Taylor's Theorem

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Theorem 1 (Taylor's Theorem with Remainder)**.** *Let n and p natural numbers. Assume that $f\colon [a,b] \to \mathbb{R}$ is such that $f^{(n-1)}$ is continuous on $[a,b$ and $f^{(n)}(x)$ exists on (a,b). Then there exists $c \in (a,b)$ such that*

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n, \tag{1}$$

where

$$R_n = \frac{(b-c)^n - p(b-a)^p}{p(n-1)!}f^{(n)}(c). \tag{2}$$

In particular, when $p = n$, we get Lagrange's form of the remainder

$$R_n = \frac{(b-a)^n}{n!}f^{(n)}(c), \tag{3}$$

and when $p = 1$, we get Cauchy's form of the remainder

$$R_n = b-a)\frac{(b-c)^{n-1}}{(n-1)!}f^{(n)}(c) \tag{4}$$

$$= \frac{(b-a)^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(a+\theta(b-a)), \text{ where } 0 < \theta := \frac{(c-a)}{(b-a)} < 1. \tag{5}$$

Proof. Consider

$$F(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!}f''(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x).$$

We have for $x \in (a,b)$,

$$F'(x) = \frac{-(b-x)^{n-1}f^{(n)}(x)}{(n-1)!}. \tag{6}$$

We now define

$$g(x) = F(x) - \left(\frac{b-x}{b-a}\right)^p F(a). \tag{7}$$

The g is continuous on $[a,b]$, differentiable on (a,b) and $g(a) = 0 = g(b)$. Hence by Rolle's theorem, there exists $c \in (a,b)$ such that $g'(c) = 0$. Using the definition of g in (7) we get

$$0 = g'(c) = F'(c) + \frac{p(b-c)^{p-1}}{(b-a)^p}F(a). \tag{8}$$

Using (6) in (8) and simplifying we get

$$\frac{(b-c)^{n-1}}{(n-1)!} f^{(n)}(c) = \frac{p(b-c)^{p-1}}{(b-a)^p} F(a). \tag{9}$$

That is, $F(a) = \frac{(b-c)^n - p(b-a)^p}{p(n-1)!} f^{(n)}(c)$. This is what we set out to prove.

Lagrange's form of the remainder is obvious. If we write $c = a + \theta(b-a)$ for some $\theta \in (0,1)$, Cauchy's form of the remainder is obtained from (2). □

Theorem 2 (Binomial Series)**.** *Let $m \in \mathbb{R}$. Define*

$$\binom{m}{0} = 1 \text{ and } \binom{m}{k} := \frac{m(m-1)\cdots(m-k+1)}{k!} \text{ for } k \in \mathbb{N}.$$

Then

$$(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k = 1 + mx + \frac{m(m-1)}{2!} + \cdots, \text{ for } |x| < 1.$$

Proof. If $m \in \mathbb{N}$, this is the usual binomial theorem. In this case, the series is finite and there is no restriction on x.

Let $m \notin \mathbb{N}$. Consider $f\colon (-1,\infty) \to \mathbb{R}$ defined by $f(x) = (1+x)^m$. For $x > -1$, we have

$$f'(x) = m(1+x)^{m-1}, \ \ldots, f^{(n)}(x) = m(m-1)\cdots(m-n+1)(1+x)^{m-n}.$$

If $x = 0$, the result is trivial as $1^m = 1$. Now for $x \neq 0$, by Taylor's theorem

$$f(x) = f(0) + xf'(0) + \cdots + R_n = \sum_{k=0}^{n-1} \binom{m}{k} x^k + R_n.$$

Therefore, to prove the theorem, we need to show that, for $|x| < 1$,

$$|f(x) - \sum_{k=0}^{n-1} \binom{m}{k} x^k| = |R_n| \to 0 \text{ as } n \to \infty.$$

To prove $R_n \to 0$, we use Lagrange's form for the case 0¡x¡1.

