$\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ iff $m = n$

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The aim of this article is to prove the theorem of the title. As a rule no first course in topology proves this result. Even if they raise the question of homeomorphism between $\mathbb{R}^m$ and $\mathbb{R}^n$ they refer to Brouwer's theorem on the invariance of domain which is proved as an application of Homology Theory. We wish to make the following elementary proof more widely known. It could be taught in any first course on Topology. There is nothing original in the following proof except the organization of the material available in the literature.

Outline of the Proof

- Definition of a simplex, standard simplex; barycentric subdivision.
- Dimension of a compact metric space. $\dim T^n \leq n$ where T^n is the standard n-simplex in $\mathbb{R}^n$.
- Sperner's lemma — restricted version (applicable only to triangulation arising from barycentric subdivision).
- Nagata's Lemma (restricted version): Let $\Delta^k(s)$ be the k-th barycentric subdivision of an n-simplex. Let $\{U_i\}_0^n$ be an open cover of s such that $U_i \subseteq s \setminus F_i$ where F_i is the face opposite to the vertex e_i. Then there exists and r and an n-simplex in $\Delta^r(s)$ which intersects each of U_i.
- $\dim s^n = n$.

Definition 1. A topological space X is said to have $\dim X \leq n$ if given any open cover $\mathcal{A}$ of X there exists an open cover $\mathcal{B}$ with the following properties:

1. For each $B \in \mathcal{B}$ there is an $A \in \mathcal{A}$ such that $B \subset A$.
2. There exists on element of X which lies in $n+1$ memvbers of $\mathcal{B}$ and no element of X lies in more than $n+1$ members of $\mathcal{B}$.

We say that X is of (covering) dimension n if n is the least integer m such that $\dim X \leq m$. If no such n exists then we write $\dim X = \infty$.

Ex. 2. Homeomorphic spaces have the same dimension.

Ex. 3. Let X be a compact metric space. We say that $\dim X \leq n$ if for every $\varepsilon > 0$ there is a finite open cover $\mathcal{A}$ of X by sets of diameter $< \varepsilon$ such that some point of X lies in $n+1$ members of $\mathcal{A}$ and no point of X lies in more than $n+1$ members of $\mathcal{A}$. *Hint:* Use Lebesgue covering lemma.

Ex. 4. Let X be a comapct metric space of dimension n. Let K be a closed subset of X. Then $\dim K \leq n$.

Definition 5. A set of vectors $\{v_0, v_1, \dots, v_k\}$ in $\mathbb{R}^n$ is said to be geometrically (or affinely) independent if the set of vectors $\{v_1 - v_0, v_2 - v_0, \dots, v_n - v_0\}$ is linearly independent. A singleton set is geometrically independent by definition.

Example 6. A set $\{v_0, v_1\}$ is geometrically indenpendent iff they are not multiples of each other. A set $\{v_0, v_1, v_2\}$ is geometrically indenpendent iff they are not collinear. A set $\{v_0, v_1, v_2, v_3\}$ is geometrically indenpendent iff they are not coplanar.

Ex. 7. A set of vectors $\{v_0, v_1, \dots, v_k\}$ in $\mathbb{R}^n$ is said to be geometrically (or affinely) independent iff for any set of real numbers λ_i with $\sum_{i=0}^k \lambda_i = 0$ and $\sum_{i=0}^k \lambda_i v_i = 0$ we have $\lambda_i = 0$ for $0 \leq i \leq k$.

Definition 8. Let $\{v_0, v_1, \dots, v_k\}$ in $\mathbb{R}^n$ be geometrically independent. The (open) *simplex* s^k is the set

$$s^k := \{x \in \mathbb{R}^n : x = \sum_{i=0}^k \lambda_i v_i, \quad \lambda_i > 0 \text{ and } \sum_i \lambda_i = 1\}.$$

We refer to k as the *algebraic dimension* of s. s is also called a k-simplex. We denote the simplex s^k by $(v_0 v_1 \dots v_k)$. v_i are called the *vertices* of s^k. The simplex σ_i spanned by $\{v_0, \dots, \hat{v}_i, \dots, v_k\}$ is called the i-th face opposite to the vertex v_i. ($\hat{v}_i$ means v_i is omitted.) More generally, one defines an r-face of the simplex $(v_0 \dots v_k)$ as the r-simplex $(v_{i_1} \dots v_{i_r})$ for $0 \leq i_1 < i_2 \cdots i_r \leq k$. If σ is an r-face of s we write $\sigma \prec s$.

Example 9. Any one simplex with vertices v_0 and v_1 is the open line segment joining $v0$ and v_1. Any two simplex spanned by three noncollinear vectors is the open interior of the triangle of which they are vertices. Any three simplex spanned by four coplanar vectors is the open tetrahedron. The faces are respectively the endpoints of the line segment, sides of the triangle and the faces of the tedrahedron.

The proof of the theorem of the title depends on the following

Theorem 10. *Let T^n be the closure of any n-simplex in $\mathbb{R}^n$. Then* $\dim T^n = n$.

Assuming the theorem let us complete the proof of the main result. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a homemorphism. Let us assume that, if possible, that $m \neq n$. Let $m < n$ without loss of generality. Let T^n be as above. Then $f(T^n)$ is a compact subset of $\mathbb{R}^m$. Hence there exists an m-simplex, say, s such that $f(T^n) \subset s \subset \overline{s}$. Since $\dim \overline{s} \leq m$ and $\dim T^n = n$, we infer that $n = \dim f(T^n) \leq m < n$ in view of Exercises 2 and 4. This contradiction shows that $m = n$.

We break the proof of Theorem 10 into two statements: $\dim T^n \leq n$ and $\dim T^n \geq n$. In the next section we establish the first and the second in the last section.

