Outline of a Topology Course

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Abstract

This is a summary of a course on General Topology, offered by me at the Department of Mathematics, University of Mumbai during the academic year 2004-2005. There are no proofs in this set of notes. Its merit, if any, lies in the choice of topics, their development and the emphasis on concrete and geometric examples and exercises. I plan to add a bit more material so that it could serve as a skeleton of a course in General Topology. Later I plan to develop this into a text-book. I would appreciate your comments and views.

This set may be used in conjunction with the following articles of mine on Topology:

1. Subspace Topology
2. Quotient Topology
3. Existence of Continuous Functions
4. Compact Spaces
5. Connected Spaces
6. Generating Topologies — A Unified View of Subspace, Product and Quotient Topologies.

My book *Topology of Metric Spaces* published by Narosa and the books *Topology* by Munkres and *Topology* by Armstrong are available in Indian edition and they may be used to fill in the details of my outline.

1. Finite sets and number of elements in a finite set

2. Countable and uncountable sets: Countable sets definition and equivalent characterizations. Applications: Countability of $\mathbb{N} \times \mathbb{N}$, $\mathbb{Q}^+$, $\mathbb{Q}$, countable union of countable sets, finite product of countable sets.

3. Uncountability of $2^{\mathbb{N}}$: Cantor's theorem: there exists no onto map from X to $P(X)$.

4. Metric Spaces: Definition. examples: absolute value metric on $\mathbb{R}$

5. Metrics in $\mathbb{R}^2$: L^1 and L^∞ metrics, called the sum and max metrics.

6. Generalizations of these metrics to function spaces:

7. Normed linear spaces. Examples: $\| \ \|_1$, $\| \ \|_\infty$ and $\| \ \|_2$ on $\mathbb{R}^n$.

   Three classes of function spaces: $B(X, \mathbb{R})$, $(C[0,1], \| \ \|_\infty)$ and $(C[0,1], \| \ \|_1)$.

8. ℓ^1, the space of sequences whose associated series are absolutely summable.

9. Open balls:
   (a) in $\mathbb{R}$
   (b) $B(0,1)$ in $\mathbb{R}^2$ with $\| \ \|_1$, $\| \ \|_2$ and $\| \ \|_\infty$.
   (c) in $\mathbb{Z}$ with the induced metric. Identify **all** open balls.
   (d) Relations between $B(x,r)$ and $B(y,s)$.
   (e) Visualizing the open balls in $C[0,1]$ under $\| \ \|_\infty$.
   (f) In an NLS, $B(x,r) = x + rB(0,1)$.

10. Open sets:
   (a) in $\mathbb{R}$: various examples and non-examples such as $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$
   (b) $\{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$ in $\mathbb{R}^2$.
   (c) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$ in $\mathbb{R}^2$.
   (d) In an NLS V, if any vector subspace W is open, then $W = V$. Application: Is $C[0,1]$ open in $BF[0,1]$, the set of bounded functions?
   (e) Is $\mathbb{R} \setminus \mathbb{Z}$ open in $\mathbb{R}$?
   (f) $B(x,r)$ is open.
   (g) $\{y \in X : d(x,y) > r\}$ is open.
   (h) What are the open sets in a finite metric space?
   (i) Can $\{h \in C[0,1] : f(x) < h(x) < g(x)\}$ for some $f, g \in C[0,1]$ be an open ball? Is it an open set?
   (j) Is the open unit ball in $(C[0,1], \| \ \|_\infty)$ open in $(C[0,1], \| \ \|_1)$?
   (k) If U is an open subset in an NLS, $(X, \| \ \|)$, then
      i. $x + U$ is open for any $x \in X$
      ii. $A + U$ is open for any $A \subset X$
      iii. αU is open for any nonzero scaler α.
   (l) Is the set $U := \{f \in C[0,1] : f(1/2) \neq 0\}$ open in $(C[0,1], \| \ \|_\infty)$?
   (m) Any open subgroup of $\mathbb{R}$ is $\mathbb{R}$.
   (n) A subset U of a metric space is open iff it is the union of a family of open balls. (As preliminary, we discussed the notion of a family of subsets of a set, their union etc.)
   (o) A subset $U \subset \mathbb{R}$ is open iff it is the union of a countable family of pair-wise disjoint open intervals.

11. Topology: definition and examples:
   (a) metric topology
   (b) discrete topology: every subset is open.
   (c) The topology on a finite metric space is discrete.
   (d) indiscrete topology: U is open iff $U = \emptyset$ or $U = X$.
   (e) cofinite topology: U is open iff $U = \emptyset$ or $X \setminus U$ is finite.

(f) cocountable topology: U is open iff $U = \emptyset$ or $X \setminus U$ is countable.

(g) VIP topology: Fix $p \in X$. U is open iff $U = \emptyset$ or $p \in U$.

(h) outcast topology: Fix $p \in X$. U is open iff $U = X$ or $p \notin U$.

(i) VIP+outcast topology: U is open iff either $p \notin U$ or U^c is finite.

12. Basis of a topological space and basis for a topology on a set.

13. Examples of bases:

    (a) $\{B(x, r) : x \in X, r > 0\}$ is a basis for the metric topology on any metric space.

    (b) $\{B(x, 1/n) : x \in X, n \in \mathbb{N}\}$ is a basis for the metric topology on any metric space.

    (c) $\mathcal{B} := \{\{x\} : x \in X\}$. What is the topology?

    (d) $\mathcal{B} := \{X\}$. What is the topology?

    (e) $\{(a, b) : a, b \in \mathbb{Q}\}$ is a basis for some topology on $\mathbb{R}$. What is the topology? Is this basis countable or uncountable?

    (f) A basis for the VIP topology is $\{p\} \cup \{\{p, q\} : q \in X, q \neq p\}$.

    (g) A basis for outcast topology is $\{X\} \cup \{\{q\} : q \in X, q \neq p\}$.

14. We can use bases to say something about the topologies on a set.

    Theorem 1. *Let X be any set. Let $\mathcal{B}_i$ be a basis for some topology $\mathcal{T}_i$ on X, for $i = 1, 2$. Then $\mathcal{T}_1 \leq \mathcal{T}_2$ iff the following holds: if $B_1 \in \mathcal{B}_1$, then $B_1 \in \mathcal{T}_2$. In particular, $\mathcal{T}_1 = \mathcal{T}_2$ iff every $B_1 \in \mathcal{B}_1$ is in $\mathcal{T}_2$ and every $B_2 \in \mathcal{B}_2$ is in $\mathcal{T}_1$.* □

15. Order Topology: partial and total orders, dictionary order on products, $\mathbb{C}$ is totally ordered **but** is not an ordered field. Intervals of the form (a, b) and rays of the form $(-\infty, a)$ and (b, ∞). Examples in $\mathbb{R}^2$: the rays $(-\infty, (1, 2))$, $((-1, 1), \infty)$ and the intervals $((-1, 1), (3, -2))$ and $((0, 0), (0, 10))$. Basis for order topology. What is the order topology on $\mathbb{R}$, on $\mathbb{Z}$, on $\mathbb{N}$ and on a finite totally ordered set?

16. When do two norms generate the same topology on a vector space? iff we can find positive constants C_1 and C_2 such that $C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1$ for all x. One then says that the norms are equivalent.

    Useful observations to prove this are:

    (a) It is enough to show that $B_1(0, 1)$ is open in $\|\ \|_2$-topology and vice-versa.

    (b) $\|x\|_2 \leq C_2 \|x\|_1$ iff $B_1(0, 1) \subset C_2 B_2(0, 1) = B_2(0, C_2)$.

17. In $\mathbb{R}^n$, the three norms $\|\ \|_1$, $\|\ \|_2$ and $\|\ \|_\infty$ are equivalent. To see this observe the following:

$$\frac{1}{n} \|x\|_1 \leq \frac{1}{\sqrt{n}} \|x\|_2 \leq \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 .$$

    Later, we shall show that all norms on $\mathbb{R}^n$ induce the same topology, that is, they are all equivalent.

18. The class of all topologies on a given set is a partially ordered set: if $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, we define $\mathcal{T}_1 \leq \mathcal{T}_2$ iff $\mathcal{T}_1 \subset \mathcal{T}_2$, as subsets of $P(X)$. The indiscrete topology is the smallest element and the discrete topology is the largest element of the class of topologies on X. The union of topologies on X need not be topology whereas the intersection of a (nonempty) family of topologies on X is again a topology.

    This allows to define the following: if $\mathcal{A}$ is an arbitrary collection of subsets of a set X, there exists a unique smallest topology on X which contains $\mathcal{A}$ and is called the topology generated by $\mathcal{A}$. We compared this with the concept of subgroup (or a vector subspace) generated by subset in a group (or in a vector space). We shall later see a practical way of looking at this topology. See Item 122.

19. Lower Limit Topology: A basis for this topology is $\{[a,b) : a,b \in \mathbb{R}, a < b\}$. Since $(a,b) = \cup_{n\geq N}[a + \frac{1}{N}, b)$ for N sufficiently large, it follows that the lower limit topology is finer than the usual topology on $\mathbb{R}$.

20. Let X be a set and $\mathcal{T}_c$ and $\mathcal{T}_f$ be respectively cocountable and cofinite topologies on X. Then the cocountable topology is finer than the cofinite topology.

    They are the same iff X is finite. To see this, we needed a result form set theory: If X is an infinite set, then there exists a set A such that $X \setminus A$ is infinitely countable.

21. Continuity: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be **topological** spaces. Let $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map and $x_0 \in X$. We say that f is continuous at x_0 if for any given open set V containing $f(x_0)$, there exists an open set U containing x_0 such that $f(U) \subset V$.

    We motivated this definition and also proved the following theorem:

    Theorem 2. *Let $f\colon (X,d) \to (Y,d)$ be a map between metric spaces. Let $x_0 \in X$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the topologies on X and Y induced buy their respective metrics. Then $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous at x_0 iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $d(x, x_0) < \delta$, we have $d(f(x), f(x_0)) < \varepsilon$.* □

22. Let $f\colon X \to Y$ be any map between two sets. Let $B \subset Y$. The set $f^{-1}(B) := \{x \in X : f(x) \in B\}$ is called the inverse image of B under f. The following are well-known facts:

    (a) If $\{B_i : i \in I\}$ is a family of subsets of Y, then

        i. $f^{-1}(\cup_{i\in I} B_i) = \cup_{i\in I} f^{-1}(B_i)$.

        ii. $f^{-1}(\cap_{i\in I} B_i) = \cap_{i\in I} f^{-1}(B_i)$.

    (b) For any set $B \subset Y$, we have $X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$.

23. Let X and Y be topological spaces. Then a map $f\colon X \to Y$ is said to be continuous on X iff it is continuous at each point $x \in X$. We proved the following

    Theorem 3. *Let X and Y be topological spaces. Then a map $f\colon X \to Y$ is continuous on X iff for every open subset $V \subset Y$, the inverse image $f^{-1}(V)$ is open in X.* □

24. We looked at the following examples:

    (a) Any constant map from a topological space to another is continuous.

    (b) The identity map from $(X, \mathcal{T}_X)$ to itself is continuous.

(c) If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X, then the identity map $I\colon (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous iff $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

(d) Let $(X, \mathcal{T}_X)$ be a topological space with the property that any map $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous. Then $\mathcal{T}_X$ is discrete and conversely.

(e) Let $(Y, \mathcal{T}_Y)$ be a topological space with the property that any map $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous. Then $\mathcal{T}_Y$ is indiscrete and conversely.

(f) The identity map from X with cocountable topology to X with cofinite topology is continuous. The other way map is continuous iff X is finite.

(g) Let X be a set with at least two elements and $p \in X$. Let V (resp. O) denote the VIP topology (resp. the outcast topology) on X with respect to p. Then

i. The identity map $I\colon (X, V) \to (X, O)$ is not continuous. However it is continuous at $x = 0$ and at no other point.

ii. The identity map $I\colon (X, O) \to (X, V)$ is not continuous at any point.

25. The identity map from $\mathbb{R}$ with the lower limit topology is continuous to $\mathbb{R}$ with the usual topology.

26. Let $\| \ \|_k$, $k = 1, 2$, be two norms on a vector space V. Then they are equivalent iff the identity map $I\colon (V, \| \ \|_1) \to (V, \| \ \|_2)$ and $I\colon (V, \| \ \|_2) \to (V, \| \ \|_1)$ are continuous.

27. Let X be an uncountable set with cocountable topology $\mathcal{T}_c$. Then the only continuous functions $f\colon (X, \mathcal{T}_c) \to \mathbb{R}$ are constants.

28. We discussed the set of points of continuity of all real valued functions on the following spaces.
(i) $\mathbb{R}$ with VIP topology with 0 as the VIP.
(ii) $\mathbb{R}$ with outcast topology with 0 as the outcast.
(iii) $\mathbb{N}$ with the topology $\mathcal{T} := \{\emptyset, \mathbb{N}\} \cup \{I_n : n \in \mathbb{N}\}$ where $I_n = \{1, 2, \dots, n\}$.

29. On any metric space X, we have lots of real valued continuous functions: $f(x) := d(x, p)$ for any fixed $p \in X$. In particular, given $p \neq q$ in X, there exists a real valued continuous function f on X such that $f(p) \neq f(q)$.

30. Let A be a nonempty subset of a metric space X. We defined $d_A(x) \equiv d(x, A) := \inf\{d(x, a) : a \in A\}$. We looked at the following examples and drew graphs for the first three:

(a) $X = \mathbb{R}$ and $A = [-1, 1]$.

(b) $X = \mathbb{R}$ and $A = \mathbb{Q}$.

(c) $X = \mathbb{R}$ and $A = \mathbb{Z}$.

(d) $X = \mathbb{R}^2$ and A is the x-axis.

(e) $X = \mathbb{R}^2$ and $A = \{(x, y) : x^2 + y^2 = 1\}$.

(f) W is a vector subspace of $\mathbb{R}^n$. *Hint:* If $\mathbb{R}^n = W \oplus W^\perp$, and if $x = w + w'$, then $d_W(x) = \|w'\| = \|x - p_W(x)\|$, where $p_W\colon \mathbb{R}^n \to W$ is the orthogonal projection.

31. We showed that for any nonempty subset A of a metric space X, the function $d_A\colon X \to \mathbb{R}$ is continuous: $d_A(x) \leq d(x,a) \leq d(x,y) + d(y,a)$, for $a \in A$. Hence $d_A(x)$ is a lower bound for the set $\{d(x,y) + d(y,a) : a \in A\}$. But then $\inf\{d(x,y) + d(y,a) : a \in A\} = d(x,y) + d_A(y)$.

32. The function $x \mapsto \|x\|$ is continuous on an NLS $(V, \|\ \|)$.

33. The functions $\pi_i\colon x \mapsto x_i$, the coordinate projections are continuous on $\mathbb{R}^n$ (with respect to any of the norms $\|\ \|_i$, $i = 1, 2, \infty$).

34. Composite of continuous functions is continuous: Let X, Y, Z be topological spaces. Let $f\colon X \to Y$ be continuous at $p \in X$ and $g\colon Y \to Z$ be continuous at $q := f(p) \in Y$. Then $g \circ f\colon X \to Z$ is continuous at p.

35. The functions $\mathbb{R}^2 \to \mathbb{R}$ given by $\sigma\colon (x,y) \mapsto x + y$ and $\mu\colon (x,y) \mapsto xy$ are continuous.

36. Let $f, g\colon X \to \mathbb{R}$ be continuous functions. Consider $\mathbb{R}^2$ with $\|\ \|$ being one of the three norms: $\|\ \|_1, \|\ \|_2, \|\ \|_{\max}$. Then the function $\varphi\colon X \to \mathbb{R}^2$ given by $\varphi(x) = (f(x), g(x))$ is continuous.

37. If f, g are continuous functions from a topological space to $\mathbb{R}$ and if $a, b \in \mathbb{R}$, then the functions $af + bg$ and fg are continuous. Hint: Use Items 34–36.

38. Any polynomial function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous.

39. The map $\rho\colon \mathbb{R}^* \to \mathbb{R}^*$ given by $\rho(x) = 1/x$ is continuous.

40. Let $f\colon X \to \mathbb{R}$ be continuous and assume that $f(x) \neq 0$ for all $x \in X$. Then $1/f\colon X \to \mathbb{R}$ is continuous.

