Outline of a Topology Course

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Abstract

This is a summary of courses on General Topology, offered by me at the Department of Mathematics, University of Mumbai during the academic year 2004-2005 and at the Department of Mathematics and Statistics, University of Hyderabad in Jan-April 2012. There are minimal number of proofs in this set of notes. Its merit, if any, lies in the choice of topics, their development and the emphasis on concrete and geometric examples and exercises. I plan to add a bit more material so that it could serve as a skeleton of a course in General Topology. Later I plan to develop this into a text-book. (So, please do not plagiarize!) I would appreciate receiving your comments and views.

This set may be used in conjunction with the following articles of mine on Topology:

1. Subspace Topology
2. Quotient Topology
3. Existence of Continuous Functions
4. Compact Spaces
5. Connected Spaces
6. Generating Topologies — A Unified View of Subspace, Product and Quotient Topologies.

Topology of Metric Spaces, 2nd edition, by S. Kumaresan is published by Narosa. The books *Topology* by Munkres and *Topology* by Armstrong are available in Indian edition. These three books may be used to fill in the details of my outline. My book is strongly recommended for pictures, geometric insights and developing a taste for topology.

1. **Finite sets.** Let X be a set. We say that it is finite if $X = \emptyset$ or if there exists a bijective map of X into an initial segment $I_n := \{k \in \mathbb{N} : 1 \leq k \leq n\}$ of $\mathbb{N}$.

   Using induction/well-ordering principle, one can show that if X has a bijection with I_m and I_n, then $m = n$. The unique n is called the number of elements in X. (For a proof, see my article on Finite sets.) The number of elements in the emptyset is 0.

2. **Countable and uncountable sets.** We say that a set X is countable if either $X = \emptyset$ or if either of the equivalent conditions are satisfied:

   (a) There is a one-one map $f\colon X \to \mathbb{N}$.

   (b) There exists an onto map $g\colon \mathbb{N} \to X$.

Applications: Countability of $\mathbb{N} \times \mathbb{N}$, $\mathbb{Q}^+$, $\mathbb{Q}$, countable union of countable sets, finite product of countable sets. (See my article on Countable and Uncountable sets, also Munkres.)

3. Uncountability of $2^{\mathbb{N}}$: Cantor's theorem: there exists no onto map from X to $P(X)$.

   We prove this by contradiction. Assume $f\colon P(X) \to X$ be onto. Consider the set $S := \{x \in X : x \notin f(x)\}$. Since f is onto there exists $a \in X$ such that $f(a) \in S$. Now exactly one of the following must happen: (i) $a \in S$ or (ii) $a \notin S$. If $a \in S$, by the very definition of S, $a \notin f(a) = S$, contradiction. Similarly (ii) cannot happen. Hence we conclude that no such f exists.

4. Metric Spaces: In most of a first course in real analysis, we just needed the notion of a distance between two real numbers to define the concept of convergent sequences or the concept of continuous functions. Motivated by this we define a *metric* or a distance function on a (nonempty) set X as a function $d\colon X \times X \to \mathbb{R}$ which satisfies the following properties:

   (a) For all $x, y \in X$, we have $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$.

   (b) $d(x,y) = d(y,x)$ for all $x, y \in X$.

   (c) For all $x, y, z \in X$, we have the traingle inequality:

   $$d(x,z) \leq d(x,y) + d(y,z).$$

5. Metrics in $\mathbb{R}^2$: L^1 and L^∞ metrics, called the sum and max metrics:

   $$\begin{aligned} d_1(x,y) &:= \sum_{k=1}^{n} |x_k - y_k| \\ d_{\max}(x,y) \equiv d_\infty(x,y) &:= \max\{|x_k - y_k| : 1 \leq k \leq n\}. \end{aligned}$$

6. Generalizations of these metrics to function spaces.

7. **Normed linear spaces.** A *norm* on a vector space V over $\mathbb{R}$ (or over $\mathbb{C}$) is a function $\| \ \| : V \to \mathbb{R}$ satisfying the following conditions:
   (i) For $x \in V$, $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$.
   (ii) For $x \in V$ and $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$ if X is vector space over $\mathbb{C}$) , we have $\|\lambda x\| = \|\lambda\| \|x\|$.
   (iii) For $x, y \in V$, we have the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$.

8. Examples of normed linear spaces:

   (a) Finite dimensional normed linear spaces: On $\mathbb{R}^n$, we have the following norms:

   $$\|x\|_1 := \sum_{k=1}^{n} |x_k| \quad \text{and} \quad \|x\|_\infty := \max\{|x_k| : 1 \leq k \leq n\}.$$

   That these are norms is easily verified.

   Another norm is the standard/Euclidean norm: $\|x\|_2 := \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2}$. We need Cauchy-Schwarz inequality to verify that this is a norm.

9. Function spaces.

   (a) Let X be any nonempty set. Let $B(X, \mathbb{R})$ denote the real vector space of all bounded real valued functions on X. Then $\| f \|_\infty := \sup\{|f(x)| : x \in X\}$ is a norm on $B(X, \mathbb{R})$.

   (b) Let $X = [0, 1]$. Let $V := C(X, \mathbb{R})$ the vector space of all continuous real valued functions on X. Then $\| f \|_1 := \int_0^1 |f(t)|\, dt$ defines a norm on V.

   (c) Since $C([0, 1], \mathbb{R}) \subset B([0, 1], \mathbb{R})$, we have another norm on V, namely, $\| f \|_\infty$.

10. ℓ^1, the space of sequences whose associated series are absolutely summable is defined as follows:

$$\ell^1 := \left\{ (z_n) : z_n \in \mathbb{R}; \sum_n |z_n| \text{ is convergent.} \right\}$$

    Then $\| z \| = \| (z_n) \| := \sum_n |z_n|$ is a norm on ℓ^1.

11. **Open balls.** Let (X, d) be a metric space. Fix $a \in X$ and $r > 0$. The *open ball* $B(a, r)$ and the closed ball $B[a, r]$ centred at a and radius r are defined by

$$B(a, r) := \{x \in X : d(x, a) < r\} \quad \& \quad B[a, r] := \{x \in X : d(x, a) \le r\}.$$

    We now look at some examples.

   (a) in $\mathbb{R}$: $B(p, r) = (p - r, p + r)$.

   (b) $B(0, 1)$ in $\mathbb{R}^2$ with $\| \ \|_1$, $\| \ \|_2$ and $\| \ \|_\infty$. Picture!
   Look at the pictures. How do we arrive at them? The "boundary" of $B(0, 1)$ is identified. In the case of $\| \ \|_1$, the boundary is defined by $|x| + |y| = 1$. Hence $B(0, 1)$ in this space is the 'region' enclosed by lines $x + y = 1$, $-(x + y) = 1$, $x - y = 1$ and $y - x = 1$. In the case of $\| \ \|_\infty$, the bounding lines are $x = 1$, $-x = 1$, $y = 1$ and $-y = 1$.

   (c) in $\mathbb{Z}$ with the induced metric. Identify **all** open balls. Answer: Any set of $2n + 1$ consecutive integers.

   (d) Relations between $B(x, r)$ and $B(y, s)$.
   If $x = y$ and $r < s$, then $B(x, r) \subseteq B(x, s)$. Equality can occur. Consider the discrete metric and $r = 1/2$ and $s = 3/4$.
   If d is discrete, and if $r > 1$ and $s > 1$, then for *any* x, y, we have $B(x, r) = B(y, s)$.

   (e) Visualizing the open balls in $C[0, 1]$ under $\| \ \|_\infty$.

   (f) In an normed linear space , $B(x, r) = x + rB(0, 1)$.

12. **Open sets in a metric space.** A subset U of a metric space is said to be *open* or d-open if for each $x \in U$, there exists $r_x > 0$ such that $B(x, r_x) \subset U$. We now look at lots of examples to build our intuition. In each of the examples, draw pictures of the sets and see whether you can enclose each of the points x in an open ball $B(x, r_x)$ contained in the given set. In most of the cases, the geometry will lead you to the 'best possible' radius r_x. This will develop your intuition to 'identify' the open sets "instantly". **Pictures!**

   (a) in $\mathbb{R}$: various examples such as open intervals, union of open intervals and non-examples such as $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$,

(b) $\{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$ in $\mathbb{R}^2$.

(c) $\{(x, y) \in \mathbb{R}^2 : x \geq 0, y > 0\}$ in $\mathbb{R}^2$.

(d) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$ in $\mathbb{R}^2$.

(e) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ in $\mathbb{R}^2$.

(f) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$.

(g) Various conic sections in $\mathbb{R}^2$.

(h) In an normed linear space V, if any vector subspace W is open, then $W = V$. Application: Is $C[0, 1]$ open in $BF[0, 1]$, the set of bounded functions?

(i) Is $\mathbb{R} \setminus \mathbb{Z}$ open in $\mathbb{R}$?

(j) The open ball $B(x, r)$ is open in any metric space.

Draw picture. If $y \in B(x, r)$, we need to find $s > 0$ such that $B(y, s) \subset B(x, r)$. Let us find such an s. Let $z \in B(y, s)$. We need to show $z \in B(x, r)$. That is we need an estimate for $d(z, x)$. The obvious estimate is $d(z, x) \leq d(z, y) + d(y, x) < s + d(y, x)$. If we can show $s + d(y, x) < r$, we are through. This suggest we choose $0 < s < r - d(x, y)$.

(k) $\{y \in X : d(x, y) > r\}$ is open. *Hint:* Modify the idea of the last sub-item.

(l) What are the open sets in a finite metric space?

(m) Can $\{h \in C[0, 1] : f(x) < h(x) < g(x)\}$ for some $f, g \in C[0, 1]$ be an open ball? Is it an open set?

