Generating Topologies— A Unified View of Subspace, Product and Quotient Topologies

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Many constructions of new topologies in point set topology arise out of the following type of questions.

(i) Let X be a topological space and Y a nonempty set. Assume that a map $f\colon X \to Y$ is given. Can one find a topology $\mathcal{T}_Y$ such that the function $f\colon X \to (Y, \mathcal{T}_Y)$ is continuous.

(ii) The situation is reversed now. Assume that X is a nonempty set and Y a topological space. We are given a map $g\colon X \to Y$. We wish to endow X with a topology $\mathcal{T}_X$ so that $g\colon (X, \mathcal{T}_X) \to Y$ becomes continuous.

Both the questions have trivial answers. In case (i), we may take the topology on Y to be the indiscrete topology $\mathcal{T}_Y := \{\emptyset, Y\}$. In case (ii), we can endow X with the discrete topology. If we look at the questions a little more closely, it transpires that we need to ask for 'the largest topology' on Y in Case (i) while in Case (ii), we should go for 'the smallest topology' on X so that the respective maps (f or g) become continuous.

These questions and their generalizations occur very naturally in various contexts in topology. In abstract setup, we have 'very natural' maps from a set into another set and one of them is topological space. We are looking for an 'optimal' topology on the set with no topology. Let us look at two such natural instances.

Example 1. Let X be a topological space. Let $\sim$ be an equivalence relation on X. Let $Y := X/\sim$ be the set of equivalence classes of $\sim$ in X. We then have a natural map, namely, the quotient map $\pi : x \mapsto [x]$, that is, each $x \in X$ is mapped to its equivalence class.

Example 2. If X is a subset of a topological space Y, we have the natural map of inclusion of X into Y, namely, g is the restriction of the identity map Y to X. Here we are looking for the 'smallest topology' on X so that g is continuous.

We now return to answer the general questions posed above.

Case (i). If $\mathcal{T}_Y$ is a topology on Y so that f is continuous, then for any $V \in \mathcal{T}_Y$, the set $f^{-1}(V)$ must be open in X. This suggests us the following definition. We say a subset $V \subset Y$ is open if and only if $f^{-1}(V)$ is open in X. If we set

$$\mathcal{T}_Y := \{V \subset Y : f^{-1}(V) \text{ is open in } X\},$$

then it is easily verified that $\mathcal{T}_Y$ is a topology on Y such that $f\colon X \to (Y, \mathcal{T}_Y)$ is continuous. It is also the largest topology with this property. For, if $\mathcal{T}'_Y$ is another such topology and if $W \in \mathcal{T}'_Y$, then $f^{-1}(W)$ must be open in X. Hence, $W \in \mathcal{T}_Y$.

Case (ii). Reasoning as in Case (i), we arrive at the following.

$$\mathcal{T}_X := \{U \subset X : \text{ there exists an open set } V \subset Y \text{ such that } U = f^{-1}(V)\}.$$

It is easy to see that $\mathcal{T}_X$ is a topology on X such that g becomes continuous. It is the smallest topology with this property. For, if $\mathcal{T}'_X$ is another topology with this property, then for any open set $V \subset Y$, the set $f^{-1}(V)$ must be in $\mathcal{T}'_X$. Thus, any arbitrary subset of $\mathcal{T}_X$ belongs to $\mathcal{T}'_X$.

Let us apply these constructions to the specific examples cited above. In Example 1, according to our construction, a set $U \subset X$ is open iff there exists an open set $V \subset Y$ such that $U = g^{-1}(V)$. Since g is Id_Y restricted to X, we see that $g^{-1}(V) = V \cap X$. Thus the open sets in X are all of the open $V \cap X$, as V varies over all open sets in Y. This topology is known as the subspace topology of X!

In Example 2, if $\pi\colon X \to X/\sim$ is the quotient map, then $V \subset Y$ is open iff $\pi^{-1}(V)$ is open in X. This topology is known as the quotient topology on the quotient set $X/\sim$.

