Existence of Continuous Functions

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If $x \neq y$ are two distinct points of a space X, is there a continuous function $f \colon X \to \mathbb{R}$ such that $f(x) \neq f(y)$? In general, this may not be true. There may not exist continuous functions on the given space other than the constants. For each pair of distinct points, if there is an $f \in C(X, \mathbb{R})$ with $f(x) \neq f(y)$, we say that the family $C(X, \mathbb{R})$ separates points. This is the reason for defining the completely regular and normal spaces which ensures plenty of continuous functions. One kind of spaces for which existence of an abundance of continuous real valued functions is assured is the class of metric spaces. We shall look at them first.

1 Case of Metric Spaces

The crucial fact here is the simple observation: If (X, d) is a metric space and $x \in X$, then the function $f_x(y) := d(x, y)$ is continuous on X. For, by triangle inequality we have

$$|f_x(y) - f_x(z)| = |d(x, y) - d(x, z)| \leq d(y, z).$$

Thus $\{f_x : x \in X\}$ is a separating family of continuous functions on X. More generally, we have

Lemma 1. *Let A be any nonempty subset of a metric space (X, d). Define $d(x, A) \equiv d_A(x) := \inf\{d(x, a) : a \in A\}$. Then $|d_A(x) - d_A(y)| \leq d(x, y)$ and hence d_A is uniformly continuous on X.*

Proof. Observe from the triangle inequality $d(x, a) \leq d(x, y) + d(y, a)$, we obtain

$$\begin{aligned} \inf_{a \in A} d(x, a) &\leq \inf_{a \in A} \left(d(x, y) + d(y, a)\right) \\ &= d(x, y) + \inf_{a \in A} d(y, a), \end{aligned}$$

so that $d_A(x) \leq d(x, y) + d_A(y)$. Thus, $d_a(x) - d_A(y) \leq d(x, y)$. Interchanging x and y yields the result. ☐

Ex. 2. $d_A(x) = 0$ iff x is a limit point of A. Hence if A is a closed set then $d(x, A) = 0$ iff $x \in A$.

Lemma 3 (Urysohn's Lemma for Metric Spaces)**.** *Let A and B be nonempty disjoint closed subsets of a metric space X. Then there exists an $f \in C(X, \mathbb{R})$ such that $0 \leq f(x) \leq 1$ for $x \in X$ and $f = 0$ on A and $f = 1$ on B.*

Proof. Note that for any $x \in X$, $d(x, A) + d(x, B) \neq 0$. For, if it were so, then $d(x, A) = 0 = d(x, B)$. Since A and B are closed $x \in A$ and $x \in B$ by the last exercise. This contradicts our hypothesis that $A \cap B = \emptyset$.

The function $f(x) := \frac{d(x,A)}{d(x,A)+d(x,B)}$ meets our requirements. □

Theorem 4 (Tietze extension theorem for metric spaces)**.** *Let Y be a closed subspace of a metric space (X, d). Let $f \colon Y \to \mathbb{R}$ be a bounded continuous function. Then there exists a continuous function $g \colon X \to \mathbb{R}$ such that $g(y) = f(y)$ for all $y \in Y$ and*

$$\inf\{g(x) : x \in X\} = \inf\{f(y) : y \in Y\}, \qquad \sup\{g(x) : x \in X\} = \sup\{f(y) : y \in Y\}.$$

Proof. Assume that $f \geq 0$. Consider the function $M_x(r) := \sup\{f(y) : y \in Y \cap B(x, r)\}$. Then, for each $x \in X$, M_x is real valued, bounded and monotonic increasing in r. Hence it is Riemann integrable as a function of r over any finite interval. Let $\delta(x) := d(x, Y)$. Note that $\delta(x) > 0$ iff $x \notin Y$. We define g by $g(x) = f(x)$ if $x \in Y$ and if $x \notin Y$,

$$g(x) := \frac{1}{\delta(x)} \int_{\delta(x)}^{2\delta(x)} M_x(r)\, dr.$$

We claim that g is continuous at any $y \in Y$. If $x \notin Y$, then

$$\min_{Y \cap B(y,3\delta)} f \leq g(x) \leq \max_{Y \cap B(y,3\delta)} f, \qquad \text{where } \delta := d(x, y).$$

Since $3\delta \to 0$ as $x \to y$ (with $x \notin Y$), it follows that, for any $\varepsilon > 0$, $|g(x) - f(y)| < \varepsilon$ if $x \notin Y$ and $\delta(x) < \delta_0$ for δ_0 sufficiently small. On the other hand, $|g(x) - f(y)| = |f(x) - f(y)| < \varepsilon$ if $x \in Y$ and $d(x, y) < \delta_1$ by continuity of f on Y. Thus g is continuous at y.