(n) Is the open unit ball in $(C[0, 1], \| \ \|_\infty)$ open in $(C[0, 1], \| \ \|_1)$?

(o) If U is an open subset in an normed linear space , $(X, \| \ \|)$, then

  i. $x + U$ is open for any $x \in X$

  ii. $A + U$ is open for any $A \subset X$

  iii. αU is open for any nonzero scaler α.

(p) Is the set $U := \{f \in C[0, 1] : f(1/2) \neq 0\}$ open in $(C[0, 1], \| \ \|_\infty)$?

(q) Any open subgroup G of $\mathbb{R}$ is $\mathbb{R}$.

For, $0 \in G$ and hence $(-\varepsilon, \varepsilon) \subset G$ for some $\varepsilon > 0$. Since G is group, for all $x, y \in (-\varepsilon, \varepsilon)$ we have $x + y \in G$, that is, $(-2\varepsilon, 2\varepsilon) \subset G$. By induction, $(-n\varepsilon, n\varepsilon) \subset G$ for $n \in \mathbb{N}$.. Now let $x \in \mathbb{R}$ be nonzero. By Archimedean property, there exists $N \in \mathbb{N}$ such that $N\varepsilon > |x|$. Hence $x \in (-N\varepsilon, N\varepsilon)$. It follows that $\mathbb{R} = \cup_{n \in \mathbb{N}}(-n\varepsilon, n\varepsilon) \subset G$.

(r) A subset U of a metric space is open iff it is the union of a family of open balls. For, if $x \in U$, there exists $r_x > 0$ such that $B(x, r_x) \subset U$. We then have an indexed family $\{B(x, r_x) : x \in U\}$ of open balls. Clearly, $U = \cup_{x \in U} B(x, r_x)$.

(s) A subset $U \subset \mathbb{R}$ is open iff it is the union of a countable family of pair-wise disjoint open intervals. (See Lemma 1.2.42 on Page 23 of my book on Metric spaces.)

13. The class $\mathcal{T}$ of open subsets of a metric space (X, d) have the following properties:

(a) $\emptyset, X \in \mathcal{T}$.

(b) If $\{U_i : i \in I\}$ is any collection of elements in $\mathcal{T}$, then $U := \cup_{i \in I} U_i \in \mathcal{T}$.

(c) If U_k, $1 \leq k \leq n$ are in $\mathcal{T}$, then $U_1 \cap U_2 \cap \cdots \cap U_n \in \mathcal{T}$.

14. **Topology: Definition and Examples.** A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X which satisfies the three conditions (a)–(c) of the last item. Elements of $\mathcal{T}$ are called open sets, to be precise $\mathcal{T}$-open.

    (a) Metric topology: Let (X, d) be a metric space. Then the collection of (d-) open subsets is a topology on the metric space. This topology is called the metric topology on the metric space.

    (b) Discrete topology: Here $\mathcal{T} = P(X)$, the power set of X. Thus, every subset is open.

    (c) The topology on a finite metric space is discrete.

    (d) Indiscrete topology: U is open iff $U = \emptyset$ or $U = X$, that is, $\mathcal{T} = \{\emptyset, X\}$.

    (e) Co-finite topology: U is open iff $U = \emptyset$ or $X \setminus U$ is finite, that is ,
    $$\mathcal{T} = \{U \subset X : \text{ Either } U = \emptyset \text{ or } X \setminus U \text{ is finite.}\}$$

    (f) Co-countable topology: U is open iff $U = \emptyset$ or $X \setminus U$ is countable, that is,
    $$\mathcal{T} = \{U \subset X : \text{ Either } U = \emptyset \text{ or } X \setminus U \text{ is countable.}\}$$

    (g) VIP topology: Fix $p \in X$. U is open iff $U = \emptyset$ or $p \in U$.

    (h) Outcast topology: Fix $p \in X$. U is open iff $U = X$ or $p \notin U$.

    (i) Outcast + co-finite topology: U is open iff either $p \notin U$ or U^c is finite.

15. A topology on $\mathbb{Z}$. Let $\mathcal{B}$ be the set of arithmetic progressions in $\mathbb{Z}$. Any element $B \in \mathcal{B}$ is of the form $a + \mathbb{Z}b$ for some nonzero b. Note that $\mathcal{B}$ is nothing other than the set of cosets of all additive (non-trivial) subgroups of $\mathbb{Z}$. An example: $2 + 5\mathbb{Z}$. We define a topology $\mathcal{T}$ on $\mathbb{Z}$ as follows: a subset $U \subset \mathbb{Z}$ is open iff for each $x \in U$, a coset of the from $x + \mathbb{Z}b \subset U$. Clearly, $\emptyset, \mathbb{Z} \in \mathcal{T}$. If $\{U_i\}$ is a collection of sets in $\mathcal{T}$ and $x \in \cup_i U_i$, then $x \in U_j$ for some j and hence there is a $b \neq 0$ such that $x \in x + \mathbb{Z}b \subset U_j \subset \cup_i U_i$. If $x \in U \cap V$ for some $U, V \in \mathcal{T}$, then there exist b, c such that $x + \mathbb{Z}b \subset U$ and $x + \mathbb{Z}c \subset V$. Clearly, $x + \mathbb{Z}\operatorname{lcm}(b, c) \subset U \cap V$. Hence $\mathcal{T}$ is a topology on $\mathbb{Z}$.

    (1) Observe that any element of $\mathcal{B}$ can be written in the form $r + \mathbb{Z}b$ where $b > 0$ and $0 \leq r < b - 1$. Hence in view of $\mathbb{Z} = \cup_{0 \leq r < b-1} r + \mathbb{Z}b$, we see that any element of $\mathcal{B}$ and its complement are both open!

    (2) Another observation is that no nonempty finite set can be open.

    As an application of these observations, we now give a topological proof of Eulcid's theorem on the infinitude of primes in $\mathbb{Z}$. We prove this by contradiction. Assume that $p_1, \ldots, p_n$ are the set of all primes. Now the only integers that are not divisible by any prime are ± 1. Hence
    $$\mathbb{Z} \setminus \{\pm 1\} = \cup_{k=1}^{n} \mathbb{Z}p_k = \cup_k U_k, \text{ say.}$$

    Let us take the complements on both sides of the above equality. The complement of left side is $\cap_k U_k^c$, a finite intersection of open sets (in view of Observation 1) and hence is open. Hence the left side, a finite set is open, a contradiction to observation 2).

16. Basis of a topological space and basis for a topology on a set.

    Basis for a topological space. Let $(X, \mathcal{T})$ be a topological space. A subset $\mathcal{B} \subset \mathcal{T}$ of open sets is said to be a basis for $\mathcal{T}$ if every element in $\mathcal{T}$ is a union of elements from $\mathcal{B}$. In other words, $\mathcal{B}$ is a basis for $\mathcal{T}$ if for any $U \in \mathcal{T}$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. The typical example of a basis is the set of all open balls for the topology on a metric space.

17. Examples of bases:

    (a) $\{B(x, r) : x \in X, r > 0\}$ is a basis for the metric topology on any metric space. The indexing set is $X \times (0, \infty)$.

    (b) $\{B(x, 1/n) : x \in X, n \in \mathbb{N}\}$ is a basis for the metric topology on any metric space. The indexing set is $X \times \mathbb{N}$.

    (c) When $X = \mathbb{R}$, we can do better than the last two bases. Consider $\mathcal{B} := \{(a, b) : a, b \in \mathbb{Q}\}$ is a basis for the standard topology on $\mathbb{R}$. Note that this basis is countable, as it is indexed by $\mathbb{Q} \times \mathbb{Q}_+$. (Why? $(a, b) = B(c, r)$ where $c = (a+b)/2 \in \mathbb{Q}$ and $r = (b - a)/2 \in \mathbb{Q}_+$.)

    (d) A basis for the VIP topology is $\{p\} \cup \{\{p, q\} : q \in X, q \neq p\}$.

    (e) A basis for outcast topology is $\{X\} \cup \{\{q\} : q \in X, q \neq p\}$.

    (f) $\mathcal{B} := \{\{x\} : x \in X\}$ is a basis for the discrete topology on a set X.

    (g) $\mathcal{B} := \{X\}$ is a basis for the indiscrete topology on a set X.

    (h) Note that $(0, 1) = (0, 1/2) \cup (1/4, 1) = \cup_{n \geq n}(0, (n - 1)/n)$. Hence there is no uniqueness while expressing an open set as a union of some elements from the basis.

18. The second notion is a basis for a topology on a set X. The question here is: given a set X and a subset $\mathcal{B} \subset P(X)$ of subsets of X, does there exist a topology $\mathcal{T}$ on X for which $\mathcal{B}$ is a basis? Suppose such a topology $\mathcal{T}$ exists. Then $X \in \mathcal{T}$ so that a first requirement is (1) $\cup_{B \in \mathcal{B}} B = X$. Also, since any $B \in \mathcal{B}$ must be in $\mathcal{T}$, $B_1 \cap B_2 \in \mathcal{T}$ for any $B_1, B_2 \in \mathcal{B}$. Hence the second condition: (2) for any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$. If these two conditions are satisfied, we define a topology $\mathcal{T}$ on X as follows:

    $$\mathcal{T} := \{U \subset X : \forall x \in U,\ \exists B \in \mathcal{B} \text{ such that } x \in B \subset U\}.$$

    It is easy to verify that $\mathcal{T}$ is a topology on X and that $\mathcal{B}$ is a basis for this topology.