Now that we have created these new objects, how do we work with them? The answer is provided by the so-called universal mapping properties. Before stating them precisely, let us see what kind of working knowledge we need about these topologies. If X and Y are given as in Case (i), and we define the topology on Y as above, the natural questions would be: if Z is a topological space and if we are given a map h either from Z to Y or from Y to Z, how do we know the map h is continuous. The Universal mapping property gives a definitive answer to precisely one of the question by transferring the onus of proving the continuity of h to that of a 'natural composite' map. Note that if $h\colon Y \to Z$ is a map, then the composite $h \circ f\colon X \to Z$ makes sense. Universal mapping property of the topology on Y says that if $h\colon Y \to Z$ is a map, then h is continuous iff the natural composite $h \circ f\colon X \to Z$ is continuous. In Case (ii), the natural composite map is from $Z \to X$ followed by the given map g from X to Y. Universal mapping property of the topology on X says that if $h\colon Z \to X$ is a map, then h is continuous iff the natural composite $g \circ h\colon Z \to Y$ is continuous. These are easily proved and we prove these and more as Theorem after generalizing the Cases (i) and (ii). Note that we keep mum on the continuity $h\colon Z \to Y$ in Case (i) and that of $h\colon Z \to X$!

Thus we have dealt with both the Cases rather easily and satisfactorily. We can generalize these questions further and at least one of them is of immense interest.

Case I. Let $\{X_i : i \in I\}$ be a family of topological spaces and Y a nonempty set. Assume that we are given maps $f_i\colon X_i \to Y$ for each $i \in I$. We are interested in finding a topology on Y so that each of the maps f_i is continuous. As pointed out earlier, we need to look for the largest topology on Y with this property. Again arguing as in Case (i), we arrive at the following collection of sets which must be declared to be open sets in Y:

$$\mathcal{O} := \{V \subset Y : f_i^{-1}(V) \text{ is open for some } i \in I\}.$$

Unfortunately this time, this collection need not be a topology on Y! Before we see how to resolve this difficulty, let us consider

Case II. Let now X be a set and Y_i be topological spaces for $i \in I$. Let $f_i\colon X \to Y_i$ be given. We look for the smallest topology on X so that the maps f_i are continuous. Proceeding as earlier, we are led to conclude that any topology on X which makes f_i continuous must contain the collection

$$\mathcal{O} := \{U \subset X : f_i^{-1}(V_i) \text{ for some open set } V_i \subset Y_i, \text{ where } i \in I\}.$$

Once again there is no guarantee that this collection is a topology on X.

Thus, in both the cases, we are faced with the following type of problem. We have a set X and we have a collection $\mathcal{O}$ of subsets of X and we are looking for the smallest topology $\mathcal{T}$ on X which will contain $\mathcal{O}$. In theory, there is a smallest topology containing $\mathcal{O}$, namely, the intersection of all topologies which contain $\mathcal{O}$. (Note that $P(X)$, the power set of X is a topology on X which contains $\mathcal{O}$ and hence we are not taking the intersection over an empty collection!) What we would like to have is a practical way of dealing with this topology. Here is a neat description of the topology.

Definition 3. Given a collection $\mathcal{O} \subset P(X)$, we say a subset $U \subset X$ is open if given $x \in U$, there exists a finite collection $G_1, \ldots, G_n$ of members of $\mathcal{O}$ such that

$$x \in G_1 \cap \cdots \cap G_n \subset U.$$

It is an easy exercise to show that the collection $\mathcal{T}$ of all open sets (according to this definition) is a topology on X which contains $\mathcal{O}$. It is also clear that it is the smallest topology that contains $\mathcal{O}$. This topology is said to be the topology generated by $\mathcal{O}$.

Remark 4. It is not hard to motivate this definition. We shall relegate the motivation to the end of this article. It is more important to see this construction of the topology in a concrete context.

Example 5. Let X_i be topological spaces, $i \in I$. Let $X := \prod_{i\in I} X_i$be the Cartesian product of the sets X_i. We then have natural maps $\pi\colon X \to X_i$, the projections on the i-th factor: $\pi_i(x) = x_i$. (Recall that

$$X := \{x\colon I \to \cup_{i\in I} X_i \text{ such that } x(i) \in X_i\}.$$

For $x \in X$, x_i stands for $x(i)$.) Then any set in the collection $\mathcal{O}$ of Case II is of the form $\pi_i^{-1}(U_i)$ where U_i is an open subset of X_i and $i \in I$. Note that $\pi^{-1}(U_i) := \prod_{j\in I} G_j$ where $G_j = X_j$ for $j \neq i$ and $G_i = U_i$. Thus a subset $U \subset X$ is open iff for each $x \in U$, we can find $i_1, \ldots, i_n \in I$ and open sets $U_{i_k} \subset X_{i_k}$ for $1 \le k \le n$ such that

$$x \in \prod_j G_j \subset U, \text{ where } G_j = X_j \text{ for } j \neq i_k, \text{ and } G_{i_k} = U_{i_k}, 1 \le k \le n.$$