41. To check continuity, it suffices to show that the inverse images of basic elements in the codomain are open in the domain:

    Lemma 4. *Let $(X_1, \mathcal{T}_i)$ be topological spaces $i = 1, 2$ and let $\mathcal{B}_2$ be a basis for $\mathcal{T}_2$. Then $f\colon (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ is continuous iff $f^{-1}(B_2) \in \mathcal{T}_1$ for all $B_2 \in \mathcal{B}_2$.* □

    Item 25 is an immediate consequence of this.

42. Any linear map from $\mathbb{R}^n$ with any one of our three standard norms to any normed linear space is continuous. In particular, any linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$ is continuous. *Hint:* Show that there exists $C > 0$ such that $\|Tx\| \leq C\|x\|$.

43. In view of Item 42, the three norms on $\mathbb{R}^n$ are equivalent.

44. Let X and Y be normed linear spaces. A linear map $T\colon X \to Y$ is continuous iff there exists a positive constant C such that $\|Tx\| \leq C\|x\|$ for all $x \in X$. *Hint:* Use ε-δ definition of continuity at 0.

    Deduce that a linear map between two NLS's is continuous iff it is continuous at 0.

45. Use Items 26 and 44 to give an easier proof of Item 16.

46. One can use functions whose continuity are known to assert that certain subsets are open.

(a) Since polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}$ are continuous

   i. The subsets $\{(x,y) \in \mathbb{R}^2 : xy \neq 0\}$, $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\}$ and $\{(x,y) \in \mathbb{R}^2 : xy \neq 1\}$ are all open.

   ii. The subset $\{(x,y) \in \mathbb{R}^2 : x^3 - 34x^2y - 28xy^2 - y^3 + 7xy - 19y + 125 \neq 0\}$ is open in $\mathbb{R}^2$.

   iii. $\mathbb{R}^3 \setminus P$, where $P := \{(x,y,z) : ax + by + cz = d\}$ is a plane, is open in $\mathbb{R}^3$.

   iv. The rectangle $R := (a,b) \times (c,d)$ is open in $\mathbb{R}^2$: $R = p_1^{-1}(a,b) \cap p_2^{-1}(c,d)$, where $p_1(x,y) = x$ etc.

   v. The set $\{f \in C[0,1] : f(1/2) \neq 0\}$ in $X := (C[0,1], \| \ \|_\infty)$ is open. *Hint:* Consider $T \colon X \to \mathbb{R}$ given by $T(f) := f(1/2)$.

(b) Let W be a vector subspace of $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus W$ is open in $\mathbb{R}^n$. *Hint:* Write $\mathbb{R}^n = W \oplus W^\perp$ and let $u_1, \ldots, u_k$ be an orthonormal basis of $W^\perp$. Then $x \in \mathbb{R}^n$ lies in W iff $\langle x, u_i \rangle = 0$ for all $1 \leq i \leq k$.

   Alternately, consider the orthogonal projection $\pi \colon \mathbb{R}^n \to W^\perp$. Then $\mathbb{R}^n \setminus W = \pi^{-1}(W^\perp \setminus \{0\})$.

(c) Let $M_{m\times n}(\mathbb{R})$ denote the set of all $m \times n$ matrices with real entries. We identify it with $\mathbb{R}^{mn}$ in an obvious way. We use any one of the standard norms on $M_{m\times n}(\mathbb{R})$. We let $M(n,\mathbb{R}) := M_{n\times n}(\mathbb{R})$. Then we have

   i. $GL(n,\mathbb{R})$, the set of all invertible matrices is open in $M(n,\mathbb{R})$.

   ii. The set of symmetric matrices, being a vector subspace, cannot be open in $M(n,\mathbb{R})$. *Hint:* See Item 10d.

   iii. Same holds true for the set of skew symmetric matrices.

47. Closed Sets: Let $(X, \mathcal{T})$ be a topological space. A set $F \subset X$ is called a closed set (or said to be closed) in X if $X \setminus F$ is open in X. Let $\mathcal{C}$ be the class of all closed subsets in X. The following are more or less immediate:

   (a) $\emptyset, X \in \mathcal{C}$.

   (b) If $\{F_i : i \in I\}$ is a family of closed sets, then their intersection $\cap_{i\in I} F_i$ is again closed.

   (c) If F_1 and F_2 are closed, then so is $F_1 \cup F_2$.

48. Examples of Closed Sets:

   (a) Any finite subset of a metric space is closed.

   (b) Any closed ball $B[x,r]$ in a metric space is closed. Hence any closed interval $[a,b]$ is closed in $\mathbb{R}$.

   (c) Any sphere $S(x,r) := \{y \in X : d(x,y) = r\}$ in a metric space is closed.

   (d) The set $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ is closed in $\mathbb{R}$.

   (e) The set $(-\infty, 0) \cup [1, \infty)$ is closed in $\mathbb{R}$ with lower limit topology but not closed in $\mathbb{R}$ with the usual topology.

   (f) The only subsets of $\mathbb{R}$ which are both open and closed are $\emptyset$ and $\mathbb{R}$.

   (g) The set $[0,1)$ is neither closed nor open in $\mathbb{R}$.

   (h) Any subset of a discrete space is open as well as closed.

(i) Any subset $A \subset \mathbb{R}^*$ is closed in $\mathbb{R}$ with VIP topology with 0 as the VIP.

(j) What are the sets which are both open and closed in $\mathbb{R}$ with VIP topology with 0 as the VIP?

(k) Any subset of $\mathbb{R}$ containing 0 is closed in $\mathbb{R}$ with the outcast topology with 0 as the outcast.

(l) What are the sets which are both open and closed in $\mathbb{R}$ with the outcast topology with 0 as the outcast?

(m) Any vector subspace of $\mathbb{R}^n$ is closed. So are its translates.

(n) The set of $n \times n$ symmetric matrices and the set of $n \times n$ skew-symmetric matrices are closed in $M(n, \mathbb{R})$.

(o) The set $GL(n, \mathbb{R})$ is not closed in $M(n, \mathbb{R})$.

(p) The set of singular matrices in $M(n, \mathbb{R})$ is closed.

(q) The set $\{f \in C[0,1] : f(1/2) = 0\}$ in $X := (C[0,1], \| \ \|_\infty)$ is closed.

(r) The sets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are neither closed nor open in $\mathbb{R}$.

49. We have the following characterization of continuity in terms of closed sets.

Theorem 5. *Let $f\colon X \to Y$ be a map between topological spaces. Then f is continuous iff $f^{-1}(B)$ is closed in X for every closed set $B \subset Y$.* □

50. As we did earlier in the case of continuity and open sets, we may use the above theorem to assert that certain subsets are closed.

(a) The set $\{(x,y) \in \mathbb{R}^2 : xy = 0\}$, $\{(x,y) \in \mathbb{R}^2 : xy = 1\}$, $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ are closed in $\mathbb{R}^2$.

(b) The closed rectangle $R := [a,b] \times [c,d]$ is closed in $\mathbb{R}^2$.

(c) The unit n-dimensional sphere $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is closed in $\mathbb{R}^{n+1}$.

(d) The set $SL(n, \mathbb{R})$ of matrices $A \in M(n, \mathbb{R})$ with determinant 1 is closed in $M(n, \mathbb{R})$.

(e) The subset of matrices whose trace is 0 is closed in $M(n, \mathbb{R})$. (Also follows from Item 48m.)

(f) The set $O(n)$ of orthogonal matrices is closed in $M(n, \mathbb{R})$. *Hint:* The maps $M(n, \mathbb{R}) \to \mathbb{R}$ given by $A \mapsto R_i(A) \cdot R_j(A) \equiv \sum_{k=1}^n a_{ik} a_{jk}$ are continuous. Here $R_i(A)$ denotes the i-th row of A.

(g) The set of singular matrices in $M(n, \mathbb{R})$ is closed.

(h) The set of nilpotent matrices in $M(n, \mathbb{R})$ is closed.

Items 47–50 were done on September 15, 2004.

51. * Let A be a subset of a topological space. The characteristic function χ_A of A is defined by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

What can you conclude about A if the function χ_A is continuous on X?

52. Analysis of the answers and remarks on the question paper of the first test.

53. Revision of closed sets and their properties. For any set A of a topological space $(X, \mathcal{T})$, the smallest closed set containing A exists. It is denoted by $\overline{A}$ and called the closure of A in X. (Compare this with the existence of the smallest topology containing a family $\{A_i : i \in I\}$ of subsets of a set X.) Note that $A \subset \overline{A}$.

54. Examples of closures:

    (a) The closure of $(a, b) \subset \mathbb{R}$ is $[a, b]$.

    (b) The closure of $\mathbb{Q}$ in $\mathbb{R}$ is $\mathbb{R}$.

    (c) The closure of an open ball $B(x, r)$ in $\mathbb{R}^n$ is the closed ball $B[x, r]$. In a general metric space, this need not be true. Consider $B(x, 1)$ and $B[x, 1]$ in a discrete metric space with at least two points.

    (d) Let $\mathbb{R}$ be given the VIP topology with 0 as the VIP. Then the closure of $A = \{0\}$ is $\mathbb{R}$. The closure of $\mathbb{R} \setminus \mathbb{Q}$ is itself. The closure of $\{a\}$ is itself if $a \neq 0$.

    (e) Investigate the case of $\mathbb{R}$ with outcast topology.

Items 52–54 were done on 05-10-2004.

55. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then $x \notin \overline{A}$ iff there exists an open set $U \ni x$ with $U \cap A = \emptyset$. Hence, $x \in \overline{A}$ iff for every open set $U \ni x$, we have $U \cap A \neq \emptyset$. This suggests the following definition.

    Definition 6. $x \in X$ is said to be a *limit point* of A if for every open set $U \ni x$, we have $U \cap A \neq \emptyset$.

    This is NOT the standard definition and hence should not be confused with the notion of cluster or an accumulation point which we shall see below. We shall follow our nomenclature only.

56. Every point of A is a limit point of A.

57. $x \in \overline{A}$ iff x is a limit point of A.

58. Let (X, d) be a metric space, $A \subset X$. Then $x \in X$ is a limit point of A iff there exists a sequence (a_n) in A such that $a_n \to x$.

59. With the notation as in the last item, $x \in \overline{A}$ or x is a limit point of A iff $d_A(x) = 0$.

60. In any normed linear space $(X, \| \ \|)$, the closure of an open ball $B(p, r)$ is $B[p, r]$. Thus, $q \in X$ is a limit point of $B(p, r)$ iff $d(p, q) \leq r$. In particular, $\overline{B(p, r)} = B[p, r]$. *Hint:* If $q \in B[p, r]$, consider the line segment $(1 - t)p + tq$, $0 \leq t \leq 1$. Draw picture. From this line segment, you can find a sequence $p_k \in B(p, r)$ which converges to q.

61. The set theoretic results about the closure operation:

    (a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

    (b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

    (c) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. Strict containment can occur.

(d) $\cup_{i\in I}\overline{A_i} \subset \overline{\cup_{i\in I}A}$. Strict containment can occur.

Definition 7. $x \in X$ is a cluster or an accumulation point of A iff for every open set $U \ni x$, the set $(U \setminus \{x\}) \cap A \neq \emptyset$, that is, any open set $U \in x$ contains a point of A other than x.

62. Intuitively, A accumulates or clusters around x. (They are like celebrities of A!) Obviously, any cluster point of A is a limit point of A, but not conversely. The notion of a cluster point is much stronger and more stringent than that of a limit point.

63. Every point of $A = \mathbb{Z} \subset \mathbb{R}$ is a limit point of A but there exists no cluster point of A in $\mathbb{R}$.

64. Consider $\mathbb{R}$ with VIP topology with 0 as the VIP. Then any nonzero real number is a cluster point of $A = \{0\}$. Zero is obviously a limit point of A but not a cluster point of A.

65. The last example also shows that the following can occur. x may be a cluster point of A, but there may exist open sets $U \ni x$ with $U \cap A$ is finite!

66. Any point in any ball (open or closed) in an NLS is a cluster point of the ball.

67. Analyse the situation in a metric space. In a metric space, if x is a cluster point of A, then every open set $U \ni x$ will contain infinitely many points of A. The proof suggested the following definition.

Definition 8. A topological space X is said to be Hausdorff iff for every pair $x, y \in X$ of distinct points, there exist open set U, V such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$.

68. Let $(X, \mathcal{T})$ be a Hausdorff (topological) space and $A \subset X$. Then $x \in X$ is a cluster point of A iff for every open set $U \ni x$, the set $U \cap A$ is infinite.

69. Any finite set in a Hausdorff space cannot have a cluster point. (Hausdorff condition is required. Look at $\mathbb{R}$ with VIP topology with zero as the VIP and $A = \{0\}$.)

Items 55–69 were done on 07-10-2004.

70. Let $(X, \mathcal{T})$ be any topological space and $A \subset X$. Then $\overline{A}$ is the union of A and the cluster points of A.

Definition 9. We say that a sequence (x_n) in a topological space $(X, \mathcal{T})$ *converges* to a point $x \in X$, if for every open set $U \ni x$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. The point x is called the limit of the sequence and (x_n) is said to be *convergent*.

71. If $(X, \mathcal{T})$ is a hausdorff (topological) space, then any convergent sequence has a unique limit.

    This need not be true in a general space. For instance, if we consider $\mathbb{R}$ with discrete topology, any sequence is convergent to any point of $\mathbb{R}$!

72. In any hausdorff space, any finite set is closed.

This need not be true in an arbitrary topological space. For instance, consider the indiscrete topology on $\mathbb{R}$.

Hence conclude: The topology of any finite hausdorff is discrete.

73. Examples of Convergent sequences:

(a) The only convergent sequences in any discrete space are eventually constant sequences.

(b) In the NLS $(B(X,\mathbb{R}), \| \|_\infty)$, a sequence (f_n) converges to $f \in B(X,\mathbb{R})$ iff f_n converges to f uniformly on X.

(c) A sequence (x_k) in $\mathbb{R}^n$ converges to $x \in \mathbb{R}^n$ iff $x_{kj} \to x_j$ as $k \to \infty$ for $1 \leq j \leq n$.

74. We analyzed the proof of Item 58 and arrived at the following conclusion:

> Let $(X, \mathcal{T})$ be a space with the following property: For every $x \in X$, there exists a countable collection of open sets $\{U_{n,x} : n \in \mathbb{N}\}$ such that
>
> (a) For every open set $U \ni x$, there exists n such that $x \in U_{n,x} \subset U$
>
> (b) $\cap_n U_{n,x} = \{x\}$.
>
> Then, $x \in X$ is a limit point of $A \subset X$ iff there exists a sequence (a_n) in A such that $a_n \to x$.

75. The foregoing item led us to the following concepts.

Definition 10. Let $(X, \mathcal{T})$ be a topological space and $p \in X$. Then by a *local base* at p, we mean a family $\{U_i : i \in I\}$ of open sets containing p with the property that if U is an open set containing p, then there exists $i \in I$ such that $x \in U_i \subset U$.

A space is said to be *first countable* if there exists a countable local base at every point $p \in X$.

76. Observe that if $(X, \mathcal{T})$ is countable, then we may assume that any local base $\{U_{n,p} : n \in \mathbb{N}\}$ is decreasing sequence.

77. We look at some examples:

(a) In $\mathbb{R}$ with standard topology, $\{(p - 1/n, p + 1/n) : n \in \mathbb{N}\}$ is a local base. Hence $\mathbb{R}$ is first countable. More generally, any metric space is first countable.

(b) If $\mathbb{R}$ is endowed with the discrete topology, then a local base at any point can be taken as $\{x\}$. Hence $\mathbb{R}$ with discrete topology is first countable.

(c) Consider $\mathbb{R}$ with VIP topology. (Convention: VIP is always 0.) Then the set $\{p, 0\}$ is a local base at any $p \in \mathbb{R}$. (If $p = 0$, then the set $\{p, 0\} = \{0\}$!) Hence $\mathbb{R}$ with VIP topology is first countable.

(d) Any indiscrete topology is first countable.

78. Let $(X, \mathcal{T})$ be a hausdorff, first countable space. Let $\{U_{p,n} : n \in \mathbb{N}\}$ be a countable local base. Then $\cap_n U_{n,p} = \{p\}$. (We did not use the full power of hausdorff condition. We could have achieved the same result with less stringent hypothesis, but we shall not worry about this!)

