Topological Groups — via Problems

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Definition 1. A *topological group* is a triple $(G, \tau, \cdot)$ such that (G, τ) is a topological space and $(G, \cdot)$ is a group. Both these structures are inter-related in the sense that the group operations are continuous, that is,
(i) the group multiplication $G \times G \to G$ given by $(x, y) \mapsto xy$ and
(ii) inversion map $G \to G$ given by $x \mapsto x^{-1}$ are continuous.

Example 2. $(\mathbb{R}^n, +)$ is a topological group with the usual topology.

Example 3. Let $GL(n, \mathbb{K})$ denote the group of all $n \times n$ invertible matrices with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Then $GL(n, \mathbb{K}) \subset M(n, \mathbb{K}) \simeq \mathbb{K}^{n^2}$ is open. $GL(n, \mathbb{K})$ is a topological group.

Example 4. The group $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is a topological group with the subspace topology.

Example 5. Let G be any group. If G is endowed with the discrete topology, then G is a topological group.

Ex. 6. If we endow an abstract group with the indiscrete topology, does it become a topological group?

Ex. 7. Show that the triple $(G, \tau, \cdot)$ is a topological group iff the map $(x, y) \mapsto xy^{-1}$ is continuous.

Ex. 8. If G_i, $1 \le i \le n$, are topological groups, then so is their product with the product topology.

Ex. 9. If H is a subgroup of a topological group, then H is a topological group with the subspace topology.

Ex. 10. How do you define a topological subgroup of a topological group?

Ex. 11. If H is a subgroup of G, then the closure $\overline{H}$ is also a subgroup.

Ex. 12. The following subgroups (of the respective groups) are topological groups with the subspace topology.
(a) Let $SL(n, \mathbb{K})$ denote the subgroup of $GL(n, \mathbb{K})$ with determinant 1.
(b) Let $O(n, \mathbb{R})$ denote the subgroup of $GL(n, \mathbb{R})$ of all orthogonal matrices.
(c) Let $U(n)$ denote the subgroup of all unitary matrices in $GL(n, \mathbb{C})$.

(d) Let $SO(n,\mathbb{R})$ and $SU(n,\mathbb{R})$ denote respectively the subgroups consisting of elements of $O(n,\mathbb{R})$ and $U(n)$ whose determinant is one.
(e) Let $GL^+(n,\mathbb{R})$ denote the subgroup of all elements with positive determinant.
Show that $O(n,\mathbb{R})$, $SO(n,\mathbb{R})$, $U(n)$ and $SU(n)$ are compact.

Ex. 13. The left translations $L_a\colon x \mapsto ax$ are homeomorphisms. So are the right translations R_a.

Ex. 14. If U is an open set, so is $U^{-1} := \{x^{-1} : x \in U\}$. If A is an arbitrary subset of G, then AU and UA are open.

Ex. 15. Let $\mathcal{U}$ denote the set of all neighborhoods of $e \in G$. Show that the topology on G is completely determined by the knowledge of $\mathcal{U}$.

Ex. 16. With the notation of the last exercise, prove that $\mathcal{U}$ has the following properties:
(i) $e \in U$ for all $U \in \mathcal{U}$.
(ii) If $U_1, U_2 \in \mathcal{U}$, then there exists $U \in \mathcal{U}$ such that $U \subset U_1 \cap U_2$. *Hint:* Use the continuity of the group multiplication at (e,e).
(iii) If $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $VV^{-1} \subset U$.
(iv) If $U \in \mathcal{U}$ and $a \in U$, then there exists $V \in \mathcal{U}$ such that $aV \subset U$.
(v) If $U \in \mathcal{U}$ and $a \in G$, there exists $V \in \mathcal{U}$ such that $aVa^{-1} \subset U$.

Ex. 17. Let G be a group. Let $\mathcal{U}$ be a collection of subsets with the properties enumerated in the previous exercise. Show that there exists a topology on G such that G becomes a topological group with this topology and the neighbourhood basis at $e \in G$ is precisely $\mathcal{U}$.

Ex. 18. Let G be a topological group. Let $\mathcal{U}$ be the neighbourhood base at e. Show that G is Hausdorff iff $\cap_{U\in\mathcal{U}} = \{e\}$.

Ex. 19. If H is a normal subgroup of G, then the closure $\overline{H}$ is also a normal subgroup.

Ex. 20. If H is an open subgroup of G, then H is closed. *Hint:* Consider the coset decomposition of G with respect to H. That is, observe that $H = G \setminus \cup_{x\notin H} xH$.

Ex. 21. If H is a closed subgroup of finite index in a topological group, then H is open.

Ex. 22. Let G be a topological group and $E \subset G$. Show that

$$\overline{E} = \cap_{U\in\mathcal{U}} UE = \cap_{U\in\mathcal{U}} EU.$$

Ex. 23. How will you define the uniform continuity of $f\colon G \to \mathbb{C}$ on any topological group? (G need not be metrizable.)

Ex. 24. Let $f\colon G \to \mathbb{C}$ be a continuous function with compact support. Show that f is uniformly continuous.

Ex. 25. Let G and H be topological groups. The a group homomorphism $f\colon G \to H$ is continuous iff it is continuous at e.