Consider next the continuity of g at $z \notin Y$. Let x be any point in X with $d(x, z) < d(z, Y)/3$. Let $\alpha := d(x, z)$. Then $2\alpha < d(x, Y)$. For, otherwise, $d(x, Y) \leq 2\alpha$ so that $d(z, Y) \leq d(z, x) + d(x, Y) < 3\alpha$. Hence $\alpha > d(z, X)/3$, contradicting our assumption on x. Since $|\delta(x) - \delta(z)| \leq d(x, z) = \alpha$ (by Lemma 1), $M_z(r) \geq M_x(r - \alpha)$ as $B(x, r - \alpha) \subset B(z, r)$

and $M_z(r) \geq 0$, we have

$$\begin{aligned}
g(x) - g(z) &= \frac{1}{\delta(x)} \int_{\delta(x)}^{2\delta(x)} M_x(r)\,dr - \frac{1}{\delta(z)} \int_{\delta(z)}^{2\delta(z)} M_z(r)\,dr \\
&\leq \frac{1}{\delta(x)} \int_{\delta(x)}^{2\delta(x)} M_x(r)\,dr - \frac{1}{\delta(x)+\alpha} \int_{\delta(x)+\alpha}^{2\delta(x)-2\alpha} M_x(r-\alpha)\,dr \\
&= \frac{1}{\delta(x)} \int_{\delta(x)}^{2\delta(x)-3\alpha} M_x(r)\,dr + \frac{1}{\delta(x)} \int_{2\delta(x)-3\alpha}^{2\delta(x)} M_x(r)\,dr \\
&\quad - \frac{1}{\delta(x)+\alpha} \int_{\delta(x)}^{2\delta(x)-3\alpha} M_x(s)\,ds, \text{ using a change of variable} \\
&= \frac{\alpha}{\delta(x)[\delta(x)+\alpha]} \int_{\delta(x)}^{2\delta(x)-3\alpha} M_x(r)\,dr + \frac{1}{\delta(x)} \int_{2\delta(x)-3\alpha}^{2\delta(x)} M_x(r)\,dr \\
&\leq \frac{4M\alpha}{\delta(x)},
\end{aligned}$$

where $M = \sup_Y f$. A similar inequality holds with x and z interchanged. Hence $g(x) \to g(z)$ as $x \to z$. One easily checks that g is as desired.

To treat the general case, let $m := \inf_Y f$. Consider $F := f - m$. Apply the first case to F to get a continuous extension G. Then $g := G + c$ is as required. □

We shall use Weierstrass approximation theorem to give a proof of Tietze theorem for $\mathbb{R}^n$.

Proof. (of Tietze theorem for $\mathbb{R}^n$.) Let us prove the result when the closed set is compact. So, we assume that $f\colon K \to \mathbb{R}$ is a continuous function on a compact subset of $\mathbb{R}^n$. By Weierstrass approximation theorem, for each $k \in \mathbb{Z}_+$, there exists a polynomial p_k such that $|f(x) - p(x)| < 2^{-k-2}$ for all $x \in K$. We let $q_0 = p_0$ and $q_k := p_k - p_{k-1}$. Then $p_k = \sum_{i=1}^k q_i$ and $\sum q_k$ converges uniformly to f on K.

Let $M := \sup\{|f(x)| : x \in K\}$. Then $|p_0(x)| \leq 2^{-2} + M$ for $x \in K$. Also, $|q_k(x)| < 2^{-k}$ for $k \geq 1$ and $x \in K$. We let

$$\begin{aligned}
h_0 &:= \max\{-2^{-2} - M, \min\{q_0, 2^{-2} + M\}\}, \\
h_k &:= \max\{-2^{-k}, \min\{q_k, 2^{-k}\}\}, \qquad \text{for } k \geq 1.
\end{aligned}$$

Then $h_k(x) = q_k(x)$ for $x \in K$, h_k is continuous on $\mathbb{R}^n$ and $|h_k(x)| \leq 2^{-k}$ for $x \in \mathbb{R}^n$ and for all k. Hence $\sum h_k$ converges uniformly on $\mathbb{R}^n$ to a continuous function h. Then h is continuous and $h(x) = f(x)$ for $x \in K$.