79. In view of Item 74 and Item 78, we have the following.

Theorem 11. *Let $(X, \mathcal{T})$ be first countable and hausdorff. Then x is a limit point of A iff there exists a sequence (a_n) in such that $a_n \to x$.*

Definition 12. We say that a topological space $(X, \mathcal{T})$ is *second countable* if there exists a countable basis for $\mathcal{T}$.

80. Clearly, any second countable space is first countable.

81. Examples and non-examples:

    (a) $\mathbb{R}$ with the standard topology is second countable. (See Item 13e.)

    (b) A discrete space X is second countable iff the set X is countable.

    (c) $\mathbb{R}$ with VIP topology is first countable but not second countable.

    (d) Any indiscrete space is second countable.

82. Think over this: What will be the counter part (in terms of open sets) of the smallest closed set containing A? We shall soon discuss this.

83. Answer to the last item: It is the largest open set contained in A. It is called the interior of A and is denoted by $\text{Int}\,(A)$.

84. Examples of interior of a set:

    (a) The interior of an open set is itself.

    (b) The interior of $[a, b] \subset \mathbb{R}$ is (a, b).

    (c) The interior of $\mathbb{Q} \subset \mathbb{R}$ is the empty set. What is $\text{Int}\,(\mathbb{R} \setminus \mathbb{Q})$?

    (d) The interior of a proper vector subspace of $\mathbb{R}^n$ is empty. Does this generalize to any NLS?

    (e) The interior of a closed ball $B[p, r]$ in any NLS is the open ball $B(p, r)$. In a general metric space, such a result is not true.

From Item 70 to up to this item were done on October 12, 2004.

    (f) Let $(X, \mathcal{T})$ be a discrete space. Then $\text{Int}\,(A) = A$ for any $A \subset X$.

    (g) Let $(X, \mathcal{T})$ be an indiscrete space. Then $\text{Int}\,(A) = \emptyset$ for any $A \subset X$, $A \neq X$.

    (h) Consider $\mathbb{R}$ with the VIP topology (VIP is 0). The interior of $\mathbb{R}^*$ is the empty set. What is $\text{Int}\,(\mathbb{Q})$ and $\text{Int}\,(\mathbb{R} \setminus \mathbb{Q})$ in this topology? More generally, if $0 \in A$, then $\text{Int}\,(A) = A$ and if $0 \notin A$, then $\text{Int}\,(A) = \emptyset$.

    (i) Consider $\mathbb{R}$ with the outcast topology (outcast is 0). The interior of any set A is $A \setminus \{0\}$.

85. A is open iff $A = \text{Int}\,(A)$.

86. Set theoretic results about the interior operation:

    (a) If $A \subset B$, then $\text{Int}\,(A) \subset \text{Int}\,(B)$.

    (b) $\text{Int}\,(A) \cup \text{Int}\,(B) \subset \text{Int}\,(A \cup B)$.

(c) $\text{Int}(A \cap B) = \text{Int}(A) \cap \text{Int}(B)$.

(d) $\cup_{i \in I} \text{Int}(A_i) \subset \text{Int}(\cup_{i \in I} A_i)$.

Definition 13. Let X be a (metric) space and $A \subset X$. A point $x \in X$ is said to be a *boundary point* of A in X if every open set that contains x intersects both A and $X \setminus A$ non-trivially. The *boundary* of A in X is the set of boundary points of A in X. We denote it by ∂A.

87. Find the boundaries of each of the following sets:

(a) $A_1 = (a, b]$.

(b) $A_2 = \mathbb{R} \setminus \{0\}$.

(c) $A = \mathbb{Q} \subset \mathbb{R}$.

(d) $\partial \emptyset = \emptyset = \partial X$ for any topological space X.

(e) The boundary of an open or closed ball in $\mathbb{R}^n$ is the sphere: $\partial B(x, r) = \partial B[x, r] = S(x, r) := \{y \in \mathbb{R}^n : d(x, y) = r\}$. Is this true in an NLS? in an arbitrary metric space?

(f) In $\mathbb{R}$ with VIP topology and $\mathbb{R}$ with outcast topology, find ∂A, where $A = \{0\}, \{x\}, \mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$. ($x$ is a nonzero real number.)

(g) Let B be an open ball in $\mathbb{R}^n$. Find the boundary of B minus a finite number of points.

(h) Let $A := \{z \in \mathbb{C} : z = re^{it}, r \in [0, 1], t \in (0, 2\pi)\}$. (Draw a picture.) Find the boundary of A.

Note: Items marked by * are the ones not discussed in the class but we may discuss them in future.

88. * A few more examples to sharpen our geometric intuition.

(a) Consider $A = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$. What is the boundary of A in $\mathbb{R}^2$?

(b) $A = U_1 \cup U_2 \cup U_3$ is the subset of $\mathbb{R}^2$ where $U_1 := \{x^2 + y^2 < 1, y > 0\}$, $U_2 := \{-1 \leq x \leq 1, y = 0\}$ and $U_3 := \{x^2 + y^2 = 1, y < 0\}$.

(c) $A = \{(x, y) : x^2 + y^2 = 1\}$.

89. Show that for any subset A of a topological space $(X, \mathcal{T})$, $\partial A = \overline{A} \cap \overline{X \setminus A}$. (This is the standard definition.

90. While trying to prove the equivalence of the definition of continuity at a point (of a function between two metric spaces) with the sequential definition, we established the following.

Theorem 14. *Let X and Y be arbitrary topological spaces and $p \in X$. Let $f \colon X \to Y$ be a map.*

1. *If f is continuous at p, then for every sequence (x_n) in X converging to p, we have $f(x_n) \to f(p)$.*

2. *Assume that X is first countable. Assume further that f has the property that for every sequence (x_n) converging to p, the sequence $(f(x_n))$ converges to $f(p)$ in Y. Then f is continuous at p.* □

From part of Item 84 till the last item were done on October 14, 2004.

Definition 15. A subset $D \subset X$ of a topological space is dense in X if for every *nonempty* open set $U \subset X$, we have $D \cap U \neq \emptyset$, that is U intersects D non-trivially.

91. Examples of dense sets:

   (a) $\mathbb{Q}$ is dense in $\mathbb{R}$. Is $\mathbb{R} \setminus \mathbb{Q}$ dense in $\mathbb{R}$? Can you think of a countable dense subset in $\mathbb{R}^2$? in $\mathbb{R}^n$?

   (b) In $\mathbb{R}$, with the lower limit topology, the sets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense.

   (c) The set $A := \{x \in \ell_1 : x_n = 0 \text{ for all } n \geq N \text{ for some } N\}$ is dense in ℓ_1.

   (d) The set D_n of all sequences $x = (x_m) \in \ell_1$ whose terms are rational and $x_k = 0$ for $k > n$. Let $D := \cup_{n \in \mathbb{N}} D_n$. Then D is a countable dense subset of ℓ_1.

   (e) The only dense subset of a discrete space X is X itself.

   (f) In an indiscrete space, any nonempty subset is dense.

   (g) The set $\{0\}$ is dense in $\mathbb{R}$ with the VIP topology. The set $\mathbb{R} \setminus \mathbb{Q}$ is not dense.

   (h) The set $\mathbb{R} \setminus \{0\}$ is dense in $\mathbb{R}$ with the outcast topology. This space cannot have a countable dense set.

   (i) * $S := \{n + m\sqrt{2} : n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. (Did you notice that $\mathbb{Z}$ and $\sqrt{2}\mathbb{Z}$ are closed and S is a sum of two closed sets?) An item for Student Seminar.

   (j) * Is $\mathbb{Q}^2$ dense in $\mathbb{R}^2$ with the order topology?

   (k) * Weierstrass approximation theorem says that the vector subspace of polynomials in the NLS $(C[0,1], \| \ \|_\infty)$ is dense. (This should be a topic for Student Seminar!)

92. $D \subset X$ is dense in a space (X, d) iff every point of X is a limit point of D.

93. $D \subset X$ is dense in the space X iff its closure $\overline{D} = X$. (This is the standard definition.)

94. In a metric space (X, d), a set A is dense in X iff for every $x \in X$ and $\varepsilon > 0$, there exists an $a \in A$ such that $d(x, a) < \varepsilon$. (Thus, A is dense in X, if we can "approximate" any point $x \in X$ to "any level of approximation" by an element of A.)

95. * Let (X, d) be a metric space. Assume that the only dense subset is X itself. Can we say something about the topology?

96. * Let A, B be two dense subsets of a space? Is $A \cup B$ dense? Is $A \cap B$ dense?

97. * If A, B are open dense subsets of a space X, is their intersection dense?

98. * Give an example of a proper open dense subset of $\mathbb{R}$.

99. * Continuation of the last item. If we write an open set $U = \dot\cup J_k$, as the disjoint union of open intervals, then we say that the "measure" or "length" of U is $\sum_k \ell(J_k)$, the sum of lengths of the intervals J_k. Given $\varepsilon > 0$, can you find an open dense subset of $\mathbb{R}$ whose length is less than or equal to ε?

100. * Let D be dense in $(X, \mathcal{T}_1)$. Is D (necessarily) dense in $(X, \mathcal{T}_2)$ where $\mathcal{T}_2$ is finer (respectively, coarser) than $\mathcal{T}_1$?

101. * Let X, Y be topological spaces. Assume that A is dense in X and $f\colon X \to Y$ is continuous and onto. Then $f(A)$ is dense in Y.

102. * The set of matrices in $M(2,\mathbb{C})$ with distinct eigenvalues is dense. In particular, the set of all diagonalizable matrices in $M(2,\mathbb{C})$ is dense. (The topology on $M(2,\mathbb{C})$ is given by $\|A\| := \max\{|a_{ij}| : 1 \le i, j \le 2\}$. This exercise requires a good background in Linear Algebra. Perhaps, we shall do it later.)

    Definition 16. A topological space is *separable* if there exists a countable dense subset.

103. Examples and non-examples of separable spaces:

    (a) $\mathbb{R}^n$ is separable.

    (b) A discrete space X is separable iff X is countable.

    (c) ℓ_1 is separable.

    (d) $\mathbb{R}$ with VIP topology is separable.

    (e) $\mathbb{R}$ with outcast topology is not separable.

    (f) Any second countable space is separable.

    (g) * Let X be infinite with cofinite topology and let A be any infinite subset of X. Then any $x \in X$ is a limit point of A. In particular, X with cofinite topology is separable.

    (h) * Is $\mathbb{R}^2$ with the order topology separable? (Recall the geometric description of basic open sets in this space. See Item 15.)

104. * Let X be uncountable with cofinite topology. Then X is not first countable but separable by Item 103g.

105. * Let X be uncountable with cocountable topology. No countable set can have a limit point and hence X is not separable.

106. * Let ℓ_∞ denote the set of all bounded real sequences. It is a normed linear space with respect to the norm $\|x\|_\infty := \sup\{|x_n| : n \in \mathbb{N}\}$. The space $(\ell_\infty, \|\ \|_\infty)$ is not separable. *Hint:* Consider the uncountable subset $\{x\colon \mathbb{N} \to \{0,1\}\}$ of ℓ_∞.

107. A metric space is separable iff it is second countable.

108. * $\mathbb{R}_\ell$, the space $\mathbb{R}$ with lower limit topology, is first countable but not second countable.

109. * Let $f, g\colon X \to Y$ be continuous and Y be hausdorff. Then the set $\{x \in X : f(x) = g(x)\}$ is closed in X.

110. * Let the hypothesis be as in the last item. Assume that A is dense in X and that $f(a) = g(a)$ for all $a \in A$. Then $f(x) = g(x)$ for all $x \in X$.

    Most of the items from Definition 15 till the last item were done on October 16, 2004.

111. Let X, Y be sets. Suppose $f\colon X \to Y$ is a bijection. Assume further that one of the sets has an extra mathematical structure such as a group, vector space, metric or a topology. Then we can transfer the structure to the other set using the bijection. We look at some specific instances.

(a) Let X be a group. Then we define $y_1 \cdot y_2$ to be $f(x_1 \cdot x_2)$ where $f(x_i) = y_i$, $i = 1, 2$. It turns out that Y is group and that $f\colon X \to Y$, by virtue of the very definition of group law on Y, is a group homomorphism (and hence an isomorphism.)

(b) * Let Y be a metric space. Then we set $d(x_1, x_2) := d(f(x_1), f(x_2))$. Then the metric space (X, d) is isometric to (Y, d).

(c) Let X be a topological space. Let $\mathcal{T}_X$ be the topology on X. We then define a topology $\mathcal{T}_Y$ on Y be declaring that $V \in \mathcal{T}_Y$ iff there exists $U \in \mathcal{T}_X$ such that $V = f(U)$. Then the map $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a homeomorphism, a term not yet defined!

112. Let $f\colon X \to Y$ be *any* map between two sets. Assume that one of them is a topological space. What we wish to do is to endow the other set with an *optimal* topology in such a way that $f\colon X \to Y$ becomes a continuous map between the spaces.

(a) Let Y be a topological space. Then if we endow X with the discrete topology, then the problem is solved! But this topology has no bearing on Y and/or on the map f! So what we require is the smallest topology on X making f continuous.

(b) Let X to be a topological space. Considerations similar to the last item suggest us that we require the largest topology on Y making f continuous.

113. These problems arise in a very natural way.

(a) Let X be a subset a topological space Y. Then we have an *obvious* or *natural* map $i\colon X \to Y$, the inclusion of X in Y, that is, the restriction of the identity on Y to X.

(b) Let X be any topological space and $\sim$ an equivalence relation on X. Then as Y, we take the quotient set $X/\sim$, that is, the set of equivalence classes. Once again, we have a natural map $\pi\colon X \to Y$, where $\pi(x)$ is the equivalence class of x.

114. More general situations may also arise. Let X be a set and Y_i be topological spaces, indexed by a set I. Assume that we are given certain maps $f_i\colon X \to Y_i$ for each $i \in I$. We again ask for a single smallest topology on X making all the maps f_i continuous. Typical instances of this phenomenon are:

(a) Let $\{X_i : i \in I\}$ be an indexed family of topological spaces. Let $X := \prod_{i\in I} X_i$. We have obvious maps $\pi_i(x) = x_i$, the i-th projection. We wish to equip X with the smallest topology such that each of the projections becomes continuous.

(b) * Let E be a set and let $X := \mathcal{F}$ be a family of functions from E to $\mathbb{R}$. Consider the evaluation maps $\varepsilon_x(f) := f(x)$ for each $x \in E$. Thus, we have a family of maps $\varepsilon_x\colon X \to \mathbb{R}$ and we want the smallest topology which will make all these maps continuous.

115. Let us deal with various cases. Let X be a set and Y be a topological space and $f\colon X \to Y$ be a map. Any topology on X which makes f continuous must contain the set $\mathcal{U} := \{f^{-1}(V) : V \in \mathcal{T}_Y\}$. It turns out this collection is already a topology and hence is the smallest topology on X, as required. (We were lucky this time!)

116. Let us look at the concrete case in Item 113a. Then the topology on X is given by

$$\mathcal{T}_X := \{i^{-1}(V) : V \in \mathcal{T}_Y\} = \{V \cap A : V \in \mathcal{T}_Y\}.$$

The topology $\mathcal{T}_X$ is called the subspace topology on Y and any $U \in \mathcal{T}_X$ is said to be open *in* X. We say that $F \subset X$ is closed in X if its complement, $X \setminus F$, *in* X is open in X.

Let us look at some examples to develop our intuition:

(a) Consider $A = [0,1] \subset \mathbb{R}$. Then the sets $[0,1/2)$, $(1/2,1]$ and $(1/2,3/4)$ are open in in A.

(b) Consider $A = \mathbb{Q} \subset \mathbb{R}$. Then the set $\{r \in \mathbb{Q} : -\sqrt{2} \leq r \leq \sqrt{2}\}$ is both open and closed in $\mathbb{Q}$.

(c) Let X be a metric space and $\emptyset \neq A \subset X$. Then we have two topologies on A: (i) one comes from the induced metric, call it δ, on A and (ii) the other is the subspace topology. They are the same.

(d) * Let $A := [0,1] \times [0,1]$. Then A has the order topology as well as the subspace topology as a subset of $\mathbb{R}^2$ with order topology. They are not the same. (Contrast this the last item.)

(e) Consider $\mathbb{R}$ with VIP topology and $A = \mathbb{R}^*$. Then the subspace topology on $\mathbb{R}^*$ is the discrete topology. The subspace topology on $\mathbb{Q}$ is the VIP topology on $\mathbb{Q}$. (Do you understand this statement?)