19. Order Topology: partial and total orders, dictionary order on products, $\mathbb{C}$ is totally ordered **but** is not an ordered field. Intervals of the form (a, b) and rays of the form $(-\infty, a)$ and (b, ∞). Examples in $\mathbb{R}^2$: the rays $(-\infty, (1, 2))$, $((-1, 1), \infty)$ and the intervals $((-1, 1), (3, -2))$ and $((0, 0), (0, 10))$. Basis for order topology. What is the order topology on $\mathbb{R}$, on $\mathbb{Z}$, on $\mathbb{N}$ and on a finite totally ordered set? **Pictures!** **Details!**

20. Lower Limit Topology: Consider $\mathcal{B} := \{[a, b) : a, b \in \mathbb{R}, a < b\}$. It is easy to see that $\mathcal{B}$ satisfies both the conditions laid out in Item 18. The topology associated with this basis is known as the lower limit topology on $\mathbb{R}$, denoted by $\mathcal{T}_L$.

When is a subset $U \subset \mathbb{R}$ open in $\mathcal{T}_L$? If for $x \in U$, we can find $[a, b) \in \mathcal{B}$ such that $x \in [a, b) \subset U$. A picture will immediately lead you to a 'better' condition: for $x \in U$, we can find $b > x$ such that $[x, b) \subset U$. In particular, any interval $(a, b) \in \mathcal{T}_L$. Hence the lower limit topology is finer than the standard topology on $\mathbb{R}$. In fact, it is strictly finer, since $[a, b)$ is open In $\mathcal{T}_L$ but not in the standard topology.

Note that no countable sub-collection of $\{[a, b) : a, b \in \mathbb{R}, a < b\}$ will serve as a basis for the lower limit topology. For, if $\{[a_n, b_n) : n \in \mathbb{N}\}$ is one such, then choose $a \in \mathbb{R}$ such that $a \neq a_n$ for $n \in \mathbb{N}$. Then the open set $[a, a + 1)$ cannot be written as a union of any such elements. Why? For, a has to be in one of them, say, $[a_k, b_k)$ Since $a_k \neq a$, it follows that $a_k < a < b_k$ so that the union will have elements from $[a_k, a)$ which are not in $[a, a + 1)$.

Question: How about the collection $\mathcal{B} = \{(a, b] : a, b \in \mathbb{R}, a < b\}$? Is it a basis for some topology on $\mathbb{R}$? If so,what will you call it?

21. The class of all topologies on a given set is a partially ordered set: if $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, we define $\mathcal{T}_1 \leq \mathcal{T}_2$ iff $\mathcal{T}_1 \subset \mathcal{T}_2$, as subsets of $P(X)$. The indiscrete topology is the smallest element and the discrete topology is the largest element of the class of topologies on X.

    The union of topologies on X need not be topology. Let $X = \{a, b, c\}$ be a three element set. Let $\mathcal{T}_1 := \{\emptyset, \{a\}, X\}$ and $\mathcal{T}_2 := \{\emptyset, \{b\}, X\}$. These are two topologies on X but their union is not a topology.

    However, the intersection of a (nonempty) family of topologies on X is again a topology, as can be easily verified.

    Compare this with analogous results from algebra: intersections of subgroups of a group is again a group, intersection of vector subspaces of a vector spaces a vector subspace, intersection of ideas in a ring is again an ideal and so on. Associated with this phenomenon is the concept of subgroup (a vector subspace, an ideal, or a submodule) generated by subset S in a group (in a vector space, in a ring, or in a module over a ring).

    These motivate us to define the following: if $\mathcal{A}$ is an arbitrary collection of subsets of a set X, there exists a unique smallest topology on X which contains $\mathcal{A}$ and is called the topology generated by $\mathcal{A}$. We shall later see a practical way of looking at this topology. See Item 142. For the time being, let us work out two examples.

    - Let X be a nonempty set with at least three elements. Let $\mathcal{S}$ be the collection of all two element subsets of X. What is the smallest topology $\mathcal{T}$ containing $\mathcal{S}$? Fix $a \in X$. We can find two distinct elements, say, $x, y \in X$ none of which is a. Then $\{a, x\}$ and $\{a, y\}$ lie in $\mathcal{S}$ and hence in $\mathcal{T}$. It follows that $\{a\} \in \mathcal{T}$. Thus,every singleton subset is in $\mathcal{T}$ and hence $\mathcal{T}$ is the discrete topology on X.

      Question: What is $\mathcal{T}$ if X has only two elements?
    - Let $\mathcal{S}$ consist of single element $A \subset X$. Then $\mathcal{T} = \{\emptyset, A, X\}$.

22. Let X be a set and $\mathcal{T}_c$ and $\mathcal{T}_f$ be respectively co-countable and co-finite topologies on X. Then the co-countable topology is finer than the co-finite topology.

They are the same iff X is finite. If X is finite, then the two topologies are the same. To see the converse, we need a result form set theory: If X is an infinite set, then there exists a set A such that $X \setminus A$ is infinite and countable.

23. We can use bases to say something about the topologies on a set.

    Theorem 1. *Let X be any set. Let $\mathcal{B}_i$ be a basis for some topology $\mathcal{T}_i$ on X, for $i = 1, 2$. Then $\mathcal{T}_1 \leq \mathcal{T}_2$ iff the following holds: if $B_1 \in \mathcal{B}_1$, then $B_1 \in \mathcal{T}_2$. In particular, $\mathcal{T}_1 = \mathcal{T}_2$ iff every $B_1 \in \mathcal{B}_1$ is in $\mathcal{T}_2$ and every $B_2 \in \mathcal{B}_2$ is in $\mathcal{T}_1$.* □

24. Continuity: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map and $x_0 \in X$. We say that f is continuous at x_0 if for any given open set V containing $f(x_0)$, there exists an open set U containing x_0 such that $f(U) \subset V$. This definition is an abstraction of the standard ε-δ definition of continuity, say, of functions $f\colon \mathbb{R} \to \mathbb{R}$. In this context, $V = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ and $U = (x_0 - \delta, x_0 + \delta)$. In fact, we have the following theorem:

    Theorem 2. *Let $f\colon (X, d) \to (Y, d)$ be a map between metric spaces. Let $x_0 \in X$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the topologies on X and Y induced buy their respective metrics. Then $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous at x_0 iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $d(x, x_0) < \delta$, we have $d(f(x), f(x_0)) < \varepsilon$.*

    Proof. Let us assume that $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous at x_0. Let $\varepsilon > 0$ be given. Then $V := B(f(x_0), \varepsilon)$ is an open set containing $f(x_0)$. Hence there exists an open set $U \ni x_0$ such that for all $x \in U$ we have $f(x) \in V$. Since U is open there exists $\delta > 0$ such that $B(x_0, \delta) \subset U$. Hence it follows $d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon$, that is, $f\colon (X, d) \to (Y, d)$ is continuous at x_0. Pictures!

    The converse is similar. Let $f\colon (X, d) \to (Y, d)$ is continuous at x_0. Assume that an open $V \ni f(x_0)$ is given. Then we can find $\varepsilon > 0$ such that $B(f(x_0), \varepsilon) \subset V$. For this $\varepsilon > 0$ by the definition of continuity in metric space context, there exists $\delta > 0$ such that $d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon$. If we let $U := B(x_0, \delta)$, then U is open, $x_0 \in U$, and for $x \in U$, we have $f(x) \in B(f(x_0 < \varepsilon) \subset V$. □

25. Let $f\colon X \to Y$ be any map between two sets. Let $B \subset Y$. The set $f^{-1}(B) := \{x \in X : f(x) \in B\}$ is called the inverse image of B under f. The following are well-known facts:

    (a) If $\{B_i : i \in I\}$ is a family of subsets of Y, then
        i. $f^{-1}(\cup_{i \in I} B_i) = \cup_{i \in I} f^{-1}(B_i)$.
        ii. $f^{-1}(\cap_{i \in I} B_i) = \cap_{i \in I} f^{-1}(B_i)$.

    (b) For any set $B \subset Y$, we have $X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$.

    Thus, "the inverse images behave well under set-theoretic operations."

26. Let X and Y be topological spaces. Then a map $f\colon X \to Y$ is said to be continuous on X iff it is continuous at each point $x \in X$.

    Let X and Y be topological spaces and $f\colon X \to Y$ be continuous at each point $x \in X$. Let $V \subset Y$ be any open subset of Y. Let $U := f^{-1}(V) = \{x \in X : f(x) \in V\}$. Let $a \in U$. By the definition of U, $f(a) \in V$. Since f is continuous at a and $V \ni f(a)$ is an open set, there exists an open set $U_a \ni a$ such that for all $x \in U_a$, we have $f(x) \in V$. This implies $U_a \subset U$. Since $a \in U$ was arbitrary, what we have shown is that for each $a \in U$, there exists an open set U_a such that $a \in U_a$ and $U_a \subset U$. In particular, $U = \cup_{a \in U} U_a$ is open. (The argument of this paragraph teaches an algorithm: in the case of a topological space if we want to show that a set U is open , we need to find an open set U_a for each $a \in U$ such that $a \in U_a$ and $U_a \subset U$. Compare this with the algorithm to show a subset of a metric space is open, Item 12r.)

We have thus shown that $f^{-1}(V)$ is open in X for each open subset $V \subset Y$ of Y.

Is the converse true? That is, if $f^{-1}(V)$ is open in X for each open subset $V \subset Y$ of Y, is f continuous on X? This is easy. Let $a \in X$ and $V \ni f(a)$ be open in Y. Then $U := f^{-1}(V)$ is open by hypothesis . Clearly $a \in U$. Also, for each $x \in U$, $f(x) \in V$, that is, f is continuous at a. Since a is arbitrary, it follows that f is continuous on X.

We have thus arrived at the following result.