Ex. 26. If G is a topological group and H is a subgroup, then the coset space G/H is endowed with the quotient topology. The quotient map $\pi\colon G \to G/H$ is an open continuous map.

Ex. 27. With the notation of the previous exercise, if we further assume that H is normal in G, then the quotient group G/H becomes a topological group with the quotient topology.

Ex. 28. Show that the quotient group G/H is Hausdorff iff H is closed in G. Is it still true if H is only a subgroup rather than a normal subgroup?

Ex. 29. When is G/H discrete?

Ex. 30. Let G be a connected topological group. Let U be a symmetric neighbourhood of e, that is, $U = U^{-1}$. Then $G = \cup_{n=1}^{\infty} U^n$. *Hint:* Observe that the union is an open subgroup.

Ex. 31. With the notation of the last exercise, assume that G is also compact. Can you sharpen the result in this case?

Ex. 32. Let H be a subgroup of a topological group G. If G/H and H are connected then G is connected.

Ex. 33. Let G be a topological group. Let G_0 denote the connected component of G containing e. Show that G_0 is a closed normal subgroup.

Ex. 34. Show that $GL^+(n,\mathbb{R})$ is connected and that it is the connected component of $GL(n,\mathbb{R})$ containing the identity. *Hint:* By induction. For, $n > 1$, consider the subgroup H consisiting fo elements of the form $g = \begin{pmatrix} 1 & v \\ 0 & h \end{pmatrix}$ where $v \in \mathbb{R}^{n-1}$ and $h \in GL^+(n-1,\mathbb{R})$.

Ex. 35. Show that $GL(n,\mathbb{C})$ is connected.

Ex. 36. How will you define the action of a topological group on a topological space X?

Definition 37. We say that a topological group G acts on a topological space X if the group G acts on X in the algebraic sense and if the group action $G \times X \to X$ given by $(g,x) \mapsto gx$ is continuous. We then say X is a G-space.

Ex. 38. Examples of such actions. $SL(2,\mathbb{R})$ acts on the upper half plane via fractional linear transformations. $O(n,\mathbb{R})$ acts on the unit sphere S^{n-1}. The group of affine transformations $f_{A,v} : x \mapsto Ax + v$ on $\mathbb{K}^n$ where A is a nonsingular linear map and $v \in \mathbb{K}^n$ is fixed. The group law is the composition of maps. This group acts on $\mathbb{K}^n$.

Ex. 39. Let G the a topological group and H a closed subgroup. Let $X := G/H$ be the quotient space. Then G acts on X via $(g, xH) \mapsto gxH$. This action is transitive.

Ex. 40. When do we say two G-spaces X and Y are G-isomorphic?

Ex. 41 (Baire's Theorem)**.** Let X be a locally compact Hausdorff space. Assume that $X = \cup_{n=1}^{\infty} F_n$ where F_n is closed for each n. Show that at least one F_n is open in X. *Hint:* Go through the proof in the case of metric spaces.

Ex. 42. Let X be a locally compact Hausdorff space. Let a locally compact Hausdorff group with a countable basis. Assume that G acts on X transitively. Let $x_0 \in X$ be fixed. Let H be the isotropy of x_0, that is, $H := \{g \in G : gx_0 = x_0\}$. Then X is "isomorphic" to the quotient space G/H as G-spaces.

Ex. 43. What is the isotropy at i when $SL(2, \mathbb{R})$ acts on the upper half plane? Same question when $O(n)$ acts on $S^{n-1} \subset \mathbb{R}^n$.

Ex. 44. Show that $O(n, \mathbb{R})/O(n-1, \mathbb{R})$ is G-isomorphic to S^{n-1}. How does $O(n-1, \mathbb{R})$ sit in $O(n, \mathbb{R})$?

Definition 45. A subgroup $\Gamma \subset G$ is called a *discrete subgroup* if it is a subgroup of G which is both closed and discrete as a subset of G.

Ex. 46. Let Γ be a discrete subgroup of $\mathbb{R}^n$. Then there exist $v_1, \ldots, v_d \in \Gamma$ such that
(i) $v_1, \ldots, v_d$ are linearly independent over $\mathbb{R}$.
(ii) Every element of Γ is uniquely written as an integral linear combination of v_j's.
(iii) d is unique though v_j's need not be.
Hint: Consider the L^1-norm on $\mathbb{R}^n$: $\|x\|_1 := \sum_{i=1}^n |x_i|$. Show that $\inf\{\|\gamma\| : \gamma \in \Gamma, \gamma \neq 0\}$ is positive and attained, say, $v_1 \in \Gamma$. If $\Gamma \neq \mathbb{Z}v_1$, extend it to a basis $\{u_1 = v_1, \ldots, u_n\}$ of $\mathbb{R}^n$. Show that

$$\inf\{\sum_{j\neq 1} |x_j| : \text{ where } \gamma \in \Gamma \setminus \{0\}, \gamma = \sum_{i=1}^n x_i u_i\}$$

is attained at some $v_2 \in \Gamma$.

Ex. 47. If $f\colon G \to H$ is a continuous homomorphism into a locally compact Hausdorff group H, then f is necessarily open.

Ex. 48. Let G be a connected group and H a discrete normal subgroup of G. Then H is contained in the center of G.