(f) * Investigate the subspace topology on $\mathbb{Q}$ considered as a subset of $\mathbb{R}$ with outcast topology.

117. Let A be nonempty and open in X. Then $U \subset A$ is open in A iff it is open in X.

118. Let $A \subset X$. Then $F \subset A$ is closed in A iff there exists a closed set C in X such that $F = A \cap C$.

* As a specific example, the set of Item 116b is open as well as closed in $\mathbb{Q}$. (Contrast this with Item 48f.)

119. Let us consider the general case in Item 114. We want the smallest topology $\mathcal{T}$ that contains all sets of of the form $f_i^{-1}(V_i)$ where V_i is open in X_i and $i \in I$. That is $\mathcal{T}$ is the smallest topology containing the family of sets $\mathcal{S} := \{f_i^{-1}(V_i) : V_i \in \mathcal{T}_i; i \in I\}$, where $\mathcal{T}_i$ is the topology on X_i.

There is no reason to believe that $f_i^{-1}(V_i) \cap f_j^{-1}(V_j)$ must be again of the form $f_r^{-1}(V_r)$ for some $r \in I$. Hence $\mathcal{S}$ may not be topology on X.

Items 111–119 were done on October 19, 2004.

120. When we wanted to look at the concrete case in Item 114a, we needed to review the concept of cartesian product.

Definition 17. Let $\{X_i : i \in I\}$ be an indexed family of sets. Then the Cartesian product $X := \prod_{i \in I} X_i$ is defined by

$$\prod_{i \in I} X_i := \{x \colon I \to \biguplus_{i \in I} X_i : x(i) \in X_i \text{ for each } i \in I\},$$

where $\biguplus_{i\in I} X_i$ stands for the disjoint union.

(a) We usually write $x \in \prod_{i\in I} X_i$ as $x = (x_i)$, where $x_i := x(i)$. We shall call x_i as the i-th coordinate of x. Let $\pi_i\colon \prod_{j\in I} X_i \to X_j$ denote the map $\pi_j(x) = x(j) = x_j$. This is called the j-th projection of X onto the j-th factor X_j.

(b) As a convention, if $I = \{1, 2, \ldots, n\}$, we identify X with $X_1 \times \cdots \times X_n$, that is, with the set of "ordered n-tuples" $(x_1, \ldots, x_n)$. Similarly, if $I = \mathbb{N}$, we identify X with $X_1 \times X_2 \times \cdots \times X_n \times \cdots$, that is the set of ordered infinite tuples $x \mapsto (x_1, x_2, \ldots, x_n, \ldots)$.

(c) If $V_j \subset X_j$, then $\pi_j^{-1}(V_j) = \prod_{i\in I} U_i$ where $U_i = X_i$ for $i \neq j$ and $U_j = V_j$. In particular, $\pi_1^{-1}(V_1) = V_1 \times X_2$ where $X = X_1 \times X_2$ etc.

121. What we requite on $X := \prod_{i\in I} X_i$ to make the projections π_i $(i \in I)$ continuous is the smallest topology that contains

$$\mathcal{S} := \{f_i^{-1}(V_i) : V_i \in \mathcal{T}_i, i \in I\}.$$

This led us to the more general problem.

122. Given a set X and a collection $\mathcal{S}$ of subsets of X, how to we describe the open sets in the smallest topology, say, $\mathcal{T}_{\mathcal{S}}$ that contains $\mathcal{S}$? (We assume, as this is the case that occurs in practice, that for every $x \in X$, there exists $S \in \mathcal{S}$ such that $x \in S$.) We did this in two steps.

(a) We wanted a base for some topology on X which will also contain $\mathcal{S}$. Clearly, $\mathcal{B} := \{S_1 \cap \cdots \cap S_n : S_j \in \mathcal{S}, n \in \mathbb{N}\}$ is a base for some topology and $\mathcal{S} \subset \mathcal{B}$.

(b) The topology $\mathcal{T}_{\mathcal{B}} := \{U \subset X : \forall x \in U, \exists B \in \mathcal{B}$ such that $x \in B \subset U\}$ is then the smallest topology that contains $\mathcal{S}$.

(c) Thus, we can rid of the intermediate $\mathcal{B}$ and define the topology directly in terms of $\mathcal{S}$. We say $U \in \mathcal{T}_{\mathcal{S}}$ iff for every $x \in U$, there exists $n \in \mathbb{N}$ such that we can find S_j, $1 \leq j \leq n$ with $x \in S_1 \cap \cdots \cap S_n \subset U$. One can again show directly that this is the smallest topology containing $\mathcal{S}$.

(d) $\mathcal{S}$ is called a *subbase* and $\mathcal{T}_{\mathcal{S}}$ is the topology generated by $\mathcal{S}$.

123. Let us look at some concrete examples:

(a) Consider $\mathcal{S} := \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$. The topology generated by $\mathcal{S}$ on $\mathbb{R}$ is the usual topology.

(b) Consider $\mathbb{R}$ and $\mathcal{S} := \{\{0, x\} : x \neq 0, x \in \mathbb{R}\}$. What is the topology on $\mathbb{R}$?

(c) Let X be a set with at least 3 elements. Let $\mathcal{S}$ be the family of two-element subsets of X. The topology generated by $\mathcal{S}$ is the discrete topology.

(d) What is the topology on $\mathbb{R}^2$, if we take the subbase consisting of all straight lines in $\mathbb{R}^2$?

(e) What is the topology on $\mathbb{R}^2$, if we take the subbase consisting of all straight lines parallel to the x-axis in $\mathbb{R}^2$?

(f) * What is the topology on $\mathbb{R}^2$, if we take the subbase consisting of all circles in $\mathbb{R}^2$?

(g) * What is the topology on $\mathbb{R}^2$, if we take the subbase consisting of all circles, with centre at the origin, in $\mathbb{R}^2$?

(h) * Consider $\mathcal{S} = \{X\}$ as a subbase on X. What topology do we get on X?

Items 120–123 were done on October 21, 2004.

124. We reviewed Item 122. We applied it to the problem posed in Item 121. Thus we arrived at the definition of *product topology* on $\prod_{i\in I} X_i$ as follows.

Definition 18. As a subbase for a topology on X, we take the set

$$\mathcal{S} := \left\{ \prod_{i\in I} U_i : \text{ where } U_i = X_i \text{ for all but finitely many } i \text{ and } U_i \text{ is open in } X_i \right\}.$$

The basis for the product topology on X is finite intersections of elements from $\mathcal{S}$. In particular, $G \subset X$ is open iff for every $x \in G$, there exists $S_1, \dots, S_n \in \mathcal{S}$ such that $x \in S_1 \cap \dots \cap S_n \subset G$. Thus, there exist $i_1, \dots, i_n \in I$ and U_{i_j} an open subset of X_{i_j} such that $x \in \prod_i V_i$ where $V_i = X_i$ for $i \neq i_j$ and $V_{i_j} = U_{i_j}$ and $x \in \prod_{i\in I} V_i \subset G$.

125. Let $\emptyset \subsetneq U_i \subsetneq X_i$, be open in X_i for $i \in I$. Then $U = \prod_{i\in I} U_i$ could never be open in X unless I is finite.

126. If I is finite, say, $I = \{1, 2, \dots, n\}$, then the basic open sets are of the form $U_1 \times \dots \times U_n$ where U_i is an arbitrary open set in X_i for each $1 \leq i \leq n$.

127. **Warning:** If, at first, we defined finite products of topological spaces with basis as in the last item, we would be tempted to use the following collection as a basis for a topology on the product $\prod_{i\in I} X_i$:

$$\mathcal{B} := \left\{ \prod_{i\in I} U_i : \text{ where } U_i \text{ is an arbitrary open set in } X_i \right\}.$$

The topology given rise to by this basis is called the box topology. Evidently, this is finer than the product topology.

128. However, the product topology on X is the smallest topology which makes all the projection maps π_i continuous. We shall always use this topology on the product sets.

129. We also saw how to draw pictures for arbitrary product spaces to gain some geometric intuition.

130. To have a feeling for the product topology, we looked at the following results/questions:

(a) The product of hausdorff spaces is hausdorff.

(b) A sequence (x_k) in the product space is convergent to an element x iff it converges coordinate-wise, that is, iff $\pi_i(x_k) \to \pi_i(x)$ for each $i \in I$.

(c) Let $A_i \subset X_i$ and $A := \prod_{i\in I} A_i$. Then $\overline{A} = \prod_{i\in I} \overline{A_i}$. In particular, if each A_i is closed, then the product $A := \prod_{i\in I} A_i$ is closed in the product space X. Contrast this with Item 125.

(d) Let D_i be dense in X_i for each i. Then $D := \prod_{i\in I} D_i$ is dense in X.

(e) Let X_i be a discrete space for each i. When is $\prod_{i\in I} X_i$ is discrete?

(f) Let X, Y be metric spaces. We have a product metric on the product $X \times Y$ given by $\delta((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), d(y_1, y_2)\}$. Thus we have two topologies on $X \times Y$, namely, the topology induced by the metric δ and the product topology (got out of the metric topologies on X and Y). We saw that these two topologies are the same. Later, we shall see an easy proof.

Investigate whether the converses (wherever applicable) are true.

Optional: Investigate how many of them are true if we equip X with the box topology. Note that if D is dense in $(X, \mathcal{T}_2)$ and if $\mathcal{T}_1$ is another topology on X with $\mathcal{T}_1$ weaker than $\mathcal{T}_2$, then D is dense in $(X, \mathcal{T}_1)$.

Remark 19. In most of the examples above, we looked at subsets of the product set X which are of the form $\prod_{i\in A_i}$, where $A_i \subset X_i$. You should be aware that not all subsets of X are of this form. For example, $S := \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$ is not a product of subsets of $\mathbb{R}$.

131. A problem similar to Item 130a: Let X be any set and $\mathcal{F}$ be a collection of real valued functions on X with the property that for any pair of distinct points $x, y \in X$, there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$. Then the smallest topology on X which makes all the functions in $\mathcal{F}$ continuous is hausdorff.

132. Let X be a topological space and $\sim$ is an equivalence relation on X. Let $Y := X/\sim$ be the quotient set, that is, the set of all equivalence classes. Let $\pi\colon X \to Y$ be the quotient map $\pi(x) := [x]$, the equivalence class of x. The largest topology on Y with respect to which π is continuous is called the quotient topology on Y. It is given by

$$\left\{V : \pi^{-1}(V) \text{ is open in } X\right\}.$$

133. It is equally important to recognize product spaces in disguise. The following are very typical of this situation.

(a) Define a topology on the set S of all real sequences such that a sequence (x_k) in S converges to $x \in S$ iff the $x_{kn} \to x_n$ as $n \to \infty$ for all k where $x_k = (x_{k1}, x_{k2}, \dots, x_{kn}, \dots)$. (Convergence = Coordinatewise convergence).

(b) Let X denote the set of all real valued functions on $\mathbb{R}$. Define a topology on X such that a sequence (f_n) of functions in X converge to a function $f \in X$ iff $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$. (Convergence = pointwise convergence of functions.)

(c) Let $I = \mathbb{N}$ and $X_i = \{0, 1\}$ for $i \in \mathbb{N}$. Then the product space $X := \prod_{i\in\mathbb{N}} X_i$ "is isomorphic to" the Cantor set. We have to introduce concepts and develop some more theory to explain this satisfactorily.

134. Contrast Item 133b with the following. Let E be any set and let $B(E, \mathbb{R})$ denote the set of all bounded real valued functions on E. If we endow this vector space with the norm $\|f\|_\infty := \sup_{x\in E} |f(x)|$, then $f_n \to f$ in this NLS iff $f_n \to f$ uniformly on E. (This is Item 73b.)

135. The analogue of isomorphism in algebra for topological spaces is the concept of homeomorphism. A *homeomorphism* $f\colon X \to Y$ between two topological spaces is a bijection such that $f\colon X \to Y$ and $f^{-1}\colon Y \to X$ are continuous.

Remember that this definition was arrived at by one of you and was not given by me!

Contrast this with the isomorphisms in algebra!

Items 124–135 were done on November 3-4, 2004.

136. We *studied* part of my article "Generation Topologies — A Unified View of Subspace, Product and Quotient Topologies". We also did the Universal mapping properties for Cases (i) & (ii) of the article.

Only Item 136 was done on November 5, 2004! The main purpose of today's work was to train you to learn from a book some mathematics taught in the class as well as something new, not done in the class.

137. Universal mapping properties were done in the general case and applied to concrete situations and interpreted.

    (a) Universal mapping property for subspace topology.

    (b) Universal mapping property for quotient topology.

    (c) Universal mapping property for product topology.

    Remark 20. The students had difficulty in the last item in describing $f^{-1}(V)$ where $f\colon Y \to X = \prod_{j\in I} X_j$ and $V := \prod_{j\in I} V_j$, a subbasic open set.

138. Examples of applications of universal mapping property:

    (a) The continuity of the map $[0, 2\pi]/\sim$ to S^1.

    (b) The continuity of $S^n \to \mathbb{P}^n(\mathbb{R})$. (This cannot be done using UMP.)

139. We showed that $\prod_{t\in\mathbb{R}} \mathbb{R}$ is not first countable. We interpreted this space as the space of functions and used geometric way of looking at basic open sets in the product topology and solved the problem.

Items 137–139 were done on November 6, 2004. Wish you a Happy Diwali!

140. We recalled the product topology on $X \times Y$ as well as some of the results which was done during the special sessions in the vacation.

    (a) Let X, Y be topological spaces. Let $A \subset X$ and $B \subset Y$. Let $\mathcal{T}_A$ denote the subspace topology on A induced from the topology on X etc. Let $\mathcal{T}_A \times \mathcal{T}_B$ (respectively, $\mathcal{T}_X \times \mathcal{T}_Y$) denote the product topology on $A \times B$, (respectively, the product topology on $X \times Y$). Let $\mathcal{T}_{A\times B}$ denote the subspace topology on $A \times B$ considered as a subset of $X \times Y$. Then $\mathcal{T}_A \times \mathcal{T}_B = \mathcal{T}_{X\times Y}$.

    (b) Let $f\colon Y \to X_1 \times X_2$ be a map from a topological space Y to the topological space $X_1 \times X_2$ with product topology. Then f is continuous iff each f_i, $i = 1, 2$, is continuous, where $f = (f_1, f_2)$ (or, $f_i = \pi_i \circ f$).

    (c) Let $\Delta(X)$ denote the diagonal $\{(x, x) : x \in X \times X\} \subset X \times X$. Then X is hausdorff iff $\Delta(X)$ is closed in $X \times X$.

Definition 21. A map $f\colon X \to Y$ between two topological spaces is a *homeomorphism* if (i) f is bijective, (ii) f is continuous and (iii) $f^{-1}\colon Y \to X$ is continuous.

141. Examples

(a) Any $f\colon \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax$ for a nonzero $a \in \mathbb{R}$ is a homeomorphism.

(b) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a homeomorphism.

(c) Any linear isomorphism of $\mathbb{R}^n$ is a homeomorphism.

(d) $[a, b] \simeq [0, 1]$. More generally, $[a, b] \simeq [c, d]$.

(e) $(-1, 1) \simeq \mathbb{R}$.

(f) $(0, 1] \simeq [1, \infty)$.

(g) $[0, 1) \simeq (0, 1]$.

(h) Any two discrete spaces are homeomorphic iff they have the same cardinality.

(i) Can $\mathbb{Q}$ be homeomorphic to $\mathbb{Z}$ with the subspace topologies (induced from $\mathbb{R}$)?

(j) Is $\mathbb{N} \simeq \mathbb{Z}$ with the subspace topologies (induced from $\mathbb{R}$)?

(k) If two metric spaces are isometric, then they are homeomorphic.

(l) $B(0, 1) \simeq \mathbb{R}^n$.

(m) $S^n \setminus \{e_{n+1}\} \simeq \mathbb{R}^n$. (We investigated this in detail!)

(n) $f\colon X \to Y$ continuous. Then the graph of f with the subspace topology of $X \times Y$ is homeomorphic to X. Applications:

   i. $\mathbb{R}$ is homeomorphic to the parabola $y = x^2$.

   ii. $\mathbb{R}^*$ is homeomorphic to the hyperbola $xy = 1$.

(o) The product space $[-1, 1] \times S^1$ is homeomorphic to a cylinder.

(p) The annulus $\{p \in \mathbb{R}^2 : 1 \le \|p\| \le 2\}$ is homeomorphic to the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 1 \le z \le 2\}$.