We now extend to result if the subset K is any arbitrary closed subset. If K is bounded the result follows from the previous paragraph. So, we assume that K is not bounded. Let $k \in \mathbb{N}$ be such that $B[0,k] \cap K$ is nonempty. Let f_k be the restriction of f to this nonempty compact set. Then there exists a continuous function h_k on $\mathbb{R}^n$ which extends f_k. Define

$$g_k(x) := \begin{cases} h_k(x), & \text{if } x \in B[0,k] \\ f(x), & \text{if } x \in K \cap B[0,k+1]. \end{cases}$$

Then g_k is continuous on the compact set $B[0,k] \cup (K \cap B[0,k+1])$. There is an extension h_{k+1} on $\mathbb{R}^n$. Let

$$g_{k+1}(x) := \begin{cases} h_{k+1}(x), & \text{if } x \in B[0,k+1] \\ f(x), & \text{if } x \in K \cap B[0,k+2]. \end{cases}$$

Continuing in this way, we obtain a sequence (g_m) whose domains are increasing to $\mathbb{R}^n$. Define $g(x) := g_m(x)$ if $x \in B[0,m]$. Then g is an extension of f. □

2 Normal Spaces

Lemma 5. *A space X is a normal space iff for each closed set F and an open set V containing F there exists an open set U such that $F \subset U \subset \overline{U} \subset V$.*

Proof. Let X be normal and F, V as above. Then F and $X \setminus V$ are disjoint closed sets. By normality of X there exist open sets U and W such that $F \subset U$ and $X \setminus V \subset W$ and $U \cap W = \emptyset$. Since $U \subset X \setminus W$ and $X \setminus W$ is closed, we see that $\overline{U} \subset X \setminus W \subset V$. Thus U is as required. The converse is left as an exercise. □

Ex. 6. Recall that a dyadic rational is a rational number of the form $p/2^n$, where p and n are integers. Show that the set of dyadic rationals is dense in $\mathbb{R}$.

Lemma 7. *urys2 Let X be a normal space. If A and B are closed subsets of X, for each dyadic rational $r = k2^{-n} \in (0,1]$, there is an open set U_r with the following properties: (i) $A \subset U_r \subset X \setminus B$, (ii) $\overline{U}_r \subset U_s$ for $r < s$.*

Proof. Let $U_1 := X \setminus B$. By the last lemma, there exist disjoint open sets V and W such that $A \subset V$ and $B \subset W$. Let $U_{1/2} = V$. Then, since $X \setminus W$ is closed, we have

$$A \subset U_{1/2} \subset \overline{U}_{1/2} \subset X \setminus W \subset X \setminus B = U_1.$$

Applying the same lemma once again to the open set $U_{1/2}$ containing A and to the open set U_1 containing $\overline{U}_{1/2}$, we get open sets $U_{1/4}$ and $U_{3/4}$ such that

$$A \subset U_{1/4} \subset \overline{U}_{1/4} \subset U_{1/2} \subset \overline{U}_{1/2} \subset A \subset U_{3/4} \subset \overline{U}_{3/4} \subset V.$$

Continuing this manner, we construct, for each dyadic rational $r \in (0,1)$, an open set U_r with the following properties:
(i) $\overline{U}_r \subset U_s$, $0 < r < s \leq 1$.
(ii) $A \subset U_r$, $0 < r \leq 1$. (iii) $U_r \subset U_1$, $0 < r \leq 1$.

More formally, we proceed as follows. We select U_r for $r = k2^{-n}$ by induction on n. Assume that we have chosen U_r for $r = k2^{-n}$, $0 < k < 2^n$, $1 \leq n \leq N-1$. To find U_r for $r = (2j+1)2^{-N}$, $0 \leq j < 2^{N-1}$, observe that $\overline{U}_{j2^{1-N}}$ and $X \setminus U_{(j+1)2^{1-N}}$ are disjoint closed sets. So once again appealing to the last lemma, we can choose an open set U_r such that

$$\overline{U}_{j2^{1-N}} \subset U_r \subset \overline{U}_r \subset U_{(j+1)2^{1-N}}.$$

These U_r's are as desired. □

Theorem 8. Urysohn's Lemma. *A space X is a normal space iff the following is true: For any two disjoint closed subsets A and B of X there exists a continuous function $f\colon X \to [0,1]$ such that $f = 0$ on A and $f = 1$ on B.*

Proof. Let U_r's be as in the last lemma. We define the function f so that the sets ∂U_r are the level sets of f for the value r. We achieve this by defining

$$f(x) = \begin{cases} 0, & x \in U_r \text{ for all } r \\ \sup\{r : x \notin U_r\}, & \text{otherwise.} \end{cases}$$

Clearly, $0 \leq f \leq 1$, $f = 0$ on A and $f = 1$ on B. We need only establish the continuity of f.