1 Barycentric Subdivision

Definition 11. A *complex* K is a collection of simplices $\{s\}$ with the following property: If $s \in K$ and $\sigma \prec s$ then $\sigma \in K$. The set $|K| = \cup_{s\in K} s$ is called the geometric realisation of the complex K and K is called a *triangulation* of $|K|$.

Definition 12. Let $s := (v_0 \dots v_k)$ be any simplex. Then the *barycenter* of s is the vector $(v_0 + \cdots + v_k)/(k+1)$. We denote this vector by $b(s)$.

Example 13. The barycenter of any 0-simplex is itself. The barycenter of $(v_0 v_1)$ is the midpoint of the line segment, that of $(v_0 v_1 v_2)$ is the centroid of the triangle and that of $(v_0 \dots v_3)$ is the centre of garvity of the tetrahedron.

Definition 14. Let a complex K be given. The first barycentric subdivision of K is the complex $\Delta^1(K)$ whose vertices are $b(s)$ where s runs through all the faces of all simplices of K. The other simplices in the newcomplex are of the form $(b(s_1)b(s_2)\dots b(s_r))$ where $s_1 \prec s_2 \cdots \prec s_r$. We leave it to the reader that $\Delta^1 K$ is indeed a complex. Recursively we define $D^r(K) := D^1(D^{r-1}(K))$, called the r-th barycentric subdivision of K.

The following simple exercise will be repeatedly used in the sequel.

Ex. 15. Let $\mu(K) := \max\{\operatorname{diam} \overline{s} : s \in K\}$ be *mesh* of the complex. Let s be any simplex. Show that $\mu(s)$ is the length of the longest side, i.e.,1-dimensional face and $\mu(\Delta^r(s)) \to 0$ as $r \to \infty$.

Theorem 16. *Let s be an n-simplex. Then* $\dim s \leq n$.

Proof. Given $\varepsilon > 0$ let us choose r sufficiently large so that the mesh of $\Delta^r(s)$ is less than $\varepsilon/2$. Then the closed simplices in $\Delta^{r-1}(s)$ is an ε-covering such that each vertex $v_i \in s$ lies in $\cap_{i=0}^n star(v_i)$. This does the job. (Details are to be given.) □

2 Sperner's Lemma and its Corollaries

We shall give a very special version of Sperner's lemma which will be sufficient for our purpose. For more general versions, see references.

Theorem 17. *Let $s = (v_0 \dots v_n)$ be a simplex. Let $\Delta^r(s)$ be the r-th barycentric subdivision of s. Let a map $f\colon V(\Delta^r(s)) \to V(s)$ be given such that $f(v) = v_i$ where v_i is a vertex of the carrier of v. (Such maps will be called Sperner maps.) Then there exists a simplex $\sigma \in D^r(s)$ such that $f(\sigma) = \{v_0, \dots, v_n\}$.*

Proof. We shall prove this for $r = 1$ by induction on n. For $n = 1$ this is clear. Let us assume that the result is true for $n - 1$. Assume without loss of generality that $f(z) = v_n$ where $z = b(s) \in \Delta^1(s)$. Then f induces a Sperner map on the $n-1$-simplex $(v_0 \dots v_{n-1})$. By induction hypothesis, we there is an $n-1$-simplex, say, $\sigma^{n-1} = (b_0 \dots b_{n-1})$ such that $f(V(\sigma)) = \{v_0, \dots, v_{n-1}\}$. Clearly the simplex $\tau = (b_0 \dots b_{n-1} v_n)$ is of the required type. Thus the result is true for $\Delta^1(s)$ for any n-simplex s.

To complete the proof we now use induction on r and the previous paragraph. Let a Sperner map $f\colon V(\Delta^r(s)) \to V(s)$ be given. Then it induces a Sperner map on $\Delta^{r-1}(s)$ by restriction. By induction there exists a $\sigma \in \Delta^{r-1}(s)$ such that $f(\sigma) = \{0, 1, \dots, n\}$. By the first part of the proof there exists a simplex $\tau \in \Delta^1(\sigma)$ such that τ is completely labelled. But then $\tau \in \Delta^r(s)$ and meets our requirement. □

3 $\dim T^n = n$

Theorem 18. *Let $s := (v_0 \dots v_n)$ be an n-simplex and $T^n := \overline{s}$. Let $\{U_i : 0 \leq i \leq n\}$ be an open cover of T such that U_i does not interset the i-th face. Then there a sufficiently large positive integer r such that there is an n-simplex σ in $\Delta^r s$ such that $\sigma \cap U_i \neq emptyset$.*

Proof. Let ε be the Lebesgue number of the covering $\{U_i\}$. We choose r so that the mesh of $\Delta^r(s)$ is less than $\varepsilon/2$. We define a Sperner map f: $f(v) = i$ if $v \in U_i$ and v_i is a vertex of the carrier of v. This is possible by our assumption on the cover. Sperner lemma gives a simplex of the required kind. □

Theorem 19. *Let the notation be as above. Then $\dim s^n \geq n$.*

Proof. To prove this we need to exhibit an open cover $\mathcal{A}$ such that any $\mathcal{B}$ is in Defintion 1 will be of order greater than or equal to $n+1$. Let us take $A_i = T^n \setminus F_i$, the complement of the i-th face. Let $\mathcal{B}$ be any open cover such that $B \leq \mathcal{A}$. After doing a little jugglery we may assume that $\mathcal{B}$ has $n+1$ members. By the last result the order of $\mathcal{B}$ is $n+1$. □

References

1. Alexandrov, *Combinatorial Topology*, vol. 1.
2. Munkres, *Topology, A First Course*.
3. Nagata, *Dimension Theory*.