(q) Let $f\colon X \to Y$ be a homeomorphism and let $A \subset X$. Then f induces homeomorphism between A and $f(A)$ (and between $X \setminus A$ and $f(X \setminus A)$).
This is a very useful fact. Typical ways of applying this are:

   i. $[0, 1)$ is not homeomorphic to $(0, 1)$.

   ii. $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.
   Both these results need connectedness at least in disguise, but can be proved at this stage using the intermediate value theorem.

(r) Homeomorphism between conic sections:

   i. A circle is homeomorphic to an ellipse.

   ii. A parabola is homeomorphic to a line.

   iii. A (rectangular) hyperbola is homeomorphic to $\mathbb{R}^*$.

   iv. A pair of intersecting lines is not homeomorphic to any of the other conic sections. More generally, a circle, a parabola, a hyperbola and a pair of intersecting lines are mutually non-homeomorphic. (We shall see a proof of this later. Meanwhile you may try to prove along this along the lines of a proof of Item 141(q)i.)

(s) A bijective continuous map need not be a homeomorphism. Examples and a non-example:

i. $\mathbb{R}$ with discrete topology and $\mathbb{R}$ with indiscrete topology.

ii. $f\colon [0,2\pi) \to S^1 \subset \mathbb{C}$ given by $f(t) = e^{it}$. (A more instructive exercise.)

iii. Any bijective continuous map of a finite topological sapce X to itself is a homeomorphism.

(t) The map $x \mapsto (x, y_0)$ of X into $X \times Y$ is a homeomorphism of X with $X \times \{y_0\}$ with the subspace topology.

(u) In any NLS, any two open balls are homeomorphic.

(v) In any NLS, any open ball is homeomorphic to the entire space.

(w) In $\mathbb{R}^n$, we have $B_\infty[0,1] \simeq B_2[0,1]$.

(x) $\mathbb{R}^m \simeq \mathbb{R}^n$ iff $m = n$. This is a highly nontrivial result and we shall not prove this is our course!

(y) Another most important way of proving that a map is a homeomorphism is to use the following result which you might have seen in TYBsc.

A bijective continuous map from a compact metric space to another metric space is a closed map and hence is a homeomorphism.

(z) The spaces $(\mathbb{R}, \mathrm{VIP})$ and $(\mathbb{R}, \mathrm{Outcast})$ are not homeomorphic.

142. Definitions of open and closed maps: A map $f\colon X \to Y$ is said to be *open* if $f(U)$ is open in Y for every U open in X. A closed map is defined similarly.

(a) A bijective continuous map is a homeomorphism iff it is an open map.
Application: The map $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a homeomorphism.

(b) A bijective continuous map is a homeomorphism iff it is an closed map.
Application: Item 141y is proved using the closed map criterion.

143. We say that property of a topological space is a *topological property* if every space Y homeomorphic to X also has the property. Examples:

(a) The space being hausdorff is a topological property.

(b) The space being first countable is a topological property.

(c) The space being second countable is a topological property.

(d) The space being separable is a topological property.

(e) Existence of a nonempty, proper subset which is both pen and closed is a topological property.

(f) Two metric spaces can be homeomorphic, but one of them could be bounded while the other is not. Hence 'being bounded' is not a topological property among metric spaces.

(g) Similarly, completeness is not a topological property among the metric spaces.

We shall see later a lot of examples of topological properties.

Items 140–142 were done on November 16, 18 and 20, 2004.

The study of topology is mainly understanding topological properties and using them to assert whether given two spaces are homeomorphic or not.

144. Definition of an open cover, of a subcover.

Definition 22. Let X be a topological space and $A \subset X$. We say that a collection $\{U_i : i \in I\}$ of subsets of X is an open cover of A if (i) each U_i is an open subset of X and (ii) $A \subseteq \cup_{i \in I} U_i$.

Given an open cover $\{U_i : i \in I\}$ of A, by a subcover of A, we mean a subfamily $\{U_i : i \in J\}$ for some subset $J \subset I$ such that $\{U_i : i \in J\}$ is an open cover of A.

We say that the given open cover admits a finite subcover, if J (in the notation above) is a finite set.

145. Examples of open covers:

(a) "Non trivial" open covers of $\mathbb{R}$:

i. $\{(-n, n) : n \in \mathbb{N}\}$.

ii. $\{(-\infty, n) : n \in \mathbb{N}\}$.

iii. $\{(-r, 2r) : r \in \mathbb{Q}^+\}$.

Do they admit finite subcovers?

(b) Nontrivial open covers of $(-1, 1)$.

(c) In any metric space, $\{B(x, r_x) : x \in X\}$ is an open cover where $r_x > 0$ is pre-assigned for $x \in X$. Such a cover arises "naturally" in the following way: Let $f : X \to \mathbb{R}$ be a continuous function. Let $\varepsilon > 0$ be given. Given $x \in X$, by the continuity of f at x, there exists $r_x > 0$ such that for all y with $d(x, y) < r_x$, we have $|f(x) - f(y)| < \varepsilon$. The collection $\{B(x, r_x) : x \in X\}$ is an open cover of X.

(d) Given a hausdorff space with at least two elements, think of a nontrivial open cover.

(e) Can you say something specific about any open cover of $\mathbb{R}$ with outcast topology?

(f) Give a "non-trivial" open cover of $\mathbb{R}$ with VIP topology.

(g) Open covers of S^n:

i. $\mathbb{R}^{n+1} \setminus \{0\}$. (This is a trivial open cover!)

ii. $U = \mathbb{R}^{n+1} \setminus \{N\}$ and $V := S^n \setminus \{S\}$, where N, S are north and south poles respectively.

iii. $U_i^{\pm} := \{x \in \mathbb{R}^{n+1} : x_i \lesseqgtr 0\}$, $1 \leq i \leq n+1$.

(h) Open cover for a discrete space.

(i) Open cover for an uncountable space with cocountable topology.

(j) Open cover for a set with cofinite topology.

146. Given an open cover $\{U_i : i \in I\}$ of $A \subset X$ by means of open subsets of X, then we have a "natural" open cover $\{V_i : i \in I\}$ by means of subsets of A which are open *in* A and conversely. (Note the indices. "Naturality" does not mean that given V_i's, the U_i's are unique!)

Definition 23. A subset A of a topological space X is said to be *compact* if given any open cover of A, we can find a finite subcover. We say that X is a compact space if X is a compact subset of X.

147. Compact spaces and compact sets in a space; a set is compact iff it is a compact space with subspace topology.

148. Examples of compact sets.

   (a) A finite set is compact. In particular, the empty set is compact.
   (b) An indiscrete space is compact.
   (c) A discrete space is compact iff it is finite.
   (d) $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ are not compact.
   (e) Any open ball in $\mathbb{R}^n$ (or in any NLS) is not compact.
   (f) $\mathbb{R}^n$ is not compact.
   (g) Any closed and bounded interval $[a, b] \subset \mathbb{R}$ is compact.
   (h) Any cube $[-R, R]^n \subset \mathbb{R}^n$ is compact.
   (i) $\mathbb{R}$ with VIP topology is not compact.
   (j) $\mathbb{R}$ with outcast topology is compact.
   (k) Any set with cofinite topology is compact.
   (l) An uncountable set with cocountable topology is not compact.
   (m) If X is compact and $Y \simeq X$, then Y is compact. Thus compactness is a topological property.
   (n) A finite union of compact sets is compact.
   (o) The intersection of two compact sets need not be compact. See, however, Item 148q.
   (p) A closed subset of a compact space is compact.
   (q) In a hausdorff space a compact subset is closed and hence the intersection of compact sets is compact in a hausdorff space.

149. Let (X, d) be a metric space. We say that $A \subset X$ is *bounded* if there exist $x_0 \in X$ and $r > 0$ such that $A \subset B(x_0, r)$. The following are easily seen results about this concept:

   (a) A is bounded iff for every $x \in X$, there exists $R > 0$ such that $A \subset B(x, R)$.
   (b) Let $(X, \| \ \|)$ be an NLS. Show that $A \subset X$ is bounded iff there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in A$.
   (c) Any finite set is bounded.
   (d) Any open or closed ball is bounded.
   (e) A is bounded iff there exists $M > 0$ such that $d(x, y) \leq M$ for all $x, y \in A$.
   (f) If $A \neq \emptyset$ and if we set $\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}$, then A is bounded iff $\operatorname{diam}(A) < \infty$. The extended real number $\operatorname{diam}(A)$ is called the diameter of A.
   (g) $\operatorname{diam}(B(x, r)) \leq 2r$ and strict inequality can occur.
   (h) In an NLS, $\operatorname{diam}(B(x, r)) = 2r$.
   (i) Any convergent sequence in a metric space is bounded.
   (j) Boundedness is not a topological property.
   (k) Which vector subspaces of an NLS are bounded subsets?

(l) Any convergent sequence in a metric space is bounded.

(m) The set $O(n)$ of all orthogonal matrices (that is, the set of matrices satisfying $AA^t = I = A^tA$) is a bounded subset of $M(n, \mathbb{R})$. Here $M(n, R)$ is considered as an NLS as in Ex. 46c.

(n) The set $SL(n, \mathbb{R})$ of all $n \times n$ real matrices with determinant 1 is not bounded in $M(n, \mathbb{R})$. (The metric is as in Ex. 46c.)

(o) The set of all nilpotent matrices in $M(n, \mathbb{R})$ is not a bounded set.

(p) Let G be a subgroup of the multiplicative group $\mathbb{C}^*$ of the non-zero complex numbers. Assume that as a subset of $\mathbb{C}$ it is bounded. Then $|g| = 1$ for all $g \in G$.

150. In a metric space any compact set is bounded. Applications:

(a) $SL(n, \mathbb{R})$ is not a compact subset of $M(n, \mathbb{R})$.

(b) The set of symmetric (respectively, the skew-symmetric) matrices is not compact in $M(n, \mathbb{R})$. So is the set of matrices with trace zero.

(c) The set of nilpotent matrices in $M(n, \mathbb{R})$ is not compact.

151. In any topological space, any convergent sequence along with its limit is a compact subset.

152. If A is a nonempty compact subset of $\mathbb{R}$, then $\sup A$ and $\inf A$ exist and they belong to A.

153. The product of two compact spaces is compact. Statement of Tychonoff's theorem with reference to my article for a proof.

Items 144–153 were done on November 23, 25, 30 and December 2, 2004.

154. Heine-Borel theorem (in $\mathbb{R}^n$): A subset $K \subset \mathbb{R}^n$ is compact iff K is closed and bounded. Applications:

(a) Among the non-degenerate conics in $\mathbb{R}^2$, only circles and ellipses are compact.

(b) The unit sphere $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is compact.

(c) $O(n, \mathbb{R})$, the set of orthogonal matrices is compact subset of $M(n, \mathbb{R})$.

(d) The subgroup $SL(n, \mathbb{R})$ is closed and unbounded. It is not a compact subset of $M(n, \mathbb{R})$.

(e) The set of nilpotent matrices in $M(n, \mathbb{R})$ is closed and unbounded. It is not a compact subset of $M(n, \mathbb{R})$.

(f) All norms on $\mathbb{R}^n$ are equivalent. Application: Any finite dimensional vector subspace of an NLS is always closed. *Hints:* If two equivalent norms $\| \ \|_1$ and $\| \ \|_2$ are given on a vector space X, then $(X, \| \ \|_1)$ is complete iff $(X, \| \ \|_2)$ is complete.

155. In general, a closed and bounded subset of a metric space need not be compact. (Standard example. For another, see Item 165h.)

156. Compact sets and maps:

(a) Assume that $f\colon X \to Y$ is continuous and that X is compact. Then $f(X)$ is compact. In particular, compactness is a topological property.

(b) $X \times Y$ is compact iff X and Y are compact.

(c) Let X be compact and Y be hausdorff. Then any continuous bijection $f\colon X \to Y$ is compact. Applications:

   i. Typical applications arise in the theory of quotient spaces: The quotient space $[0, 2\pi]/\sim$ is homeomorphic to S^1.

   ii. Let f be any map (not assumed to be continuous) from a compact hausdorff space X to another such space Y. Assume that the graph of f is closed as a subset of the product space $X \times Y$. Then f is continuous.

   iii. Let X be a set with two distinct topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. Assume that $\mathcal{T}_1 \subset \mathcal{T}_2$ and further that $(X, \mathcal{T}_2)$ is compact hausdorff. Then $(X, \mathcal{T}_1)$ is compact but not hausdorff.

(d) Let X be compact and Y be a metric space. Then any continuous map $f\colon X \to Y$ is bounded. The converse is not true, in general. See Item 27. For metric spaces, the converse is true. (For a proof, see my article on Compact Spaces.)

(e) Let X be compact. Then any continuous function $f\colon X \to \mathbb{R}$ attains its bounds. Applications:

   i. Let X be compact and $f\colon X \to \mathbb{R}$ be continuous. Assume that $f(x) > 0$ for all $x \in X$. Then there is a $\delta > 0$ such that $f(x) \geq \delta$ for all $x \in X$.

   ii. Let K be a compact and C a closed subsets of a metric space X such that $K \cap C = \emptyset$. Then $d(K, C) > 0$.

   iii. Let K be a nonmepty compact subset of an NLS X. Then there exists $x \in K$ such that $\|y\| \leq \|x\|$ for all $y \in K$.

(f) Let X and Y be metric spaces. Assume that X is compact. Then any continuous map $f\colon X \to Y$ is uniformly continuous.

Definition 24. Given an open cover $\{U_i : i \in I\}$ of a metric space (X, d), we say that a positive number δ is a *Lebesgue number* of the cover, if for any subset $A \subset X$ whose diameter is less than δ, there exists $i \in I$ such that $A \subset U_i$.

Remark 25. If δ is a Lebesgue number of the cover and $0 < \delta' \leq \delta$, then δ' is also a Lebesgue number of the given open cover.

157. In general, an open cover may not have a Lebesgue number. Let $X = (0, 1)$ with the usual metric. Let $U_n := (1/n, 1)$. Then $\{U_n : n \in \mathbb{N}\}$ is an open cover of X. Does there exist a Lebesgue number for this cover?

Theorem 26 (Lebesgue Covering Lemma)**.** *Let (X, d) be a compact metric space. Let $\{U_i : i \in I\}$ be an open cover of X. Then a Lebesgue number exists for this cover.* □

158. Use the last theorem to prove Item 156f. Note that the proofs of Item 156f and Lebesgue covering lema are also similar.

159. Definition of FIP: A family of subsets $\{F_i : i \in I\}$ of a set X is said to have the *finite intersection property*, (FIP, in short), if every finite collection of members of the family has a nonempty intersection. Examples:

(a) Let X be any set and (F_n) be a decreasing sequence of nonempty subsets of X. Then $\{F_n : n \in \mathbb{N}\}$ enjoys FIP.

(b) Let X be noncompact. Then there exists an open cover $\{U_i : i \in I\}$ of X which does not admit a subcover. Consider the family of closed sets $\{F_i : i \in I\}$ where $F_i := X \setminus U_i$. This family of closed sets has F.I.P.

160. A topological space is compact iff every family of closed sets with FIP has a nonempty intersection.

161. Cantor intersection theorem: Let X be any topological space. Let (K_n) be a decreasing sequence of nonempty compact subsets of X. Then $\cap_n \mathbb{K}_n \neq \emptyset$.

Definition 27. A subset A of a metric space (X, d) is said to be *totally bounded* if for any given $\varepsilon > 0$, there exist a finite number of points $x_1, \ldots, x_n \in X$ such that $A \subset \cup_{k=1}^{n} B(x_k, \varepsilon)$.

162. Examples, non-examples and properties of totally bounded sets.

(a) Any compact subset of a metric space is totally bounded.

(b) If B is totally bounded and $A \subset B$, then A is totally bounded.

(c) If A is totally bounded, so is its closure $\overline{A}$.

(d) Any totally bounded subset is bounded. The converse is not true. (Standard example!)

(e) Any bounded subset of $\mathbb{R}$ is totally bounded. In fact, any bounded subset of $\mathbb{R}^n$ is totally bounded.

Items 154–162 were done on December 7 and 9, 2004.