(Note that $\prod_j G_j = \cap_k \pi_{i_k}^{-1}(U_{i_k})$.) This topology on X is known as the product topology. To be able to work with this, we must have a universal mapping property of this topology. What is it? The only natural composite maps are of the form $Y \to X$ followed by $X \to X_i$. Thus, we may predict that $f\colon Y \to X$ is continuous iff $\pi_i \circ f\colon Y \to X_i$ are continuous for all $i \in I$. These adumbrations are formulated precisely in the next theorem and explained in detail in the couple of remarks that follow the theorem. The topologies constructed above in Cases (i)-(ii) and Case (II) are referred to as the topologies generated by the respective maps whose continuity was sought after.

Theorem 6 (Universal Mapping Property)**.**
(1.) *Let* $f : X \to Y$ *be a map from a set* X *to a topological space* Y*. Let* X *be given the topology generated by* f*. Then a function* $h: Z \to X$ *is continuous iff* $f \circ h: Z \to Y$ *is continuous.*

(2.) *Let* $g: X \to Y$ *be a map from a topological space* X *to a set* Y*. Let* Y *be endowed with the topology generated by* g*. Then a map* $h: Y \to Z$ *is continuous iff the map* $h \circ g: X \to Z$ *is continuous.*

(3.) *Let* $\pi_i : X \to X_i$ *be maps from the set* X *to topological spaces* X_i *for* $i \in I$*. Let* X *be given the topology generated by* π_i*'s. Then a map* $h: Y \to X$ *is continuous iff the maps* $\pi_i \circ h: Z \to X_i$ *are continuous.*

Proof. Let us prove (1) as a sample, as the proofs are all similar and easy. To prove the nontrivial part, let us assume that the map $f \circ h$ is continuous. Let $U \subset X$ be open. We need to show that $h^{-1}(U)$ is open in Z. By the very definition of the topology on X, there exists an open set $V \subset Y$ such that $U = f^{-1}(V)$. Now

$$h^{-1}(U) = h^{-1}(f^{-1}(V)) = (f \circ h)^{-1}(V),$$

which is open by the continuity of $f \circ h$.

To prove (2), let W be open in Z. We need to show that $h^{-1}(W)$ is open in Y. By the continuity of $h \circ g$, the set

$$(h \circ g)^{-1}(W) = g^{-1}(h^{-1}(W))$$

is open in X. By the definition of topology on Y, the subset $h^{-1}(W)$ is open.

To prove (3), we observe that it is enough to show that $h^{-1}(U)$ is open for

$$U \in \mathcal{O} := \{U : \pi_i^{-1}(V_i) \text{ for an open } V_i \subset X_i \text{ for some } i \in I\}.$$

(Prove this. Or, see Remark 11.) If $U = \pi_i^{-1}(V)$, we have $h^{-1}(U) := h^{-1}\left(\pi_i^{-1}(U)\right) = (\pi_i \circ h)^{-1}(U)$ is open, by the continuity of $\pi_i \circ h$. $\square$

Remark 7. The most important thing to observe in the theorem is that the problem of establishing continuity of a map either from or to a newly constructed space is reduced to showing the continuity of a 'natural composite map' between the 'known spaces'. Go back to the statement and understand this remark. Also, go through the next remark.

Remark 8. Let us explicate the theorem in the concrete situations.

If $f: X \to Y$ is the inclusion map (that is, the restriction of the identity of Y to X) of a subset X into a topological space Y, then (1) of the theorem says that a map $h: Z \to X$ is continuous iff we think of h as a map form the space Z to Y (taking values only in X) is continuous.