Theorem 3. *Let X and Y be topological spaces. Then a map $f: X \to Y$ is continuous on X iff for every open subset $V \subset Y$, the inverse image $f^{-1}(V)$ is open in X.* □

27. The theorem of the last item leads us to the following result:

    Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on the same set X. Then $\mathcal{T}_1 \leq \mathcal{T}_2$ iff the identity map $I: (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is continuous. In particular, $\mathcal{T}_1 = \mathcal{T}_2$ iff the identity maps $I: (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ and $I: (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ are continuous. (This is same as saying that the identity map is a homeomorphism, a concept to be defined in Item 136.)

28. We looked at the following examples:

    (a) Any constant map from a topological space to another is continuous.

    (b) The identity map from $(X, \mathcal{T}_X)$ to itself is continuous.

    (c) If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X, then the identity map $I: (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous iff $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

    (d) Let $(X, \mathcal{T}_X)$ be a topological space with the property that any map $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous. Then $\mathcal{T}_X$ is discrete and conversely.

    (e) Let $(Y, \mathcal{T}_Y)$ be a topological space with the property that any map $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous. Then $\mathcal{T}_Y$ is indiscrete and conversely.

    (f) The identity map from X with co-countable topology to X with co-finite topology is continuous. The other way map is continuous iff X is finite.

    (g) Let X be a set with at least two elements and $p \in X$. Let V (resp. O) denote the VIP topology (resp. the outcast topology) on X with respect to p. Then

        i. The identity map $I: (X, V) \to (X, O)$ is not continuous. However it is continuous at $x = 0$ and at no other point.

        ii. The identity map $I: (X, O) \to (X, V)$ is not continuous at any point.

29. The identity map from $\mathbb{R}$ with the lower limit topology is continuous to $\mathbb{R}$ with the usual topology.

30. Let $\| \ \|_k$, $k = 1, 2$, be two norms on a vector space V. Then they are equivalent iff the identity map $I: (V, \| \ \|_1) \to (V, \| \ \|_2)$ and $I: (V, \| \ \|_2) \to (V, \| \ \|_1)$ are continuous.

31. Let X be an uncountable set with co-countable topology $\mathcal{T}_c$. Then the only continuous functions $f\colon (X, \mathcal{T}_c) \to \mathbb{R}$ are constants.

32. We discussed the set of points of continuity of all real valued functions on the following spaces.
   (i) $\mathbb{R}$ with VIP topology with 0 as the VIP.
   (ii) $\mathbb{R}$ with outcast topology with 0 as the outcast.
   (iii) $\mathbb{N}$ with the topology $\mathcal{T} := \{\emptyset, \mathbb{N}\} \cup \{I_n : n \in \mathbb{N}\}$ where $I_n = \{1, 2, \dots, n\}$.

33. On any metric space X, we have lots of real valued continuous functions: $f(x) := d(x, p)$ for any fixed $p \in X$. In particular, given $p \neq q$ in X, there exists a real valued continuous function f on X such that $f(p) \neq f(q)$.

34. Let A be a nonempty subset of a metric space X. We defined $d_A(x) \equiv d(x, A) := \inf\{d(x, a) : a \in A\}$. Identify as much as possible d_A for the subsets below and draw their graphs. Picture!

   (a) $X = \mathbb{R}$ and $A = [-1, 1]$.
   (b) $X = \mathbb{R}$ and $A = \mathbb{Q}$.
   (c) $X = \mathbb{R}$ and $A = \mathbb{Z}$.
   (d) $X = \mathbb{R}^2$ and A is the x-axis.
   (e) $X = \mathbb{R}^2$ and $A = \{(x, y) : x^2 + y^2 = 1\}$.
   (f) W is a vector subspace of $\mathbb{R}^n$. *Hint:* If $\mathbb{R}^n = W \oplus W^\perp$, and if $x = w + w'$, then $d_W(x) = \|w'\| = \|x - p_W(x)\|$, where $p_W\colon \mathbb{R}^n \to W$ is the orthogonal projection.

35. We claim that for any nonempty subset A of a metric space X, the function $d_A\colon X \to \mathbb{R}$ is continuous.

   $d_A(x) \leq d(x, a) \leq d(x, y) + d(y, a)$, for $a \in A$. Hence $d_A(x)$ is a lower bound for the set $\{d(x, y) + d(y, a) : a \in A\}$. But then $\inf\{d(x, y) + d(y, a) : a \in A\} = d(x, y) + d_A(y)$.

36. The function $x \mapsto \|x\|$ is continuous on an normed linear space $(V, \|\ \|)$. Note that $\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$ so that $\|x\| - \|y\| \leq \|x - y\|$. Interchanging x and y we get
$$|\,\|x\| - \|y\|\,| \leq \|x - y\|.$$
   This establishes the (uniform) continuity of the norm function. Note that this has the continuity of modulus/absolute value as a special case.

37. The functions $\pi_j\colon x \mapsto x_j$, the coordinate projections are continuous on $\mathbb{R}^n$ (with respect to any of the norms $\|\ \|_i$, $i = 1, 2, \infty$):
$$|\pi_j(x) - \pi_j(a)| = |x_j - a_j| \leq \|x - a\|, \quad 1 \leq j \leq n.$$

38. Composite of continuous functions is continuous: Let X, Y, Z be topological spaces. Let $f\colon X \to Y$ be continuous at $p \in X$ and $g\colon Y \to Z$ be continuous at $q := f(p) \in Y$. Then $g \circ f\colon X \to Z$ is continuous at p.

   Let $W \subset Z$ be an open set such that $(g \circ f)(p) = g(q) \in W$. Since g is continuous at q, there exists an open $V \ni q$ such that $g(V) \subset W$. Since f is continuous at p and $V \ni f(p) =$, there exists an open $U \ni x$ such that $f(U) \subset V$. Clearly, $(g \circ f)(U) \subset W$.

39. Let $f\colon X \to \mathbb{R}$ be a continuous function. Then $|f|\colon X \to Y$ defined by $|f|(x) := |f(x)|$ is continuous on X. For, it is the composite of two continuous functions $|f| = |\ | \circ f$, where $|\ |\colon \mathbb{R} \to \mathbb{R}$ is the modulus function, $|\ |(x) := |x|$.

40. Let X be a topological space. Let $\mathbb{R}^n$ be given the metric topology arising form the standard Euclidean metric. Let $f\colon X \to \mathbb{R}^n$. Then we can write $f(x) = (f_1(x), \ldots, f_n(x))$. Note that $f_j(x) = \pi_j \circ f$ where π_j is the projection as in Item 37.

    We claim that f is continuous iff each $f_j\colon X \to \mathbb{R}$, $1 \le j \le n$, is continuous. Assume that f is continuous. Since $f_j = \pi_j \circ f$, it follows from Items 37–38 that f_j is continuous.

    Now the converse. Fix $a \in X$. Let $V \subset \mathbb{R}^n$ be open containing $f(a)$. Let $\varepsilon > 0$ be such that $B(f(a), \varepsilon) \subset V$. By continuity of f_j at a, there exists an open set $U_j \subset X$ such that $a \in U_j$ and $f_j(U_j) \subset B(f_j(a), \varepsilon/\sqrt{n})$, $1 \le j \le n$. Then $U := \cap_{j=1}^n U_j$ is an open set which contains a and is such that $f(x) \in B(f(a), \varepsilon)$ for all $x \in U$:

    $$d(f(x), f(a))^2 = \sum_{j=1}^n (f_j(x) - f_j(a))^2 < n(\varepsilon^2/n) = \varepsilon^2.$$

    Hence for $x \in U$, we have $f(x) \in B(f(a), \varepsilon) \subset V$, that is, f is continuous at a.

41. Let $f, g\colon X \to \mathbb{R}$ be continuous functions. Consider $\mathbb{R}^2$ with $\|\ \|$ being one of the three norms: $\|\ \|_1, \|\ \|_2, \|\ \|_{\max}$. Then the function $\varphi\colon X \to \mathbb{R}^2$ given by $\varphi(x) = (f(x), g(x))$ is continuous.

    This is a special case of the last item.

42. The functions $\mathbb{R}^2 \to \mathbb{R}$ given by $\alpha\colon (x, y) \mapsto x + y$ and $\mu\colon (x, y) \mapsto xy$ are continuous.

    To establish the continuity of these function we use Theorem 2 in Item 24.

    Let $(a, b) \in \mathbb{R}^2$. Let $\varepsilon > 0$ be given. Assume $\delta > 0$ serves. We estimate

    $$\begin{aligned} |\alpha(x, y) - \alpha(a, b)| = |(x + y) - (a + b)| &= |(x - a) + (y - b)| \\ &\le |x - a| + |y - b| \\ &\le d((x, y), (a, b)) + d((x, y), (a, b)). \end{aligned}$$

    If $d((x, y), (a, b)) < \delta$, the above estimate suggests that we take $2\delta < \varepsilon$.

    Let $\varepsilon > 0$ be given. Assume $\delta > 0$ serves. We may assume that $0 < \delta < 1$. If $d((x, y), (a, b)) < \delta$, then $|x - a| < \delta < 1$ and $|y - b| < \delta < 1$. Hence $|y| \le |y - b| + |b| < 1 + |b|$. We now estimate

    $$\begin{aligned} |\mu(x, y) - \mu(a, b)| = |xy - ab| = |xy - ay + ay - ab| &\le |y||x - a| + |a||y - b| \\ &\le (1 + |b|)|x - a| + |a||y - b| \\ &< M 2\delta, \end{aligned}$$

    where $M = \max\{1 + |b|, |a|\}$. If we choose $\delta < \frac{\varepsilon}{2M}$, as well as $\delta < 1$, the estimates above establish $|\mu(x, y) - \mu(a, b)| < \varepsilon$.