163. Characterization of compact metric spaces.

Theorem 28. *Let X be a metric space. Then the following are equivalent.*
1. X is compact.
2. X is complete and totally bounded.
3. (Bolzano-Weierstrass property.) *Every infinite subset of X has a cluster point in X.*
4. (Sequential compactness.) *Every sequence in X has a convergent subsequence.* □

164. *Applications of 2nd characterization:

(a) Arzela-Ascoli theorem as a characterization of compact subsets of $(C(X), \| \ \|_\infty)$, where X is a compact metric space. (Perhaps statement only.)

(b) A subset $A \subset \ell_1$ is compact iff A is closed, bounded and is such that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{n \geq N} |x_n| < \varepsilon$ for **all** $x \in A$.

165. Applications of (perhaps the most useful) 4th characterization.

(a) Any continuous map from a compact metric space to another metric space is bounded.

(b) Any continuous real valued function on a compact metric space attains its bounds.

(c) Let K be a nonempty compact subset of $\mathbb{R}$. Show that $\sup K, \inf K \in K$. Deduce the last item from this.

(d) Let A, B be disjoint compact subsets of a metric space. Then there exist $a \in A, b \in B$ such that $d(A, B) = d(a, b)$, and hence $d(A, B) > 0$. (This result is false, if we assume that A and B closed.)

(e) Let K be a compact subset and C a closed set in $\mathbb{R}^n$. If $K \cap C = \emptyset$, then there exist $x \in K$ and $y \in C$ such that $d(x, y) = d(K, C)$.

(f) Let K, C be as in the last item. Then $K + C$ is closed in $\mathbb{R}^n$.

(g) Let X, Y be compact metric spaces. Then $X \times Y$ is compact.

(h) Let X denote the NLS of all bounded real valued functions on $[0, 1]$ under the sup norm $\| \ \|_\infty$. Then the closed unit ball in X is closed and bounded but not compact.

Items 163–165 were done on December 13 and 14, 2004.

166. **Connected Spaces.** Look at

(a) $\mathbb{R}$, an interval,

(b) a circle, a parabola, an ellipse, two intersecting lines, a disk, a circle, a parabola or an ellipse along with a tangent line at one of its points in $\mathbb{R}^2$,

(c) a plane, a sphere, a ball in $\mathbb{R}^3$.

All of them seem to be in a "single piece." Consider now

(a) $\{-1, 1\}$, $\mathbb{Z}$, $(-1, 0) \cup (0, 1)$ in $\mathbb{R}$,

(b) two (distinct) parallel lines, a hyperbola, two disjoint open disks in $\mathbb{R}^2$,

(c) two distinct parallel planes, the set consisting of the unit ball $B(0, 1)$ along with the plane $x = 2$.

All of these seem to have more than one piece.

Definition 29. A topological space X is said to be *connected* if the only subsets of X which are both open and closed are $\emptyset$ and X. If there exists a subset $\emptyset \neq A \neq X$ which is both open and closed, then the space is said to be *disconnected* or not connected.

We say that a subset A of a topological space X is connected (or a connected subset of X), if A is a connected space with the subspace topology.

167. If X is not connected, say $\emptyset \neq A \neq X$ is both open and closed, then $B := X \setminus A$ is such that $\emptyset \neq B \neq X$ and it is both open and closed. Hence, X is disconnected iff there exist .... (Complete this sentence.) Thus X has two "pieces" A and B!

168. A topological space X is connected iff it has the following property: If U and V are nonempty open sets such that $X = U \cup V$, then $U \cap V \neq \emptyset$.

169. * A subset A is connected iff the following condition is satisfied: If U and V are open subsets of X such that $U \cap A$ and $V \cap A$ are nonempty and $A \subset U \cup V$, then $U \cap V \cap A \neq \emptyset$.

170. We now give some examples. (More examples will follow once we prove a powerful characterization of connected spaces. See Items 171–172.)

(a) $\mathbb{R}$ is connected. See Item 48f. Similar proof shows that any interval is connected.

(b) $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are not connected. See Item 116b.

(c) Any discrete space with more than one element is disconnected.

(d) Any indiscrete space is connected.

(e) Is the empty set connected?

171. The following theorem is a powerful characterization of connected spaces. The theorem remain true if we take Z to be any discrete space with at least two elements, for instance, $\mathbb{Z}$ itself.

Theorem 30. *Consider $Z := \{\pm 1\} \subset \mathbb{R}$ with subspace topology. A topological space is connected iff any continuous map $f \colon X \to Z$ is a constant.* □

172. Applications of the last theorem.

(a) Any interval is connected. Use intermediate value theorem.

(b) A subset of $\mathbb{R}$ is connected iff it is an interval. As one can give a direct proof of this, we have the intermediate value theorem as a corollary.

(c) Let $M(n, \mathbb{R})$ denote the set of all $n \times n$ matrices of real numbers. Then $GL(n, \mathbb{R}) := \{A \in M(n, \mathbb{R}) : \det(A) \neq 0\}$ is not connected.

(d) $O(n, \mathbb{R}) := \{A \in GL(n, \mathbb{R}) : AA^t = I\}$ is not connected.

(e) Let X be a topological space. Let A and B be two connected subsets of X such that $A \cap B \neq \emptyset$. Then $A \cup B$ is connected. Generalize this.

(f) Let X be a connected topological space and $g \colon X \to Y$ be a continuous map. Then $g(X)$ is connected. Applications:

i. Any line segment in an NLS is connected.

ii. The circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected. Similarly, the ellipse and parabola are connected.

iii. $GL(n, \mathbb{R})$ is not connected.

iv. $O(n, \mathbb{R})$ is not connected.

v. $SO(2, \mathbb{R}) := \{A \in O(2, \mathbb{R}) : \det A = 1\}$ is connected.

vi. A convex set, more generally any star-shaped set, in an NLS is connected.

(g) Let X be such that every pair of points of X lies in a connected subset. Then X is connected. Applications:

i. $\mathbb{R}^2 \setminus \{0\}$ is connected.

ii. $\mathbb{R}^2 \setminus \{(n, 0) : n \in \mathbb{Z}\}$ is connected.

(h) Let A be a connected subset of a space X. Let $A \subset B \subset \overline{A}$. Then B is connected. Application:

i. Consider the set $L := \{(t, 0) : t \in [0, 1]\}$, $A_n := \{(1/n, y) : y \in [0, 1]\}$ for $n \in \mathbb{N}$ and $A_0 := \{(0, y) : y \in [0, 1]\}$. Then $E := L \cup (\cup_n A_n)$ is connected and so its closure, $E \cup A_0$ is connected. Hence the set $E \cup \{(0, 1)\}$ is connected. ($X := E \cup A_0$ is known as the comb space.)

(i) Let X be the union of open disk in $\mathbb{R}^2$ along with the tangent line $x = 1$. It is connected.

(j) The open unit disk in $\mathbb{R}^2$ along with any subset of its boundary is connected. (This is geometrically 'obvious'.)

(k) Let $\{A_i : i \in I\}$ be a collection of connected subsets of a space X with the property that for all $i, j \in I$ we have $A_i \cap A_j \neq \emptyset$. Then $A := \cup_i A_i$ is connected. Applications:

  i. $B(0, 1)$ in in any NLS is connected.

  ii. Any vector subspace in an NLS (in particular $\mathbb{R}^n$) is connected.

  iii. Any hyperplane in an NLS (or $\mathbb{R}^n$) is connected.

(l) Let X and Y be topological spaces. Then the product space $X \times Y$ is connected iff both X and Y are connected.

(m) The sphere $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is connected. Applications:

  i. * $\mathbb{R}^n \setminus \{0\}$ is connected.

  ii. A cylinder in $\mathbb{R}^3$ is connected.

  iii. An annular region $\{x \in \mathbb{R}^n : r < \|x\| < R\}$ is connected.

173. The set of two distinct parallel lines in $\mathbb{R}^2$ is not connected.

174. Connectedness can be used to settle questions on homeomorphisms:

  (a) The set of irrational numbers in $\mathbb{R}$ with subspace topology is not homeomorphic to $\mathbb{R}$.

  (b) A hyperbola cannot be homeomorphic to $\mathbb{R}$.

  (c) $\mathbb{R}$ cannot be homeomorphic to $\mathbb{R}^2$.

  (d) A pair of intersecting lines cannot be homomorphic to a parabola.

175. A finite metric space is connected iff is a singleton.

176. Let X be connected and $f : X \to \mathbb{R}$ be a continuous non-constant function. Show that $f(X)$ is uncountable.

177. Let X be a connected metric space with at least two elements. There X "has at least as many elements as $\mathbb{R}$." In particular, X is uncountable.

178. What are all the continuous functions from $f : \mathbb{R} \to \mathbb{R}$ that take only rational values?

179. Are there continuous functions $f : \mathbb{R} \to \mathbb{R}$ that take irrational values at rational numbers and rational values at irrational numbers?

180. Let $f : [a, b] \to \mathbb{R}$ be continuous. "Identify" the image $f([a, b])$.

181. * Let f be a one-one continuous function on an interval. Then f is monotone.

182. What are all the continuous functions from a connected space to (i) a discrete space, (ii) a finite hausdorff space?

183. Let $f\colon X \to Y$ be a continuous map from a connected space X onto a finite hausdorff space? What can you conlcude about Y?

184. Let X be a topological space. Assume that $\{A_i : i \in I\}$ is a family of connected subsets of X. Let L be another connected subset such that $L \cap A_i \neq \emptyset$ for all $i \in I$. Show that $L \cup (\cup_{i\in I} A_i)$ is a connected subset of X.

Definition 31. Let X and Y be topological spaces and $f\colon X \to Y$ be a map. We say that f is *locally constant* if for each $x \in X$, there exists an open set U_x containing x with the property that f is a constant on U_x.

185. Show that any locally constant function is continuous.

186. Let $U \subset \mathbb{R}^n$ be a nonempty open set. Let $f\colon U \to \mathbb{R}$ be a differentiable function with derivative 0. Then f is locally constant. (It need NOT be a constant function!)

187. Let X be connected and Y be hausdorff. Then any locally constant function $f\colon X \to Y$ is a constant function on X. (This is a typical way in which connectedness hypothesis is used. Learn this proof well.)

188. In Item 186, if we further assume that U is connected, then f is a constant.

Items 166–188 were done on December 15 and 16, 2004.

Definition 32. A continuous map $\alpha\colon [a, b] \to X$ to a topological space X is called a *path*. Since any two intervals are homeomorphic, it is a standard practice to assume that $a = 0$ and $b = 1$. The point $p := \alpha(0)$ is called the initial point and $q := \alpha(1)$ is called the terminal point of the path α. We also say that p is path connected to q by the path α.

189. It is important not to identify the path α with its image $\alpha([0, 1])$ in X. (It is called the trace of α. Mnemonic: the trains could be different but the tracks may be the same.) The paths $\alpha, \beta\colon [0, 1] \to \mathbb{R}^2$ given by $\alpha(t) = (t, 0)$ and $\beta(t) = (t^3, 0)$ have the same trace.

190. Two point p and q may be connected by more than one path. Think of at least 3 different paths connecting $(-1, 0)$ to $(0, 1)$ in $\mathbb{R}^2$.

191. If $\alpha\colon [0, 1] \to X$ is a path connecting p to q, then $\beta\colon [0, 1] \to X$ given by $\beta(t) = \alpha(1-t)$ is a path connecting q to p. It is called the reverse of α.

192. A set theoretic exercise: Let $X := \cup_{i\in I} A_i$. Let $f\colon X \to Y$ be any map. Let $V \subset Y$. Let f_i denote the restriction of f to A_i: $f_i := f\,|_{A_i}$. Then

$$f^{-1}(V) = \cup_{i\in I} f_i^{-1}(V) \cap A_i.$$

193. Gluing Lemma: (There are two different ways of looking at this.)

Lemma 33. *Let X, Y be topological spaces. Assume that $\{A_i : i \in I\}$ is a family of subsets of X whose union is X. Assume further that $f_i := f\,|_{A_i} \colon A_i \to Y$ is continuous for each $i \in I$. Then*

1. *f is continuous if each A_i is open.*
2. *f is continuous if each A_i is closed and I is finite.* □

194. If x and y are path-connected and y and z are path-connected in a space, then x and z are path connected.

195. X is path connected iff there exists $p \in X$ such that any point $x \in X$ is path connected to p.

196. Any path connected space is connected.

197. The converse is not true. Two examples:

   (a) Comb space: Let $L := \{(x, 0) : 0 \leq x \leq 1\}$ and $A_n := \{(1/n, y) : 0 \leq y \leq 1\}$, for $n \in \mathbb{N}$. Let $P : \{(0, 1)\}$. Then $L \cup (\cup_{n \in \mathbb{N}} A_n)$ is connected and its closure contains $X := L \cup (\cup_{n \in \mathbb{N}} A_n) \cup \{P\}$. Hence $X$is connected. It is not path connected. If possible, let γ be path joining P to $Q = (1, 0) \in X$. Let $t_0 := \sup\{t \in [0, 1] : \gamma(t) = P\}$. Note that $\gamma_1(t_0) = 0$ and $\gamma_1((t_0, 1]) \subset \{1/n : n \in \mathbb{N}\}$. Hence $\gamma_1((t_0, 1]) = 1$, by connectedness. Thus $\gamma_1(t_0) = 1$, a contradiction.

   (b) * Topologist's sine curve. (See my book 'Topology of Metric Spaces'.)

198. Any open subset of an NLS is connected iff it is path connected.

Going through this proof, we are led to the concept of locally path connected spaces. First of all a definition.

Definition 34. Let X be a topological space and $x \in X$. A subset U is called a *neighbourhood* of x in X if there exists an open set G such that $x \in G \subset U$. Example: $[0, 1)$ is a neighbourhood of any $x \in (0, 1)$ but not of $x = 0$.

199. A set in a topological space is open iff it is a neighbourhood of each of its points.

200. **Locally P spaces**

General Philosophy: Let P be a topological property. We say that a space X is *locally* P (or enjoys P locally) if for each $x \in X$ and an open set $U \ni x$, there exists a neighbourhood N of x where N has the property P.

Definition 35. Let X be a topological space. Then X is said to be *locally path connected* if for each $x \in X$ and an open set $U \ni x$, there exists a path connected neighbourhood N of x such that $N \subset U$.

Now you can similarly define *locally connected* and *locally compact* spaces.

201. The proof of Item 198 yields the following result: An open set in a locally path connected space is connected iff it is path-connected.

202. An important remark: In general X may have property P but it may not be locally P. For instance, the complete comb space is connected but not locally connected. (Look for a connected neighbourhood of the point $(0, 1)$.) Similarly, there exists a compact space (Item 213c) which is not locally compact. (Do NOT get confused with the 'bad' definition of Munkres and hence his "note" that any compact space is locally compact!)

203. (Connected) Components. In a topological space X, the relation $x \sim y$ if there exists a connected set A with $x, y \in A$ is an equivalence relation. The equivalence classes are called the *connected components* or components of X. The following are immediate:

(a) If C is a component, then C is a closed connected set.

(b) Any component C is a maximal connected set in the sense that if A is connected and $C \subset A$, then $C = A$.

(c) If C is a component, $x \in C$ and if A is a connected set with $x \in A$, then $A \subset C$.

204. Examples of components:

(a) The only component of a connected space X is X.

(b) The components of a discrete space are the singleton sets.

(c) The components of $\mathbb{Q}$ are the singleton sets. (Note that the topology on $\mathbb{Q}$ is not discrete topology. We gave two proofs of this. One is direct use of subspace topology and another used existence of non trivial convergent sequences.)

(d) What are the components of $\mathbb{R}$ with VIP topology? with outcast topology?

205. If $f\colon X \to Y$ is a homeomorphism, then f induces a natural bijective correspondence between the components of X and those of Y: If C is a component of X, then $f(C)$ is a component of Y. Application: The pair of intersecting lines is not homeomorphic to $\mathbb{R}$. (If they are, remove a point from $\mathbb{R}$ and its image in the other set. Count the components.)

206. Path components are defined in an obvious way. If P_x (resp. C_x) is the path component containing $x \in X$, then $P_x \subseteq C_x$.

207. A space X is locally connected iff the components of any open subset (with subspace topology) are open in X. In particular, the components of X are open.

208. The components in a locally path connected space are open.

209. Let U be an open subset of a locally path connected space. Then U is connected iff it is path-connected.

210. In a locally path connected space, the components and path components are the same.

211. Locally Compact Spaces:

212. The following are descendants of Item 148q.

(a) Let K be a compact subset of a hausdorff space X and $x \notin K$. Then there exist disjoint open sets U and V such that $x \in U$ and $K \subset V$.