Let X be a topological space, $\sim$ be an equivalence relation on X and $Y = X/\sim$ be the quotient set with the quotient map $\pi: X \to Y$. Then a map $h: Y \to Z$ form the quotient space Y to a space Z is continuous iff the map $h \circ \pi: X \to Z$ is continuous.

Let $X := \prod_{i \in I} X_i$ is the Cartesian product of topological spaces with the product topology as in Example 5. Then a map $h\colon Z \to X$ can be written as $h(z) = (h_i(z))$ where the *coordinate maps* $h_i(z) := \pi_i \circ h(z)$. Thus $h\colon Z \to X$ is continuous iff the coordinate maps h_i are continuous.

To complete the story we should say something about the way $\mathcal{B}$ was got out of $\mathcal{O}$. We start with the following definition.

Definition 9. Let $(X, \mathcal{T})$ be a topological space. Then a collection $\mathcal{B}$ of open sets is called a *base* for the topology if it satisfies the following two conditions:
(i) for each $U \in \mathcal{T}$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.
(ii) Given $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

A typical and motivating example: the family of open sets $\{B(x,r) : x \in X, r > 0\}$ is a base for the metric topology on X

This definition leads to the following question. Given a set X and a collection $\mathcal{B} \subset P(X)$ of subsets of X when is $\mathcal{B}$ a base for *some* topology on X? Since condition (i) of the definition is any way will be used to declare open sets, the decisive condition is (ii). We state this as a proposition.

Proposition 10. *Let X be a nonempty set and $\mathcal{B}$ be a collection of subsets of X. Then there exists a topology $\mathcal{T}$ on X for which $\mathcal{B}$ is a base if $\mathcal{B}$ satisfies the following condition:*

For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

Proof. Declare $U \subset X$ to be open if for each $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. It is easy to check that the collection of open sets is a topology $\mathcal{T}$ on X and that $\mathcal{B}$ is a base for $\mathcal{T}$. $\square$

If a family $\mathcal{O}$ of subsets of a set X is given and if we are looking for the smallest topology on X which contains $\mathcal{O}$, we need only find a family $\mathcal{B}$ which could be base for a topology on X. The family

$$\mathcal{B} := \{B_1 \cap B_2 \cap \cdots \cap B_n : B_i \in \mathcal{O}, n \in \mathbb{N}\}$$

is a base for some topology on X. Go back to Definition 3 and try to see how we side-stepped the intermediate construction of $\mathcal{B}$ and defined the topology straight away starting from $\mathcal{O}$.

Remark 11. If $h\colon Z \to X$ is any map from a topological space Z to X, to show that h is continuous, it suffices to verify that $h^{-1}(B)$ is open for $B \in \mathcal{O}$. If $B \in \mathcal{B}$ is of the form $B = B_1 \cap \cdots \cap B_n$ with $B_i \in \mathcal{O}$ then $h^{-1}(B) = \cap_i h^{-1}(B_i)$ is the intersection of a finite number of open sets and hence is open in Z. Now any open set U in X is an arbitrary union of members from $\mathcal{B}$. For, $U = \cup_{x \in U} B_x$ where for each $x \in U$, $B_x \in \mathcal{B}$ is chosen so that $x \in B_x \subset U$. Clearly, $h^{-1}(U) = \cup_x h^{-1}(B_x)$ is open in Z.

Remark 12. The universal mapping property is the most important to deal with the newly constructed topologies. For instance, in the case of quotient space $Y = X/\sim$, giving a map $h\colon Y \to Z$ is the same as giving a map $\tilde{h}\colon X \to Z$ which is constant on the equivalence classes. Hence the continuity of h is the same as that of $\tilde{h}$. As a concrete example, let $X = \mathbb{R}$ and

$x \sim y$ iff $x - y \in \mathbb{Z}$. Let $S := \{z \in \mathbb{C} : |z| = 1\}$ be with the subspace topology. We have a natural map $\tilde{h} \colon X \to S$ given by $\tilde{h}(t) := e^{2\pi i t}$. Then $\tilde{h}$ gives rise to map $h \colon Y \to S$ which is a bijection from well-known properties of the exponential map. Also, h is continuous since $\tilde{h}$ is so. Since any continuous map from a compact space to a Hausdorff space is a closed map, h is a homeomorphism. Thus Y is the circle in $\mathbb{C}$! For more such applications, see my article on Quotient spaces.