43. If f, g are continuous functions from a topological space to $\mathbb{R}$ and if $a, b \in \mathbb{R}$, then the functions $af + bg$ and fg are continuous. Hint: Use Items 38–42.

    Thus the set $C(X, \mathbb{R})$ of all real valued continuous functions on a topological space is a vector space over $\mathbb{R}$. It is also a commutative ring with identity, in fact, an algebra over $\mathbb{R}$.

44. Given two real numbers a, b we wish to find a "formula" for $\max\{a, b\}$ and $\min\{a, b\}$. Given a, b, their mid point is $(a+b)/2$. To reach the maximum of these two, we need to move to the right for half of the distance between them, that is, we need to add $|a-b|/2$ to their mid point. Similar analysis can be done for minimum. Hence we arrive at the following formulas:

$$\max\{a, b\} = \frac{(a+b)+|a-b|}{2} \quad \text{and} \quad \min\{a, b\} = \frac{(a+b)-|a-b|}{2}.$$

45. If $f, g\colon X \to \mathbb{R}$ are two continuous functions on a topological space X, then $\max\{f, g\}$ and $\min\{f, g\}$ are continuous. This follows from Items 44, 39 and 43.

46. Any polynomial function $f\colon \mathbb{R}^n \to \mathbb{R}$ is continuous. This follows from Item 37 and 45. Examples of polynomial functions on $\mathbb{R}^2$ and $\mathbb{R}^3$ are $p(x, y) = 3x^2+y^2-xy^2+6x-7y+10$, $q(x, y, z) = z^{10} - 9y^2 + 17xyz^3 + 2012$ etc.

47. The map $\rho\colon \mathbb{R}^* \to \mathbb{R}^*$ given by $\rho(x) = 1/x$ is continuous. Look at the estimate:

$$|\rho(x) - \rho(y)| \leq \frac{|x-y|}{|xy|} \leq \frac{2|x-y|}{|x^2|},$$

if we restrict y in such a way that $|x - y| < |x|/2$.

48. Let $f\colon X \to \mathbb{R}$ be continuous and assume that $f(x) \neq 0$ for all $x \in X$. Then $1/f\colon X \to \mathbb{R}$ is continuous. For, $1/f$ is the composition $\rho \circ f$, where ρ is as in the last item.

49. Any linear map from $\mathbb{R}^n$ with any one of our three standard norms to any normed linear space is continuous. In particular, any linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$ is continuous.

    More generally, any linear map $T\colon \mathbb{R}^n \to X$, where X is any normed linear space is (uniformly) continuous.

    For let $\{e_i : 1 \leq i \leq n\}$ be the standard basis of $\mathbb{R}^n$. Then for any $x = (x_1, \ldots, x_n) = \sum_i x_i e_i \in \mathbb{R}^n$ we have

$$\begin{aligned} \|Tx\| = \left\| T\left(\sum_{i=1}^n x_i e_i\right)\right\| &\leq \sum_{i=1}^n |x_i| \, \|Te_i\| \\ &\leq M \sum_{i=1}^n \|x\|, \text{ where } M := \max\{\|Te_i\| : 1 \leq i \leq n\} \\ &= Mn \, \|x\|. \end{aligned}$$

    Note that $|x_i| \leq \|x\|$ where $\| \ \|$ could be either $\| \ \|_1$, $\| \ \|_2$ or $\| \ \|_{\max}$. Hence $\|Tx - Ty\| = \|T(x-y)\| \leq Mn \, \|x-y\|$ so that T is Lipschitz and hence uniformly continuous.

50. Let $M_{m\times n}(\mathbb{R})$ denote the set of all $m \times n$ matrices with real entries. We identify it with $\mathbb{R}^{mn}$ using an obvious linear isomorphism:

$$X = (x_{ij}) \mapsto (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{m1}, \ldots, x_{mn}).$$

    We use any one of the standard norms on $M_{m\times n}(\mathbb{R})$. We let $M(n, \mathbb{R}) := M_{n\times n}(\mathbb{R})$. Then we have

(a) The 'transpose' map $X \mapsto X^T$ from $M(n, \mathbb{R})$ to itself is continuous. For, the map is $(x_{11}, x_{12}, \ldots, x_{n1}, \ldots, x_{nn}) \to (x_{11}, x_{21}, \ldots, x_{1n}, \ldots, x_{nn})$. The coordinate maps are $f_{ij}(X) = x_{ji}$ and hence are continuous. (See Item 40.)

(b) The 'trace' map $X \mapsto \mathrm{Tr}(X)$ is continuous from $M(n, \mathbb{R})$ to $\mathbb{R}$. Observe that it is a linear map.

(c) The determinant map $\det\colon M(n, \mathbb{R}) \to \mathbb{R}$, defined by $X \mapsto \det(X)$, is a "polynomial function" and hence is continuous. When $n = 2$ and the matrix is $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(X) = ad - bc$. For general n, recall the formula for the determinant (Laplace expansion) as an alternating sum, $\det(X) := \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) x_{1\sigma(1)} \cdots x_{n\sigma(n)}$.

(d) One can use functions whose continuity are known to assert that certain subsets are open.

   i. Since polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}$ are continuous

      A. The subsets $\{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$, $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\}$ and $\{(x, y) \in \mathbb{R}^2 : xy \neq 1\}$ are all open.

      B. The subset $\{(x, y) \in \mathbb{R}^2 : x^3 - 34x^2y - 28xy^2 - y^3 + 7xy - 19y + 125 \neq 0\}$ is open in $\mathbb{R}^2$.

      C. $\mathbb{R}^3 \setminus P$, where $P := \{(x, y, z) : ax + by + cz = d\}$ is a plane, is open in $\mathbb{R}^3$.

      D. The rectangle $R := (a, b) \times (c, d)$ is open in $\mathbb{R}^2$: $R = p_1^{-1}(a, b) \cap p_2^{-1}(c, d)$, where $p_1(x, y) = x$ etc.

      E. The set $\{f \in C[0, 1] : f(1/2) \neq 0\}$ in $X := (C[0, 1], \|\ \|_\infty)$ is open. *Hint:* Consider $T\colon X \to \mathbb{R}$ given by $T(f) := f(1/2)$.

   ii. Let W be a vector subspace of $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus W$ is open in $\mathbb{R}^n$. *Hint:* Write $\mathbb{R}^n = W \oplus W^\perp$ and let $u_1, \ldots, u_k$ be an orthonormal basis of $W^\perp$. Then $x \in \mathbb{R}^n$ lies in W iff $\langle x, u_i \rangle = 0$ for all $1 \leq i \leq k$.
   Alternately, consider the orthogonal projection $\pi\colon \mathbb{R}^n \to W^\perp$. Then $\mathbb{R}^n \setminus W = \pi^{-1}(W^\perp \setminus \{0\})$.

   iii. $GL(n, \mathbb{R})$, the set of all invertible matrices is open in $M(n, \mathbb{R})$.

   iv. The set of symmetric matrices, being a vector subspace, cannot be open in $M(n, \mathbb{R})$. *Hint:* See Item 12h.

   v. Same holds true for the set of skew symmetric matrices.

51. To check continuity, it suffices to show that the inverse images of basic elements in the codomain are open in the domain:

    Lemma 4. *Let $(X_i, \mathcal{T}_i)$ be topological spaces $i = 1, 2$ and let $\mathcal{B}_2$ be a basis for $\mathcal{T}_2$. Then $f\colon (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ is continuous iff $f^{-1}(B_2) \in \mathcal{T}_1$ for all $B_2 \in \mathcal{B}_2$.* □

    Item 29 is an immediate consequence of this.

52. Consider $M(n, \mathbb{R})$ the set of all $n \times n$ real matrices. Then the map Repetition: Item 50

$$\varphi\colon A \mapsto (a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{n1}, \ldots, a_{nn})$$

    is a linear isomorphism of $M(n, \mathbb{R})$ onto $\mathbb{R}^{n^2}$. We use this to transfer the Euclidean norm on $\mathbb{R}^{n^2}$ to $M(n, \mathbb{R})$ as follows: $\|A\|^2 := \|\varphi(A)\|^2 = \sum_{i,j=1}^n |a_{ij}|^2$.

    Show that the map $M(n, \mathbb{R}) \times M(n, \mathbb{R}) \to M(n, \mathbb{R})$ given by $\mu(X, Y) = XY$, the matrix product is continuous.

53. Let X and Y be normed linear spaces. A linear map $T: X \to Y$ is continuous at $0 \in X$ iff there exists a positive constant C such that $\|Tx\| \leq C \|x\|$ for all $x \in X$. *Hint:* Use ε-δ definition of continuity at 0.

    Deduce that a linear map between two normed linear space 's is continuous iff it is continuous at 0.

54. When do two norms $\| \ \|_j$, $j = 1, 2$ generate the same topology on a vector space X? They do iff the identity maps $I: (X, \| \ \|_1) \to (X, \| \ \|_2)$ and $I: (X, \| \ \|_2) \to (X, \| \ \|_1)$ are continuous. (Why?) By the last item, this means that we can find positive constants C_1 and C_2 such that $C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1$ for all $x \in X$. We thus arrive at the following result.

    Two norms $\| \ \|_j$, $j = 1, 2$ generate the same topology on a vector space X iff positive constants C_1 and C_2 such that $C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1$ for all $x \in X$. We then say that the two norms $\| \ \|_1$ and $\| \ \|_2$ are *equivalent.*

55. In $\mathbb{R}^n$, the three norms $\| \ \|_1$, $\| \ \|_2$ and $\| \ \|_\infty$ are equivalent. This follows from Item 49. It follows also from the observation:

$$\frac{1}{n} \|x\|_1 \leq \frac{1}{\sqrt{n}} \|x\|_2 \leq \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 .$$

    Later, we shall show that all norms on $\mathbb{R}^n$ induce the same topology, that is, they are all equivalent.