(b) Let A and B be disjoint compact subsets of a hausdorff space. Then there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

(c) Let X be a compact hausdorff space. Let A and B be disjoint closed subsets of X. Then there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$. (Thus a compact hausdorff space is *normal*.) Another example of a normal space is any metric space.

213. Examples of locally compact spaces:

(a) $\mathbb{R}$, $\mathbb{R}^n$ are locally compact.

(b) $\mathbb{Q}$ is not locally compact.

(c) A compact space need not be locally compact. Example: Consider $\mathbb{Q}$ with the usual topology, adjoin an extra element, say ∞. The neighbourhoods of $x \in \mathbb{Q}$ are either the neighbourhoods of x in $\mathbb{Q}$ or ∞ added to the standard neighbourhoods. The neighbourhoods of ∞ are complements in $\mathbb{Q}$ of a finite subset of F along with ∞.

(d) An NLS is locally compact iff it is finite dimensional. (One way is easy; the proof of the other is omitted.)

(e) A locally compact metric space need not b complete. (A trivial example is $(0, 1)$! I took quite sometime to think of this!

Theorem 36. *The following are equivalent for a hausdorff space:*

1. *X is locally compact.*
2. *For every $x \in X$ and a neighbourhood U of x, there exists an open set V such that $x \in V$, $\overline{V}$ is compact and $\overline{V} \subset U$.*
3. *Each $x \in X$ has a compact neighbourhood.* □

See Figure 1 for an idea to prove (3) implies (1).

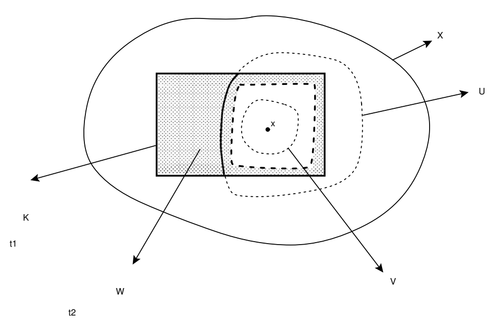

Figure 1: Locally Compact Space

Since locally compact spaces such as $\mathbb{R}^n$ arise quite often, whenever we say X is locally compact, we shall assume that X is hausdorff also.

214. Local compactness is a topological property. In fact, more is true: Let $f : X \to Y$ be a continuous open map of a locally compact space X onto Y. Then Y is locally compact.

215. A closed (respectively open) subspace of a locally compact space is locally compact.

216. One point compactification: Given a locally compact noncompact hausdorff space X, let $X_\infty := X \cup \{\infty\}$ where $\infty \notin X$. Let $\mathcal{T}$ denote the topology on X. Consider

$$\mathcal{T}_\infty := \mathcal{T} \cup \{V \subset X_\infty : X_\infty \setminus V \text{ is compact. }\}.$$

Then
(i) $\mathcal{T}_\infty$ is a hausdorff topology on X_∞.
(ii) The subspace topology on X is $\mathcal{T}$.
(iii) $(X_\infty, \mathcal{T}_\infty)$ is compact.
(iv) X is dense in X_∞.

217. Let X be noncompact, locally compact hausdorff space. Let Y be a compact hausdorff space. Assume that there exists $pq \in Y$ and a homeomorphism $f\colon X \to Y \setminus \{q\}$. Then the one point compactification X_∞ of X is homeomorphic to Y.

218. Examples:

   (a) $\mathbb{R}^n \cup \{\infty\} = S^n$.

   (b) Let $x\colon \mathbb{N} \to X$ be a sequence in X. Then $x_n \to x_\infty$ iff the function $x\colon \mathbb{N}_\infty X$ defined by $x(n) = x_n$ and $x(\infty) = x_\infty$ is continuous at ∞.

   Application: Use this to give another solution of Item 130b.

Items 189–218a were done on December 17–18 and 21–24, 2004. Wish you a Very Happy New Year!

219. * Functions vanishing at infinity: Let X be a locally compact hausdorff space. A continuous function $f\colon X \to \mathbb{R}$ is said to *vanish at infinity* if for any given $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ for $x \notin K$. (We can also define continuous function vanishing at ∞ for functions taking values in an NLS in an obvious way.)

   A continuous function $f\colon X \to \mathbb{R}$ vanishes at infinity iff it extends to a continuous function $f_\infty\colon X_\infty \to \mathbb{R}$ with $f_\infty(\infty) = 0$.

   (a) Let $f\colon X \to \mathbb{R}$ be given. Its *support* is by definition the **closure** of the set $\{x \in X : f(x) \neq 0\}$, that is,

   $$\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}.$$

   We say that f has compact support if the support of f is compact. Evidently, any continuous function with compact support vanishes at infinity.

   (b) What are the entire functions $f\colon \mathbb{C} \to \mathbb{C}$ which vanish at infinity?

   Definition 37. A subset $A \subset X$ of a topological space is said to be *nowhere dense* in X, if given any nonempty open set U, we can find a nonempty open subset $V \subset U$ such that $A \cap V = \emptyset$.

   This definition is equivalent to the standard one found in all text-books: A is nowhere dense in X iff the interior of the closure of A in X is empty: $\operatorname{Int}(\overline{A}) = \emptyset$.

220. Examples of nowhere dense sets:

   (a) Let V be any proper vector subspace of $\mathbb{R}^n$. More generally,

   (b) Any proper vector subspace of an NLS.

   (c) * The set of zeros of any polynomial map $\mathbb{R}^n \to \mathbb{R}$.

221. Baire Category theorem:

   Theorem 38 (Baire Category Theorem)**.** *Let (X, d) be a complete metric space.*

   (1) Let U_n be open dense subsets of X, for $n \in \mathbb{N}$. Then $\cap_n U_n$ is dense in X.

   (2) Let F_n be nonempty closed subsets of X such that $X = \cup_n F_n$. Then at least one of F_n's has nonempty interior. In other words, a complete metric space cannot be a countable union of nowhere dense closed subsets.

(A complete set of notes was given for this. Also, refer to my article "Applications of Baire Category Theorem" in MTTS-notes.)

222. Applications:

 (a) $\mathbb{R}^n$ cannot written as the union of a countable family of its proper vector subspaces. In particular, $\mathbb{R}^2$is not the union of a countable family of lines through the origin.

 (b) No infinite dimensional complete normed linear space can be countable dimensional. (Algebraic sense!)

 (c) There can exist no metric d on $\mathbb{Q}$ such that d induces the usual topology on $\mathbb{Q}$ and $(\mathbb{Q}, d)$ is complete.

 (d) Let (X, d) be complete and $f_n \colon X \to \mathbb{R}$ be a sequence of continuous functions. Assume that $f_n \to f$ pointwise on X. Then the set $A := \{x \in X : f \text{ is continuous at } x\}$ is dense in X. (Our proof given in the notes distributed was a beautiful application of both the versions of Baire's theorem.)

223. An amusing exercise: Let (x_n) be any sequence of real numbers. Show that the set $\{x \in \mathbb{R} : x \neq x_n, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$. Hence conclude that $\mathbb{R}$ is uncountable.

224. Show that $\mathbb{Q}$ cannot be written as the intersection of a countable family of open subsets of $\mathbb{R}$.

225. Locally closed sets: A subset A of a topological space is *locally closed* if for every $a \in A$, there exists an open set U_a in X such that $a \in U_a$ and $U_a \cap A$ is closed **in** U_a.

 (a) A characterization of locally closed sets: $A \subset X$ is locally closed iff there exist an open set U and a closed set C such that $A = U \cap C$.

 (b) The characterizations gives us easy examples of locally closed sets: $[0, 1)$ is neither closed nor open in $\mathbb{R}$ but is locally closed in $\mathbb{R}$.

226. Separation axioms. They deal with separating various kinds of disjoint objects by means of disjoint open sets that contain the given objects. The prominent ones are given below.

 (a) Hausdorff spaces: Given two distinct points $x \neq y$, if we can find open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

 (b) Regular spaces: Given a point x and a closed set F with $x \notin F$, there exist open sets U and V such that $x \in U$ and $V \subset V$ with $U \cap V = \emptyset$.

 (c) Normal spaces: Given two disjoint closed sets A, B, there exist open sets U, V such that $A \subset U$ and $B \subset V$ with $U \cap V = \emptyset$.

 (d) Completely regular spaces: Given two disjoint (nonempty) closed sets, we can find disjoint a continuous function $f \colon X \to \mathbb{R}$ such that $f = 0$ on A and $f = 1$ on B.

 (e) Clearly, a completely regular space is regular. How about completely hausdorff and completely normal spaces? These could be the spaces the objects under question are separated by means of continuous real valued functions. Make precise definitions.

 These spaces will be useful for analysts since they assure that there is an 'abundant' supply of real valued continuous functions on the given space!

227. Some standard examples and facts concerning the above concepts:

(a) Examples of regular spaces.

i. Any metric space is regular.

ii. Any locally compact hausdorff space is regular.

(b) Examples of normal spaces.

i. Any metric space is normal. We gave two proofs of this. One of them is based on Urysohn's lemma for metric spaces. See Item 227e.

ii. Any compact hausdorff space is normal.

(c) A normal space in which all singleton sets are closed is regular.

(d) * The most important result about normal spaces is the Urysohn's lemma.

Theorem 39 (Urysohn's Lemma)**.** *Let A, B be disjoint non-empty closed subsets of a normal space. Then there exists a continuous function $f\colon X \to [0,1]$ such that $f = 0$ on A and $f = 1$ on B.* □

(e) We proved Urysohn's lemma in the case of a metric space. Look at $f(x) := \frac{d(x,A)}{d(x,A)+d(x,B)}$.

(f) Note that Urysohn's lemma says that a space is normal iff it is completely normal.

(g) * A key fact needed for Urysohn's lemma is the following observation.

Proposition 40. *Let X be a normal space. Assume that a closed subset A is contained in an open set U. Then there exists an open set V such that $A \subset V \subset \overline{V} \subset U$.* □

228. Quotient spaces. A complete set of notes is available in the form of an article in MTTS Notes.

(a) We recalled concept of quotient topology. Let X be a set and $\sim$ be an equivalence relation on X. Let $X/\sim$ be the quotient set or the set of equivalence classes of $\sim$. Let $\pi\colon X \to X/\sim$ be the *quotient map* defined by $\pi(x) = [x]$, the equivalence class of x. The quotient topology on $X/\sim$ is the set of $V \subset X/\sim$ such that $\pi^{-1}(V)$ is open in X.

(b) Let X be a topological space and $\sim$ an equivalence relation on X. Then the quotient topology on $X/\sim$ is the largest topology for which the natural quotient map $\pi\colon X \to X/\sim$ is continuous.

(c) The theorem below, though easy, is the 'only' result needed to check the continuity of maps from quotient spaces to others.

Theorem 41 (Universal Mapping Property)**.** *Let $\pi\colon X \to X/\sim$ be a quotient map. A map $f\colon X/\sim \to Y$ is continuous iff $f \circ \pi$ is continuous.* □

(d) The next theorem tells us how to generate quotient spaces.

Theorem 42. *Let $f\colon X \to Y$ be continuous. Let $\sim$ be the equivalence relation on X defined by $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. Then there exists a continuous function $g\colon X/\sim \to Y$ such that $f = g \circ \pi$.* □

(e) The next result is the most important tool we employ to identify the quotient spaces. If we have some guess that the quotient space $X/\sim$ is homeomorphic to Y, we try to find a surjective continuous map $f\colon X \to Y$ such that the equivalence relation defined by f is $\sim$ and such that f is either open or closed.

Theorem 43. *Let $f\colon X \to Y$ be an open (or closed) continuous surjective map. Then Y is homeomorphic to the quotient space of X obtained by identifying each level set of f to a point.*

(f) Illustrations of the use of the above result.

i. The quotient space obtained from $[0,1]$ got by identifying the end points 0 and 1 is S^1.

ii. The quotient space got by identifying two of the opposite sides of a rectangle is homeomorphic to a cylinder.

iii. The quotient space of S^1 obtained by identifying the diametrically opposite points is again S^1!

iv. The quotient space of the unit square identifying the corresponding points on the vertical sides is homeomorphic to the cylinder $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 = 1\}$.

v. The quotient space of the unit square identifying the corresponding points of the horizontal sides as well as the points on the vertical sides is homeomorphic to $S^1 \times S^1$, a torus (a vada or a cycle tube).

vi. For any space X and a subset A of X, the space X/A stands for the quotient space of X with respect to the equivalence: $x_1 \sim x_2$ iff $x_1 = x_2$ or $x_1, x_2 \in A$. Thus X/A is the space obtained from X by collapsing A to a single point. Example: $D^n := B[0,1] \subset \mathbb{R}^n$. Then $D^n/S^{n-1} \simeq S^n$.

vii. Let F be a closed subset of a compact Hausdorff space X. The quotient space obtained from X by identifying F to a single point is homeomorphic to the one-point compactification of $X \setminus F$.

viii. The last item may be used to prove that $D^n/S^{n-1} \simeq S^n$.

ix. If $X = S^1 \times [0,1]$ is the cylinder and $A = S^1 \times \{0\}$ is the bottom circle, then $X/A \simeq D^2$.

(g) We say that an equivalence relation $\sim$ on X is open if whenever $U \subset X$ is open in X so is its *saturation* $[U] := \{x' \in X : x' \sim x \text{ for some } x \in U\}$.

Proposition 44. *An equivalence relation $\sim$ on X is open iff the quotient map $\pi\colon X \to X/\sim$ is open.* □

(h) Hausdorffness of quotient spaces. The following result is the most useful (sufficient) condition on $\sim$ that ensures the quotient space is hausdorff.

Theorem 45. *Let $\sim$ be an open equivalence relation on X. Assume that the relation $R := \{(x,y) \in X \times X : x \sim y\}$ is closed as a subset of $X \times X$. Then $X/\sim$ is hausdorff.* □

(i) Projective spaces over $\mathbb{R}$. Let $X := \mathbb{R}^{n+1} \setminus \{0\}$. The relation on X defined by $x \sim y$ iff $x = ty$ for some nonzero $t \in \mathbb{R}$ is a equivalence. The quotient $X/\sim$ is known as the n-dimensional projective space over the reals. It is denoted by $\mathbb{P}^n(\mathbb{R})$. The following are some of the properties of $\mathbb{P}^n(\mathbb{R})$.

i. $\mathbb{P}^n(\mathbb{R})$ is a compact Hausdorff space.

ii. $\mathbb{P}^n(\mathbb{R})$ is homeomorphic to the quotient of S^n with respect to the relation on S^n: $x \sim y$ iff $x = \pm y$.
In this we had to deal with the continuity of a map *into* the quotient space. Go through the proof again. It shows the typical way in which the continuity of a map $f: Y \to X/\sim$ into a quotient space is dealt with. (Universal mapping property cannot deal with this situation.) The trick was to write f as the composite of a continuous map $g: Y \to X$ followed by the quotient map $\pi: X \to X/\sim$.

iii. The one dimensional projective space is homeomorphic to S^1.

(j) Two very popular and important examples of quotient spaces.

i. Möbius Strip. On the unit square X we define the equivalence relation as follows:

$$(x, y) \sim (x', y') \iff (x, y) = (x', y') \text{ or } \{x, x'\} = \{0, 1\} \text{ and } y = 1 - y'.$$

Thus two points of opposite vertical sides are identified *cross-wise*. The quotient space is known as the Möbius strip.

ii. Klein's bottle. Let X be the unit square. Define an equivalence relation on X whose nontrivial relations are given by

$$(0, y) \sim (1, y) \quad \text{and} \quad (x, 0) \sim (1 - x, 1).$$

The quotient space is called the Klein's bottle.

Definition 46. Let X be a topological space. A *loop* in X is a path $\alpha: [0, 1] \to X$ with $\alpha(0) = \alpha(1)$. We say that α s a loop based at $\alpha(0)$.

Recall that if $\alpha, \beta: [0, 1] \to X$ are paths such that $\alpha(1) = \beta(0)$, then their *join* $\alpha * \beta$ is defined by

$$\alpha * \beta(t) := \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq 1/2 \\ \beta(2t - 1) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Then, $\alpha * \beta$ is continuous (by gluing lemma) and we say that it is got by concatenation.

Standard Notation in homotopy theory: Let $I = [0, 1]$.

Definition 47. Let X, Y be topological spaces. Let $f, g: X \to Y$ be continuous maps. We say that they are homotopic if there exists a continuous map $F: X \times I \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$. We say that $f_t(x) := F(x, t)$ for $t \in I$ and $x \in X$.