$$|R_n| = |\frac{x^n}{n!} f^{(n)}(\theta x)| = |\binom{m}{n} x^n (1+\theta x)^{m-n}| < |\binom{m}{n} x^n|,$$

if $n > m$, since $0 < \theta < 1$. Letting $a_n := |\binom{m}{n} x^n|$, we see that $a_{n+1}/a_n = x|\frac{m-n}{n+1}| \to x$. Since $0 < x < 1$, the ratio test says that the series $\sum_n a_n$ is convergent. In particular, the n-th term $a_n \to 0$. Since $|R_n| < a_n$, it follows that $R_n \to 0$ when $0 < x < 1$.

Let us now attend to the case when $-1 < x < 0$. If we try to use the Lagrange form of the remainder we obtain the estimate

$$|R_n| < |\binom{m}{n} x^n (1+\theta x)^{m-n}| < |\binom{m}{n} x^n (1-\theta)^{m-n}|$$

if $n > m$. This is not helpful as $(1-\theta)^{m-n}$ may shoot to infinity if θ goes near 1. Let us now try Cauchy's form.

$$\begin{aligned} |R_n| &= |mx^n \binom{m-1}{n-1} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} (1+\theta x)^{m-1}| \\ &\leq |mx^n \binom{m-1}{n-1} (1+\theta x)^{m-1}|. \end{aligned} \tag{10}$$

Now, $0 < 1 + x < 1 + \theta x < 1$. Hence $|(1+\theta x)^{m-1}| < C$ for some $C > 0$. Note that C is independent of n but dependent on x.

It follows that $|x^n \binom{m-1}{n-1}| \to 0$ so that $R_n \to 0$. This completes the proof of the theorem. □

Example 3. As another application of Taylor's theorem with Lagrange form of the remainder, we now establish the sum of the standard alternating series is $\log 2$.

$$\log 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}.$$

Consider the function $f\colon (-1,\infty) \to \mathbb{R}$ defined by $f(x) := \log(1+x)$. We then have By a simple induction argument, we see that

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}.$$

Hence the Taylor series of f around 0 is

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$

We now wish to show that the series is convergent at $x = 1$. This means that we need to show that the sum of the series at $x = 1$ is convergent. We therefore take $a = 0$, $b = 1$ in Taylor's theorem and show that the remainder term (in Lagrange's form) $R_n \to 0$. For each $n \in \mathbb{N}$, there exists $c_n \in (0,1)$ such that

$$R_n := \frac{f^{(n)}(c_n)}{n!} = \frac{(-1)^{n-1}(n-1)!}{n!(1+c_n)^n}.$$

We have an obvious estimate:

$$|R_n| = |\frac{(-1)^{n-1}(n-1)!}{n!(1+c_n)^n}| \leq |\frac{1}{n(1+c_n)^n}| \leq \frac{1}{n}.$$

Hence $\log 2 = f(1) = \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!} 1^n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

We now prove Taylor's theorem with the integral form of the remainder. In practice it is most often easier to estimate integrals.

Theorem 4 (Taylor's Theorem with Integral Form of the Remainder)**.** *Let f be function on an interval J with $f^{(n)}$ continuous on J. Let $a, b \in J$. Then*

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1} + R_n \tag{11}$$

where

$$R_n = \int_a^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t)\, dt. \tag{12}$$

Proof. We begin with

$$f(b) = f(a) + \int_a^b f'(t)\, dt.$$

We apply integration by parts formula $\int_a^b u dv = uv|_a^b - \int u' dv$ to the integral where $u(t) = f'(t)$ and $v = -(b-t)$. (Note the *non-obvious* choice of v!) We get

$$\int_a^b f'(t) dt = -f'(t)(b-t)|_a^b + \int_a^b f''(t)(b-t)\, dt.$$

Hence we get

$$f(b) = f(a) + f'(a)(b-a) + \int_a^b f''(t)(b-t)\, dt.$$

We again apply integration by parts to the integral where $u(t) = f''(t)$ and $v(t) = -(b-t)^2/2$. We obtain

$$\int_a^b f''(t)(b-t)\, dt = f''(t)\frac{(b-t)^2}{2}|_a^b + \int_a^b f^{(3)}(t)((b-t)^2/2)\, dt.$$