56. Closed Sets: Let $(X, \mathcal{T})$ be a topological space. A set $F \subset X$ is called a closed set (or said to be closed) in X if $X \setminus F$ is open in X. Let $\mathcal{C}$ be the class of all closed subsets in X. The following are more or less immediate:

    (a) $\emptyset, X \in \mathcal{C}$.

    (b) If $\{F_i : i \in I\}$ is a family of closed sets, then their intersection $\cap_{i \in I} F_i$ is again closed.

    (c) If F_1 and F_2 are closed, then so is $F_1 \cup F_2$.

57. Examples of Closed Sets:

    (a) $\emptyset$ and X are both open and closed in any topological space.

    (b) $\mathbb{Z}$ is closed in $\mathbb{R}$.

    (c) There exist sets which are neither open nor closed: $[0, 1)$, $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$ with usual topology,

    (d) Any finite subset of a metric space is closed.

    (e) Any closed ball $B[x, r]$ in a metric space is closed. Hence any closed interval $[a, b]$ is closed in $\mathbb{R}$.

    (f) Any sphere $S(x, r) := \{y \in X : d(x, y) = r\}$ in a metric space is closed.

    (g) The set $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ is closed in $\mathbb{R}$.

    (h) The set $(-\infty, 0) \cup [1, \infty)$ is closed in $\mathbb{R}$ with lower limit topology but not closed in $\mathbb{R}$ with the usual topology.

(i) The only subsets of $\mathbb{R}$ which are both open and closed are $\emptyset$ and $\mathbb{R}$.

Let A be both open and closed in $\mathbb{R}$. Assume that A is not empty. We need to prove $A = \mathbb{R}$. Let $a \in A$. Since a is open there exists $r > 0$ such that $(a - r, a + r) \subset A$. Consider

$$E := \{c \in \mathbb{R} : c > a, (a - \varepsilon, c) \subset A\}.$$

Then $a + \varepsilon \in E$. If $\sup E = \infty$, then it follows that $(a - \varepsilon, \infty) \subset A$. Assume $\sup E = \alpha \in \mathbb{R}$. Now either $\alpha \in A$ or $\alpha \notin A$.

If $\alpha \in A$, since A is open there exists $\delta > 0$ such that $(\alpha - \delta, \alpha + \delta) \subset A$. Since $\alpha - \delta < \alpha = \sup E$, there exists $c \in E$ such that $(a - \varepsilon, c) \subset A$. Clearly, $(a - \varepsilon, \alpha + \delta) = (a - \varepsilon, c) \cup (\alpha - \delta, \alpha + \delta) \subset A$. Hence $\alpha + (\delta/2) \in E$, contradiction to $\alpha = \sup E$.

If $\alpha \notin A$, then $\alpha \in \mathbb{R} \setminus A$, an open set. Hence there exists $\delta > 0$ such that $(\alpha - \delta, \alpha + \delta) \subset \mathbb{R} \setminus A$. As earlier, there exists $c \in E$ such that $\alpha - \delta < c$. Hence the interval $(\alpha - \delta, c)$ lies in both A and its complement, a contradiction. Thus we conclude that $\sup E = \infty$ so that $(a - \varepsilon, \infty) \subset A$. Similarly, we can conclude $(-\infty, a + \varepsilon) \subset A$ and hence $A = \mathbb{R}$.

(j) The set $[0, 1)$ is neither closed nor open in $\mathbb{R}$.

(k) Any subset of a discrete space is open as well as closed.

(l) Any subset $A \subset \mathbb{R}^*$ is closed in $\mathbb{R}$ with VIP topology with 0 as the VIP.

(m) What are the sets which are both open and closed in $\mathbb{R}$ with VIP topology with 0 as the VIP?

(n) Any subset of $\mathbb{R}$ containing 0 is closed in $\mathbb{R}$ with the outcast topology with 0 as the outcast.

(o) What are the sets which are both open and closed in $\mathbb{R}$ with the outcast topology with 0 as the outcast?

(p) Any vector subspace of $\mathbb{R}^n$ is closed. So are its translates.

Let V be a vector subspace of $\mathbb{R}^n$. Let $\mathbb{R}^n = V \oplus V^\perp$ be the orthogonal decomposition. Then $x \in \mathbb{R}^n$ lies in V iff $v \cdot u \equiv \langle x, u \rangle = 0$ for all $u \in V^\perp$. The map $f_u \colon \mathbb{R}^n \to \mathbb{R}$ given by $f_u(x) := x \cdot u$ is linear and hence by Item 49, it is continuous. Hence the kernel $f_u^{-1}(0)$ is a closed subset of $\mathbb{R}^n$. Since $V = \cap_{u \in V^\perp} f_u^{-1}(0)$ is the intersection of closed sets, V is closed.

(q) The set of $n \times n$ symmetric matrices and the set of $n \times n$ skew-symmetric matrices are closed in $M(n, \mathbb{R})$.

(r) The set $GL(n, \mathbb{R})$ is not closed in $M(n, \mathbb{R})$.

(s) The set of singular matrices in $M(n, \mathbb{R})$ is closed.

(t) The set $\{f \in C[0, 1] : f(1/2) = 0\}$ in $X := (C[0, 1], \| \ \|_\infty)$ is closed.

(u) The sets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are neither closed nor open in $\mathbb{R}$.

58. We have the following characterization of continuity in terms of closed sets.

Theorem 5. *Let $f \colon X \to Y$ be a map between topological spaces. Then f is continuous iff $f^{-1}(B)$ is closed in X for every closed set $B \subset Y$.* □

59. As we did earlier in the case of continuity and open sets, we may use the above theorem to assert that certain subsets are closed.

   (a) The set $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$, $\{(x, y) \in \mathbb{R}^2 : xy = 1\}$, $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ are closed in $\mathbb{R}^2$.
   (b) The closed rectangle $R := [a, b] \times [c, d]$ is closed in $\mathbb{R}^2$.
   (c) The unit n-dimensional sphere $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is closed in $\mathbb{R}^{n+1}$.
   (d) The set $SL(n, \mathbb{R})$ of matrices $A \in M(n, \mathbb{R})$ with determinant 1 is closed in $M(n, \mathbb{R})$.
   (e) The subset of matrices whose trace is 0 is closed in $M(n, \mathbb{R})$. (Also follows from Item 57p.)
   (f) The set $O(n)$ of orthogonal matrices is closed in $M(n, \mathbb{R})$. *Hint:* The maps $M(n, \mathbb{R}) \to \mathbb{R}$ given by $A \mapsto R_i(A) \cdot R_j(A) \equiv \sum_{k=1}^{n} a_{ik}a_{jk}$ are continuous. Here $R_i(A)$ denotes the i-th row of A.

      Or, use the fact that the map $F \colon A \mapsto (A, A^T)$ composed with $(A, B) \to AB$ is continuous. Then $O(n, \mathbb{R})$ is the inverse image $F^{-1}(I)$.
   (g) The set of singular matrices in $M(n, \mathbb{R})$ is closed.
   (h) The set of nilpotent matrices in $M(n, \mathbb{R})$ is closed.

60. Let A be a subset of a topological space. The characteristic function χ_A of A is defined by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

   What can you conclude about A if the function χ_A is continuous on X?

61. Go back to Item 21. If you understand the principle in work in it, you would have foreseen what follows. For any set A of a topological space $(X, \mathcal{T})$, the smallest closed set containing A exists. It is denoted by $\overline{A}$ and called the closure of A in X. (Compare this with the existence of the smallest topology containing a family $\{A_i : i \in I\}$ of subsets of a set X.) Note that $A \subset \overline{A}$.

62. Examples of closures:

   (a) The closure of $(a, b) \subset \mathbb{R}$ is $[a, b]$.
   (b) The closure of $\mathbb{Q}$ in $\mathbb{R}$ is $\mathbb{R}$.
   (c) The closure of an open ball $B(x, r)$ in $\mathbb{R}^n$ is the closed ball $B[x, r]$. In a general metric space, this need not be true. Consider $B(x, 1)$ and $B[x, 1]$ in a discrete metric space with at least two points.
   (d) Let $\mathbb{R}$ be given the VIP topology with 0 as the VIP. Then the closure of $A = \{0\}$ is $\mathbb{R}$. The closure of $\mathbb{R} \setminus \mathbb{Q}$ is itself. The closure of $\{a\}$ is itself if $a \neq 0$.
   (e) Investigate the case of $\mathbb{R}$ with outcast topology.

63. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then $x \notin \overline{A}$ iff there exists an open set $U \ni x$ with $U \cap A = \emptyset$. Hence, $x \in \overline{A}$ iff for every open set $U \ni x$, we have $U \cap A \neq \emptyset$. This suggests the following definition.

64. $x \in X$ is said to be a *limit point* of A if for every open set $U \ni x$, we have $U \cap A \neq \emptyset$.

    This is NOT the standard definition and hence should not be confused with the notion of cluster or an accumulation point which we shall see below. We shall follow our nomenclature only.

65. Consider the lower limit topology $\mathcal{T}_L$ on $\mathbb{R}$. Let $A = [a, b)$. Is b in the closure of A?

66. Consider $\mathbb{R}^2$ with order topology. Let $Q := \{(x, y) \in \mathbb{R}^2 : x > 0 \ \& \ y > 0\}$ be the first quadrant. What is $\overline{Q}$? Points of the other three quadrants are not in the closure. Any point $(a, 0)$ with $a > 0$ is in $\overline{Q}$ while $(0, 0)$ is not.