The map F is called a homotopy from f to g and we write $f \overset{F}{\simeq} g$.

If $f(a) = g(a)$ for all $a \in A \subset X$ and if the homotopy F is such that $F(a, t) = f(a)$ for all $t \in I$ and $a \in A$, we say that f is homotopic to to g relative to A. We denote this by $f \overset{F}{\simeq} g$ rel A.

If α and β are paths in X with the same initial and terminal points, then saying that α is homotopic to β relative to $\{0, 1\}$ is the same as saying that all the intermediate paths $\alpha_t(s) := F(s, t)$ have the same initial and terminal points, that is, they satisfy $F(0, t) = \alpha(0)$ and $F(1, t) = \alpha(1)$.

229. Examples:

(a) Let $C \subset \mathbb{R}^n$ be convex. Let $f, g \colon X \to C$ be continuous maps. Then the map $F(x,t) := (1-t)f(x) + tg(x)$ is a homotopy from f to g. If f ad g agree on a set $A \subset X$, then F is a homotopy relative to A.

(b) Let $f, g \colon X \to S^n$ be continuous maps such that $f(x) \neq -g(x)$ for $x \in X$. Then the map

$$F(x,t) := \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

is a homotopy from f to g.

(c) The map $f \colon S^1 := \{z \in \mathbb{C} : |z| = 1\} \to S^1$ defined by $f(z) = -z$ is homotopic to the identity map $g(z) = z$.

(d) Let $f \colon X \to S^n$ be a continuous map which is not onto. Then it is null-homotopic, that is, homotopic to a constant map.

(e) Consider $X := \{p \in \mathbb{R}^2 : 1 \leq \|p\| \leq 2\}$. Let α be 'the inner circle' and β be the ellipse lying in X and circumscribing α. Assume that they both start and end at $(0,1)$. They are homotopic in X. (Note that X is not convex.)

230. The relation of homotopy between the continuous maps from a space X to another space Y is an equivalence relation.

For, if $f \overset{F}{\simeq} g$ and $g \overset{G}{\simeq} h$, then

$$H(x,t) := \begin{cases} F(x,2t) & 0 \leq t \leq 1/2 \\ G(x,2t-1) & 1/2 \leq t \leq 1, \end{cases}$$

is a homotopy from f to h.

231. The relation of homotopy between the continuous maps from a space X to another space Y relative to a subset $A \subset X$ is an equivalence relation among maps that agree on A.

232. Homotopy behaves well with respect to composition of maps.

(a) Let $f, g \colon X \to Y$ be homotopic relative to a set $A \subset X$ via the homotopy F. Let $h \colon Y \to Z$ be a map. Then $h \circ f \overset{h \circ F}{\simeq} h \circ g$ relative to A.

(b) Let $f \colon X \to Y$ be a map. Assume that $g, h \colon Y \to Z$ are homotopic relative to $B \subset Y$ via a homotopy G. Then $g \circ f \overset{F}{\simeq} h \circ f$ relative to $f^{-1}(B)$, where $F(x,t) := G(f(x),t)$.

Definition 48. Fix a base point $p \in X$. Let α be a loop at p. The equivalence class $\langle \alpha \rangle$ of all loops based at p homotopic to α relative to $\{0,1\}$ is called a *homotopy class*. The collection of homotopoy classes of loops at p is denoted by $\pi_1(X,p)$.

233. Construction of the fundamental group. We make $\pi_1(X,p)$ into a group as follows. For $\langle \alpha \rangle, \langle \beta \rangle \in \pi_1(X,p)$, we let $\langle \alpha \rangle * \langle \beta \rangle := \langle \alpha * \beta \rangle$.

(a) The above multiplication is well-defined.

For, $\alpha \overset{F}{\simeq}$ and $\beta \overset{G}{\simeq} \beta'$, then $\alpha * \beta \overset{H}{\simeq} \alpha * \beta'$ where $H(s,t) := \begin{cases} F(2s,t) & 0 \leq s \leq 1/2 \\ G(2s-1,t) & 1/2 \leq s \leq 1. \end{cases}$

(b) The multiplication is associative.

First of all, we compute

$$\begin{aligned} ((\alpha * \beta) * \gamma)(s) &= \begin{cases} \alpha(4s) & 0 \leq s \leq 1/4 \\ \beta(4s-1) & 1/4 \leq s \leq 1/2 \\ \gamma(2s-1) & 1/2 \leq s \leq 1 \end{cases} \\ (\alpha * (\beta * \gamma))(s) &= \begin{cases} \alpha(2s) & 0 \leq s \leq 1/2 \\ \beta(4s-2) & 1/2 \leq s \leq 3/4 \\ \gamma(4s-3) & 3/4 \leq s \leq 1 \end{cases}. \end{aligned}$$

Define $f\colon I \to I$ by setting

$$f(s) := \begin{cases} 2s & 0 \leq s \leq 1/4 \\ s + \dfrac{1}{4} & 1/4 \leq s \leq 1/2 \\ (s+1)/2 & 1/2 \leq s \leq 1 \end{cases}.$$

Since $f(0) = 0$ and $f(1) = 1$, we see that $f \simeq 1_I$, that is, f is homotopic to the identity map 1_I of I relative to $\{0,1\}$. We have

$$\begin{aligned} (\alpha * \beta) * \gamma &= (\alpha * (\beta * \gamma)) \circ f \\ &\simeq (\alpha * (\beta * \gamma)) \circ 1_I \\ &= \alpha * (\beta * \gamma). \end{aligned}$$

(c) Existence of the identity. Let $e = e_p$ denote the constant loop at p: $e(t) = p$ for $0 \leq t \leq 1$. Then $\langle e \rangle$ serves as the identity for the multiplication. Again, proceeding as earlier, we have

$$\begin{aligned} e * \alpha(s) &= \begin{cases} e(2s) & 0 \leq s \leq 1/2 \\ \alpha(2s-1) & 1/2 \leq s \leq 1 \end{cases} \\ e * \alpha &= \alpha \circ f \\ \text{where } f(s) &= \begin{cases} 0 & 0 \leq s \leq 1/2 \\ 2s-1 & 1/2 \leq s \leq 1. \end{cases} \end{aligned}$$

Thus we have

$$e * \alpha = \alpha \circ f \simeq \alpha \circ 1_I \text{ rel } I = \alpha.$$

Similarly, one shows that $\alpha * e \simeq \alpha$.

(d) Existence of inverse. The inverse of $\langle \alpha \rangle$ is $\langle \alpha^{-1} \rangle$, where α^{-1} is the reverse path defined by $\alpha^{-1}(s) := \alpha(1-s)$.

i. The inverse s well-defined. If $\alpha \overset{F}{\simeq} \beta$ relative to $\{0,1\}$, then $\alpha^{-1} \overset{G}{\simeq} \beta^{-1}$ relative to $\{0,1\}$ where $G(s,t) := F(1-s,t)$.

ii. We show that $\alpha * \alpha^{-1} = \alpha \circ f$ where

$$f(s) = \begin{cases} 2s & 0 \le s \le 1/2 \\ 2 - 2s & 1/2 \le s \le 1. \end{cases}$$

Now, $f \simeq g$ relative to $\{0, 1\}$ where $g(s) = 0$ for $0 \le s \le 1$. Hence,

$$\alpha * \alpha^{-1} = \alpha \circ f \simeq \alpha \circ g \text{ rel } \{0, 1\} = e.$$

One similarly, shows that $\alpha^{-1} \circ \alpha \simeq e$.

(e) Explicit homotopies can also be given. (Of what use?)

i. Existence of identity.

- $\alpha * e \simeq \alpha$ via

$$H(s,t) := \begin{cases} \alpha(\frac{2t}{s+1}) & s \ge 2t - 1 \\ p & s \le 2t - 1. \end{cases}$$

- $e * \alpha \simeq \alpha$ via

$$H(s,t) = \begin{cases} p & s \ge 2t \\ \alpha(\frac{2t-s}{2-s}) & s \le 2t \end{cases}$$

ii. Existence of inverse. $\alpha * \alpha^{-1} \simeq e$ via

$$H(s,t) = \begin{cases} \alpha(2t) & s \ge 2t \\ \alpha(s) & s \le 2t \text{ and } s \le 2 - 2t \\ \alpha(2 - 2t) & s \ge 2 - 2t \end{cases}$$

iii. Associativity. $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$ via

$$H(s,t) = \begin{cases} \alpha(\frac{4t}{s+1}) & 4t - 1 \le s \\ \beta(4t - s - 1) & 4t - 2 \le s \le 4t - 1 \\ \gamma(\frac{4t-2s}{2-s} - 1) & s \le 4t - 2. \end{cases}$$

I have not verified these, simply copied from a book!

234. Let α, β be two paths such that $\alpha(1) = \beta(0)$. Then proceeding as in the last item, we show the following, as the same homotpies work as they take care of the end points!

(a) If $\alpha' \simeq \alpha$ relative to $\{0, 1\}$ and If $\beta' \simeq \beta$ relative to $\{0, 1\}$, then $\alpha * \beta \simeq \alpha' * \beta'$ relative to $\{0, 1\}$.

(b) If α, β, γ are paths such that $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ make sense, then

$$\alpha * (\beta * \gamma) \simeq (\alpha * \beta) * g \text{ relative to } \{0, 1\}.$$

(c) We have $\alpha \circ \alpha^{-1} \simeq e_{\alpha(0)}$ relative to $\{0, 1\}$ and $\alpha^{-1} \circ \alpha \simeq e_{\alpha(1)}$ relative to $\{0, 1\}$.

235. If X is path connected, then $\pi_1(X, p)$ is isomorphic to $\pi_1(X, q)$ for $p, q \in X$. This isomorphism depends on the choice of path joining p and q.

Definition 49. Let $p\colon E \to B$ be a continuous map. An open subset $U \subset B$ is said to be *evenly covered* by p if $p^{-1}(U)$ is the union $\cup_i V_i$ of disjoint open subsets V_i of E such that the restriction p_i of p to V_i is a homeomorphism of V_i onto U.

We say that p is a *covering map* if (i) p is onto and (ii) each $b \in B$ has an open neighbourhood U_b which is evenly covered by p.

The set $p^{-1}(b)$ is called the *fibre* over b.

The sets V_i are called *sheets* of $p^{-1}(U)$.

E is called the total space and B, the base of the covering map p.

236. Properties of a covering map.

(a) Any covering map is open.
(b) Each of the fibres $p^{-1}(b)$ is discrete.
(c) Each $b \in B$ has an open neighbourhood U such that $p^{-1}(U)$ is homeomorphic to $p^{-1}(b) \times U$.

237. Examples.

(a) The exponential map $p\colon \mathbb{R} \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$ is a covering.
(b) The quotient map $\pi\colon S^n \to \mathbb{P}^n(\mathbb{R})$ is a covering.
(c) Products of covering maps is again a covering map. (precise statement?)
(d) Consider the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^*$. The open set $U := \mathbb{C}^*$ is not evenly covered by exp.

    In fact, an open set $U \subset \mathbb{C}^*$ is evenly covered by the exponential map iff there exists a continuous logarithm L on U, that is, a continuous map $L\colon U \to \mathbb{C}$ such that $\exp(L(z)) = z$ for all $z \in U$.

    Note however that $\exp\colon \mathbb{C} \to \mathbb{C}^*$ is a covering map.

Definition 50. Let $p\colon E \to B$ a covering map. Let $f\colon X \to B$ be continuous map. Then a map $g\colon X \to E$ such that $p \circ g = f$ is called a *lift* of f. One has the following commutative diagram. (Figure?)

238. Uniqueness of lifts.

Theorem 51. *Let $p\colon E \to B$ be a covering map and X a connected space. Let $f\colon X \to B$ be a map. If $g, h\colon X \to E$ are lifts of f such that $g(x) = h(x)$ for some $x \in X$, then $g = h$.* □

239. Path lifting lemma.

Theorem 52. *Let $p\colon E \to B$ be a covering map. Let $c\colon I \to B$ be a path. Let $e_0 \in E$ be such that $p(e_0) = c(0)$. then there exists a unique path $\gamma\colon I \to E$ such that $\gamma(0) = e_0$ and $p\colon \gamma = c$.* □

240. A Version of homotopy lifting lemma:

Theorem 53. *Let $p\colon E \to B$ be a covering map. Let $F\colon I \times I \to B$ be a continuous map. Let $e_0 \in p^{-1}(F(0,0))$. Then there exists a unique lift $G\colon I \times I \to E$ of F such that $G(0,0) = e_0$.* □

241. Let (E, e) and (B, b) be topological spaces with base points e and b respectively. Let $p\colon E \to B$ be a covering map. If c is a loop at b and γ is its lift through e, we cannot conclude that γ is a loop at e but $p(\gamma(1)) = b$, that is, $\gamma(1) \in p^{-1}(b)$. Example: Consider the spaces $(\mathbb{R}, 0)$ and $(S^1, 1)$. A lift of $c(t) = e^{2\pi i t}$ is $\gamma(t) = t$ in $\mathbb{R}$.

242. Let c_0 and c_1 be homotopic loops at b with F as a a homotopy. We thus get a lift $G\colon I \times I \to E$ of F such that $G(0,0) = e$ and $p(G(s,t)) = c_t(s)$, for $(s,t) \in I \times I$. Let $\gamma_t(s) := G(s,t)$. Then all these paths start at e and have the same end point $\gamma_0(1)$.

    As a corollary, if $\langle c \rangle \in \pi_1(B, b)$ and γ is a lift of c through e, then

$$\pi_1(B, b) \to \pi^{-1}(b) \text{ defined by } \varphi\colon \langle c \rangle \mapsto \gamma(1) \tag{1}$$

    is well-defined.

243. Simply connected space. We say a path-connected topological space X is *simply connected* if $\pi_1(X, x)$ is trivial for some (and hence for any) $x \in X$. Examples:

    (a) Any convex subset of $\mathbb{R}^n$ is simply connected.

    (b) The parabola $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$ is not convex but simply connected.

    (c) We shall show below (Item 246) that S^n for $n \geq 2$ is simply connected.

244. Let $p\colon (E, e) \to (B, b)$ be a covering map. Assume that E is simply connected. Then the map defined in (1) is a bijection of $\pi_1(B, b)$ with $\pi^{-1}(b)$.

    As a corollary (under the above hypothesis), for any $q \in \pi^{-1}(b)$, if we let γ_y be a path joining e to y, then given a loop c at p, we have a unique $q \in \pi^{-1}(b)$ such that c is homotopic to $p \circ \gamma_y$.

245. Applications.

    (a) Fundamental group of $\mathbb{P}^n(\mathbb{R})$ $(n \geq 2)$. For $n \geq 2$, $\pi_1(\mathbb{P}^n(\mathbb{R}), [e_1])$ is isomorphic to $\mathbb{Z}_2$.

    (b) Fundamental group of S^1 is isomorphic to $\mathbb{Z}$. The following are the main steps.

        i. Given $\langle c \rangle \in \pi_1(S^1, 1)$, $\varphi(\langle c \rangle) \in \mathbb{Z}$. We call the integer the index of c.

        ii. The map $\langle c \rangle \mapsto \varphi(\langle c \rangle)$ is a group homomorphism of $\pi_1(S^1, 1)$ to $\mathbb{Z}$.

246. Let X be a space, U, V be simply connected open subsets of X such that (i) $X = U \cup V$ and (ii) $U \cap V$ is path connected. Then X is simply connected.

    Application. S^n is simply connected for $n \geq 2$.

247. Applications of the index of loops in S^1.

    (a) No retraction theorem. There is no continuous map $f\colon B^2 \to S^1$ such that $f(z) = z$ for $z \in S^1$. □

(b) Brouwer fixed point theorem. Any continuous map of B^2 to itself has a fixed point. $\square$

(c) Borsuk-Ulam theorem. Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map. Then there exist antipodal points $\pm v \in S^2$ such that $f(v) = f(-v)$. $\square$

This has a physical interpretation.

(d) Ham-Sandwich theorem. Let A, B, C be bounded connected open subsets of $\mathbb{R}^3$. Then there exists a plane in $\mathbb{R}^3$ that divides each of the sets into two subsets of equal volume. $\square$

Proof of this relied on some intuitively obvious facts on volumes.

(e) Fundamental theorem of algebra. $\square$

For proofs, you may refer to my relevant articles in *Expository Articles*.