to the constraint $f(x, y) := xy - c = 0$.

    (d) Find the extrema of the function $g(x, y) = xy$ subject to the constraint $f(x, y) := x + y - c = 0$.

    (e) Find the extrema of the function $g(x, y) = x^2 + y^2$ subject to the constraint $f(x, y) := (x/a)^2 + (y/b)^2 - 1 = 0$.

    (f) Find the extrema of the function $g(x, y) = \|p - x\|^2$ subject to the constraint $f(x) := x \cdot a - d = 0$, where $a \in \mathbb{R}^n$ is a unit vector, $p \in \mathbb{R}^n$ fixed, and $d \in \mathbb{R}$.

22. In each of the examples of the last item, the (constrained) extrema of the function g occurs at point at which both the level sets f and g meet tangentially. Reformuating this in terms of the normals, this means that at a point of extrema, we have $\operatorname{grad} g$ is a scalar multiple of the gradient of f. This is the essence of the method of Lagrange multipliers.