67. Every point of A is a limit point of A.

68. $x \in \overline{A}$ iff x is a limit point of A. (This is true because of our definition of a limit point. See Item 75.)

    For, let $x \in \overline{A}$ and $U \ni x$ be open. If $U \cap A = \emptyset$, then $A \subset X \setminus U$, a closed set and hence $\overline{A} \subset X \setminus U$. But $x \in \overline{A}$ and $x \notin X \setminus U$, a contradiction. Hence x is a limit point of A.

    Conversely, if x is a limit point of A and $x \notin \overline{A}$, then $x \in U := X \setminus \overline{A}$, an open set. But $U \cap A \subset U \cap \overline{A} = \emptyset$. Hence x is a not a limit point of A, a contradiction.

69. Let (X, d) be a metric space, $A \subset X$. Then $x \in X$ is a limit point of A iff there exists a sequence (a_n) in A such that $a_n \to x$.

70. With the notation as in the last item, $x \in \overline{A}$ or x is a limit point of A iff $d_A(x) = 0$.

71. In any normed linear space $(X, \| \ \|)$, the closure of an open ball $B(p, r)$ is $B[p, r]$. Thus, $q \in X$ is a limit point of $B(p, r)$ iff $d(p, q) \leq r$. In particular, $\overline{B(p, r)} = B[p, r]$.

    If $q \in B[p, r]$, consider the line segment $(1 - t)p + tq$, $0 \leq t \leq 1$. Draw picture. All points with $0 \leq t < 1$ are in $B(p, r)$. From this line segment, you can find a sequence $p_k \in B(p, r)$ which converges to q. Or, consider $B(q, \varepsilon)$ for $\varepsilon > 0$. Then for any $0 < t < 1$, we have

    $$d(q, (1 - t)p + tq) = \|(1 - t)(q - p)\| = (1 - t)r < \varepsilon,$$

    if t is near to 1. Thus, any open set containing q contains points of $B(p, r)$ other than q,

72. The set theoretic results about the closure operation:

    (a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.
    (b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
    (c) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. Strict containment can occur.
    (d) $\cup_{i \in I} \overline{A_i} \subset \overline{\cup_{i \in I} A}$. Strict containment can occur.

    (a) follows from the fact that any closed set that contains B will contain A. In particular, the smallest closed set $\overline{B}$ that contains B will contain A. Hence $\overline{A}$, the smallest closed set containing A will be contained in $\overline{B}$.

    (b). Since LHS is the smallest closed set containing $A \cup B$, and since $\overline{A}|cup\overline{B}$ is a closed set containing $A \cup B$, it follows that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. Let $x \in \overline{A} \cup \overline{B}$. Assume WLOG that $x \in \overline{B}$. Then for any open set $U \ni x$, we have $\emptyset \neq U \cap B \subset U \cap (A \cup B)$. That is, x is a limit point $A \cup B$ and hence $x \in \overline{A \cup B}$.

(c) Since $\overline{A} \cap \overline{B}$ is a closed set containing $A \cap B$, it follows that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. An instance of the strict containment, consider $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$.

(d) Consider $\mathbb{Q} = \cup_{x \in \mathbb{Q}} \{x\}$.

73. $x \in X$ is a *cluster* or an *accumulation point* of A iff for every open set $U \ni x$, the set $(U \setminus \{x\}) \cap A \neq \emptyset$, that is, any open set $U \in x$ contains a point of A other than x.

74. Intuitively, A accumulates or clusters around x. (They are like celebrities of A!) Obviously, any cluster point of A is a limit point of A, but not conversely. The notion of a cluster point is much stronger and more stringent than that of a limit point.

75. Let $(X, \mathcal{T})$ be any topological space and $A \subset X$. Then $\overline{A}$ is the union of A and the cluster points of A. (Compare and contrast this with Item 68.)

76. Every point of $A = \mathbb{Z} \subset \mathbb{R}$ is a limit point of A but there exists no cluster point of A in $\mathbb{R}$.

    Since $\overline{\mathbb{Z}} = \mathbb{Z}$, this examples also shows that 'limit point' cannot be replaced by 'cluster point' in Item 68.

77. Consider $\mathbb{R}$ with VIP topology with 0 as the VIP. Then any nonzero real number is a cluster point of $A = \{0\}$. Zero is obviously a limit point of A but not a cluster point of A.

78. The last example also shows that the following can occur. x may be a cluster point of A, but there may exist open sets $U \ni x$ with $U \cap A$ is finite!

79. Any point in any ball (open or closed) in an normed linear space is a cluster point of the ball. The idea in Item 71 proves this.

80. Analyze the situation in a metric space. In a metric space, if x is a cluster point of A, then every open set $U \ni x$ will contain infinitely many points of A. The proof suggested the following definition.

81. A topological space X is said to be *Hausdorff* iff for every pair $x, y \in X$ of distinct points, there exist open set U, V such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. That is, any two distinct points can be "separated by open sets."

    We also say that a topology $\mathcal{T}$ on a set X is Hausdorff if the space $(X, \mathcal{T})$ is Hausdorff.

82. Let $(X, \mathcal{T})$ be a Hausdorff (topological) space and $A \subset X$. Then $x \in X$ is a cluster point of A iff for every open set $U \ni x$, the set $U \cap A$ is infinite.

    We prove this by contradiction. Let x be a cluster point of A and assume that there exists an open set U such that $x \in U$ and $U \cap A$ is finite. Let $(U \setminus \{x\}) \cap A = \{a_1, \dots, a_n\}$. Since X is Hausdorff, for each j, $1 \leq j \leq n$, the exists an open set $U_j \ni x$ and $V_j \ni a_j$ such that $U_j \cap V_j = \emptyset$, $1 \leq j \leq n$. Then $U = \cap_{j=1}^{n} U_j$ is an open set such that $U \cap A$ is at most $\{x\}$.

83. A finite set in a Hausdorff space cannot have a cluster point. (Hausdorff condition is required. Look at $\mathbb{R}$ with VIP topology with zero as the VIP and $A = \{0\}$.) If a subset A of a Hausdorff space X has a cluster point, then A is infinite.

    But there exists an infinite set in a Hausdorff space which has no cluster point. Look at $\mathbb{Z}$ in $\mathbb{R}$.

84. Let us now look at some examples of Hausdorff spaces.

    (a) Any metric space is Hausdorff. For if $x_1, x_2 \in (X, d)$ are distinct, then $d(x_1, x_2) > 0$. Let $r = d(x_1, x_2)/2$. Then $B(x_j, r)$ is an open set containing x_j and $B(x_1, r) \cap B(x_2, r) = \emptyset$. For, x is a common point, then

    $$d(x_1, x_2) \le d(x_1, x) + d(x, x_2) < r + r < d(x_1, x_2),$$

    a contradiction. In particular, $\mathbb{R}^n$ with the standard metric and normed linear spaces are Hausdorff.

    (b) Any discrete topology is Hausdorff.

    (c) The indiscrete topology on a set X with at least two elements is not Hausdorff.

    (d) $(\mathbb{R}, \mathcal{T}_V)$ with 0 VIP is not Hausdorff.

    (e) If we have $\mathcal{T}_1 \le \mathcal{T}_2$ and $\mathcal{T}_1$ is Hausdorff, so is $\mathcal{T}_2$. As special cases, we have the following.

        i. The order topology on $\mathbb{R}^2$ with dictionary order is Hausdorff.

        ii. The lower limit topology on $\mathbb{R}$ is Hausdorff.

    (f) Let $f\colon X \to Y$ be a 1-1 continuous function. If Y is Hausdorff, so is X. Let x_1, x_2 be distinct elements of X. Then $f(x_1)$ and $f(x_2)$ are distinct elements of Y and hence there exist disjoint open sets $V_j \ni f(x_j)$. Consider $U_j := f^{-1}(V_j)$, $j = 1, 2$.

85. We now give an example of a Hausdorff space in which two disjoint closed sets cannot be separated by open sets.

    Let $X = \mathbb{R}$. For any fixed $p \in \mathbb{R}$ and $m \in \mathbb{N}$, let $B_{p,m} := \{p + km : k \in \mathbb{Z}_+\}$. Let $\mathcal{T}$ be the set of all subsets $U \subset \mathbb{R}$ such that for any $p \in U$, there exists $m \in \mathbb{N}$ such that $B_{p,m} \subset U$. Then it is easy to check that $\mathcal{T}$ is a topology on $\mathbb{R}$.

    We claim that it is Hausdorff. Consider $p \ne q$ in $\mathbb{R}$. If $p - q$ is not an integer, then $B_{p,m} \cap B_{q,m} = \emptyset$ for any $m \in \mathbb{N}$. For, otherwise, if z is a common element, then $z = p + km = q + kn$. It follows that $p - q = k(n - m)$, an integer — a contradiction.

    If $p - q = m \in \mathbb{Z}$,say, then the basic open sets $B_{p,2m}$ abd $B_{q,2m}$ separate p and q. (Verify this.)

    Fix $p \in \mathbb{R}$ and $m \in \mathbb{N}$. We claim that each element of $\{p - km : k \in \mathbb{N}\}$ is a cluster point of $B_{p,m}$. Let $q = p - km$. Consider a basic open set $B_{q,n} \ni q$. The element $q + mkn \in B_{q,n}$. Since

    $$q + kmn = p - km + kmn = p + mk(n - 1) \in B_{p,m},$$

    the claim follows.