Let $x \in X$ be such that $0 < f(x) < 1$. Let $\varepsilon > 0$. Choose dyadic rationals r and s in $(0,1)$ such that $f(x) - \varepsilon < r < f(x) < s < f(x) + \varepsilon$. Then $x \notin U_t$ for dyadic rationals $t \in (r, f(x))$. By (i), $x \notin \overline{U}_r$. On the other hand $x \in U_s$. Hence $W = U_s \setminus \overline{U}_r$ is an open neighbourhood of x. If $y \in W$, then from the definition of f we see that $r \leq f(y) \leq s$. In particular, $|f(y) - f(x)| < \varepsilon$ for $y \in W$. Thus f is continuous at x. The cases when $f(x) = 0$ or 1 are easier and left to the reader. □

Lemma 9. *Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces. Let $T\colon \mathbf{X} \to \mathbf{Y}$ be a bounded linear map. Assume that for $y_0 \in \mathbf{Y}$ there exist constants M and $r \in (0,1)$ such that there exists $x \in \mathbf{X}$ such that $\|x\| \leq M\|y_0\|$ and $\|y_0 - Tx\| \leq r\|y\|$. Then there exists $z \in \mathbf{X}$ such that $Tz = y_0$ with $\|z\| \leq M/(1-r)$.*

Proof. Let $y \in \mathbf{Y}$ be given. We may assume without loss of generality that $\|y\| = 1$. Given $y \in \mathbf{Y}$ let $z_1 = x$ as given in the lemma. For $y_0 = y - Tz_1$, we can find a $z_2 \in \mathbf{X}$ such that $\|z_2\| \leq M\|y - Tz_1\| \leq Mr$ and $\|y - Tz_1 - Tz_2\| \leq r\|y - Tz_1\| \leq r^2$. Proceeding by induction, we get a sequence (z_n) in $\mathbf{X}$ such that (i) $\|z_n\| \leq mr^{n-1}$ and (ii) $\|y - \sum_{i=1}^n Tz_i\| \leq r^n$. The series $\sum_{n=1}^\infty z_n$ converges to an element $z \in \mathbf{X}$. We have $Tz = y_0$. □

Theorem 10 (Tietze Extension Theorem)**.** *Let X be a normal space and Y a closed subset of X. Let $f \in \mathbf{Y} := C_b(Y, \mathbb{R})$. Then there exists a $g \in \mathbf{X} := C_b(X, \mathbb{R})$ such that $g(y) = f(y)$ for all $y \in Y$ and $\sup\{g(x) : x \in X\} = \sup\{f(y) : y \in Y\}$.*

Proof. Let $T\colon \mathbf{X} \to \mathbf{Y}$ denote the restriction map $g \mapsto g_{|_Y}$. We show that T satisfies the hypothesis of the previous lemma. Without loss of generality, assume that $|f(y)| \leq 1$ for all $y \in Y$. Let $A := f^{-1}([-1, -1/3])$ and $B := f^{-1}([1/3, 1])$. Then A and B are closed in Y and hence in X. By Urysohn's lemma, there exists a $g \in \mathbf{X}$ such that $|g(x)| \leq 1/3$ for $x \in X$ and $g = -1/3$ on A and $g = 1/3$ on B. One easily checks that $\|Tg - f\|_{\mathbf{X}} \leq 1/3$. If we take $M = 1/3$ and $r = 2/3$, then T satisfies the previous lemma. Note that the assertion about the equality of the norms is also obtained. □

Ex. 11. Let X be a normal space and F a closed subset. Assume that $f\colon F \to (-R, R)$ be a continuous function. Then f can be extended to a continuous function from X to $(-R, R)$. *Hint:* You may need Urysohn's lemma.

Ex. 12. Let X be a normal space and F a closed subset. Assume that $f\colon F \to \mathbb{R}$ be a continuous function. Then f can be extended to a continuous function from X to $\mathbb{R}$. *Hint:* $\mathbb{R}$ is homeomorphic to $(-1, 1)$.

Ex. 13. Assuming Tietze extension theorem, prove Urysohn's lemma.

Ex. 14. Let A be a closed subset of a normal space X. Let $f\colon A \to S^n$ be continuous. Show that there exists an open set $U \supset A$ (U depends on f) and an extension g of f to U.

Ex. 15. Show that with the notation of Exer. 14 that f may not extend to all of X. *Hint:* What happens (i) if $n = 0$ and X is connected or (ii) if $X := B[0,1] \subset \mathbb{R}^{n+1}$, $A := S^n$ and f is the identity?