Hence

$$f(b) = f(a) + f'(a)(b-a) + f''(a)\frac{(b-a)^2}{2} + \int_a^b f^{(3)}(t)\frac{(b-t)^2}{2}\, dt.$$

Assume that the formula for R_k is true:

$$R_k = \int_a^b \frac{(b-t)^{k-1}}{(k-1)!} f^{(k)}(t)\, dt.$$

By induction we let

$$u(t) = f^{(k)}(t) \text{ and } v(t) = -(b-t)^{k-1}$$

and apply integration by parts. We get

$$\int_a^b f^{(k)}(t)\frac{(b-t)^{k-1}}{(k-1)!}\, dt = f^{(k)}(t)\frac{(b-a)^k}{k!} + \int_a^b \frac{(b-t)^k}{k!} f^{(k+1)}(t)\, dt.$$

By induction the formula for R_n is obtained. ☐

Recall the mean value theorem for the Riemann integral:

Theorem 5. *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then there exists $c \in (a,b)$ such that*

$$(b-a)f(c) = \int_a^b f(t)\, dt, \text{ that is, } f(c) = \frac{1}{b-a}\int_a^b f(t)\, dt.$$

We use this to deduce as a corollary Cauchy's form the remainder in the Taylor's theorem. integrals. Applying the mean value theorem for integrals to (12), we conclude that there exists $c \in (a, b)$ such that

$$R_n = (b-a)\frac{(b-c)^{n-1}}{(n-1)!} f^{(n)}(c),$$

which is Cauchy's form of the remainder.

Example 6. We now apply the integral form of the remainder to arrive at the result (Theorem 2) on binomial series.

Assume that m is not a non-negative integer. Then $a_n := \binom{m}{n} \neq 0$. Since

$$\frac{a_{n+1}}{a_n} = \frac{m-n}{n+1} \to 1,$$

the binomial series

$$(1+x)^m = 1 + \sum_{n=1}^{\infty} \binom{m}{n} \frac{x^n}{n!}$$

has radius of convergence 1. Similarly, the series $\sum_n n\binom{m}{n} x^n$ is convergent for $|x| < 1$. Hence

$$n\binom{m}{n} x^n \to 0 \text{ for } |x| < 1. \tag{13}$$

We now estimate the remainder term using (13). We have, for $0 < |x| < 1$,

$$\begin{aligned} R_n &= \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} n! \binom{m}{n} (1+t)^{m-n}\, dt \\ &= \int_0^x n\binom{m}{n} \left(\frac{x-t}{1+t}\right)^{n-1} (1+t)^{n-1}\, dt. \end{aligned} \tag{14}$$

We claim that

$$|\frac{x-t}{1+t}| \leq |x| \text{ for } -1 < x \leq t \leq 0 \text{ or } 0 \leq t \leq x < 1.$$

Write $t = sx$ for some $0 \leq s \leq 1$. Then

$$|\frac{x-t}{1+t}| = |\frac{x-sx}{1+st}| = |x||\frac{1-s}{1+ts}| \leq |x|,$$

since $1 + sx \geq 1 - s$. Thus the integrand in (14) is bounded by

$$|n\binom{m}{n} \left(\frac{x-t}{1+t}\right)^{n-1} (1+t)^{n-1}\, dt| \leq n|\binom{m}{n}||x|^{n-1}(1+t)^{m-1}.$$

Therefore, we obtain

$$|R_n(x)| \leq n|\binom{m}{n}||x|^{n-1} \int_{-|x|}^{|x|} (1+t)^{m-1}. \leq Cn|\binom{m}{n}||x|^{n-1},$$

which goes to 0 in view of (14).

This completes the proof of the fact that the binomial series converges to $(1+x)^m$.