Consider now the two disjoint sets $F_1 := \{1\}$ and $F_2 := \{x \in \mathbb{R} : x \leq 0\}$. F_1 is closed since the space is Hausdorff. F_2 is also closed, since its complement is open. For, note that for any $p > 0$ and $m \in \mathbb{N}$, $U_{p,m} \subset (0, \infty)$.

We claim that they cannot be separated by open sets. Assume the contrary. Let $U_1 \supset F_1$ and $U_2 \supset F_2$ be open sets separating them. Then there exists a basic open set $B_{1,m} \subset U_1$. Now $1 - 2m$ is a cluster point of $B_{1,m}$. But no point of F_2 can be cluster point of U_1 since $F_2 \subset U_2$ and $U_2 \cap F_1 \subset U_2 \cap U_1 = \emptyset$.

Thus we have an example of a Hausdorff space in which two distinct points can be separated by open sets but not any two disjoint closed sets.

86. This examples is from Munkres. Consider $\mathbb{R}$ with the topology $\mathcal{T}_K$ whose basic open sets are open intervals (a, b) and open intervals $(a, b) \setminus K$ where $K := \{1/n : n \in \mathbb{N}\}$. Then $\{0\}$ and K are disjoint closed subsets which cannot be separated by open sets.

87. We say that a sequence (x_n) in a topological space $(X, \mathcal{T})$ *converges* to a point $x \in X$, if for every open set $U \ni x$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. The point x is called the limit of the sequence and (x_n) is said to be *convergent*.

88. If $(X, \mathcal{T})$ is a Hausdorff (topological) space, then any convergent sequence has a unique limit.

This need not be true in a general space. For instance, if we consider $\mathbb{R}$ with indiscrete topology, any sequence is convergent to any point of $\mathbb{R}$!

89. Consider the sequence $(1/n)$ in $\mathbb{R}$ with co-finite topology. Then $1/n \to x$, for any $x \in \mathbb{R}$! (Co-finite topology on $\mathbb{R}$ is not the discrete topology.)

90. In any Hausdorff space, any finite set is closed. Details!

This need not be true in an arbitrary topological space. For instance, consider the indiscrete topology on $\mathbb{R}$. Or, the set $\{x, 0\}$, $x \neq 0$, in $\mathbb{R}$ with VIP topology with VIP=0.

Hence conclude: The topology of any finite Hausdorff is discrete. (See also Item 12l.)

91. Examples of Convergent sequences:

(a) The only convergent sequences in any discrete space are eventually constant sequences.

For, let $x_n \to x$. Then $\{x\} \ni x$ is open so there exists $N \in \mathbb{N}$ such that $\geq N \implies x_k \in \{x\}$. Thus $x_k = x$ for $k \geq N$.

(b) In the normed linear space $(B(X, \mathbb{R}), \| \, \|_\infty)$, a sequence (f_n) converges to $f \in B(X, \mathbb{R})$ iff f_n converges to f uniformly on X.

Assume that $f_n \to f$ uniformly on X. Let $\varepsilon > 0$ be given. Choose N such that for all $k \geq N$, and $x \in X$, we have $|f(x) - f_k(x)| < \varepsilon/2$. Hence $\sup_{x \in X} |f(x) - f_k(x)| \leq \varepsilon/2 < \varepsilon$. That is, $\| f_k - f \|_\infty < \varepsilon$ for $k \geq N$ and hence f_k converge to f in the norm. Other way implication is easier.

(c) A sequence (x_k) in $\mathbb{R}^n$ converges to $x \in \mathbb{R}^n$ iff $x_{kj} \to x_j$ as $k \to \infty$ for $1 \leq j \leq n$.

92. We analyzed the proof of Item 69 and arrived at the following conclusion:

Let $(X, \mathcal{T})$ be a space with the following property: For every $x \in X$, there exists a countable collection of open sets $\{U_{n,x} : n \in \mathbb{N}\}$ such that

(a) For every open set $U \ni x$, there exists n such that $x \in U_{n,x} \subset U$

(b) $\cap_n U_{n,x} = \{x\}$.

Then, $x \in X$ is a limit point of $A \subset X$ iff there exists a sequence (a_n) in A such that $a_n \to x$.

93. The foregoing item led us to the following concepts.

94. Let $(X, \mathcal{T})$ be a topological space and $p \in X$. Then by a *local base* at p, we mean a family $\{U_{p,i} : i \in I\}$ of open sets containing p with the property that if U is an open set containing p, then there exists $i \in I$ such that $x \in U_{p,i} \subset U$.

    A typical example to keep in mind: $\{B(p, r) : r > 0\}$ is a local base at p in a metric space X.

95. A space is said to be *first countable* if there exists a countable local base at every point $p \in X$.

96. Observe that if $(X, \mathcal{T})$ is first countable, then we may assume that a local base $\{U_{p,n} : n \in \mathbb{N}\}$ at p is decreasing sequence. For, if $\{V_{p,n}\}$ is a local base at p, consider $U_{p,n} := V_{p,1} \cap \cdots \cap V_{p,n}$.

97. We look at some examples:

    (a) In $\mathbb{R}$ with standard topology, $\{(p - \frac{1}{n}, p + \frac{1}{n}) : n \in \mathbb{N}\}$ is a local base at p. Hence $\mathbb{R}$ is first countable. More generally, $\{B(p, 1/n) : n \in \mathbb{N}\}$ is a local base at p in any metric space. Hence any metric is first countable.

    (b) If $\mathbb{R}$ is endowed with the discrete topology, then a local base at x can be taken as $\{x\}$. Hence $\mathbb{R}$ with discrete topology is first countable.

    (c) Consider $\mathbb{R}$ with VIP topology. (Convention: VIP is always 0.) Then the set $\{p, 0\}$ is a local base at any $p \in \mathbb{R}$. (If $p = 0$, then the set $\{p, 0\} = \{0\}$!) Hence $\mathbb{R}$ with VIP topology is first countable.

    (d) Any indiscrete topology is first countable.

98. Let $(X, \mathcal{T})$ be a Hausdorff, first countable space. Let $\{U_{p,n} : n \in \mathbb{N}\}$ be a countable local base. Then $\cap_n U_{n,p} = \{p\}$. (We do not need the full power of Hausdorff condition. We could have achieved the same result with less stringent hypothesis, but we shall not worry about this!)

99. In view of Item 92 and Item 98, we have the following.

    Theorem 6. *Let $(X, \mathcal{T})$ be first countable and Hausdorff. Then x is a limit point of A iff there exists a sequence (a_n) in such that $a_n \to x$.*

100. We say that a topological space $(X, \mathcal{T})$ is *second countable* if there exists a countable basis for $\mathcal{T}$.

101. Clearly, any second countable space is first countable.

102. Examples and non-examples:

(a) $\mathbb{R}$ with the standard topology is second countable. (See Item 17c.)

(b) A discrete space X is second countable iff the set X is countable.

(c) $\mathbb{R}$ with VIP topology is first countable but not second countable. Why? Consider the basis $\{\{x, 0\} : x \in \mathbb{R}\}$. If $\{B_n : n \in \mathbb{N}\}$ is a countable basis, then for $x \in \{x, 0\}$ there will be $n(x) \in \mathbb{N}$ such that $x \in B_{n(x)} \subset \{x, 0\}$. Since $B_{n(x)}$ will always contain $\{0, x\}$, it follows that $B_{n(x)} = \{0, x\}$. But the family $\{\{x, 0\} : x \in \mathbb{R}\}$ is uncountable where as $\{B_n : n \in \mathbb{N}\}$ is countable.

(d) The outcast topology on $\mathbb{R}$ is first countable but not second countable.

(e) Any indiscrete space is second countable.

103. Think over this: What will be the counter part (in terms of open sets) of the smallest closed set containing A? It is the largest open set contained in A. It is called the *interior* of A and is denoted by $\mathrm{Int}\,(A)$.

104. Examples of interior of a set:

(a) The interior of an open set is itself.

(b) The interior of $[a, b] \subset \mathbb{R}$ is (a, b).

(c) The interior of $\mathbb{Q} \subset \mathbb{R}$ is the empty set. What is $\mathrm{Int}\,(\mathbb{R} \setminus \mathbb{Q})$?

(d) The interior of a proper vector subspace of $\mathbb{R}^n$ is empty. Does this generalize to any normed linear space ?

(e) The interior of a closed ball $B[p, r]$ in any normed linear space is the open ball $B(p, r)$. In a general metric space, such a result is not true.

(f) Let $(X, \mathcal{T})$ be a discrete space. Then $\mathrm{Int}\,(A) = A$ for any $A \subset X$.

(g) Let $(X, \mathcal{T})$ be an indiscrete space. Then $\mathrm{Int}\,(A) = \emptyset$ for any $A \subset X$, $A \neq X$.

(h) Consider $\mathbb{R}$ with the VIP topology (VIP is 0). The interior of $\mathbb{R}^*$ is the empty set. What is $\mathrm{Int}\,(\mathbb{Q})$ and $\mathrm{Int}\,(\mathbb{R} \setminus \mathbb{Q})$ in this topology? More generally, if $0 \in A$, then $\mathrm{Int}\,(A) = A$ and if $0 \notin A$, then $\mathrm{Int}\,(A) = \emptyset$.

(i) Consider $\mathbb{R}$ with the outcast topology (outcast is 0). The interior of any set A is $A \setminus \{0\}$.

105. A is open iff $A = \mathrm{Int}\,(A)$.

106. Set theoretic results about the interior operation:

(a) If $A \subset B$, then $\mathrm{Int}\,(A) \subset \mathrm{Int}\,(B)$.

(b) $\mathrm{Int}\,(A) \cup \mathrm{Int}\,(B) \subset \mathrm{Int}\,(A \cup B)$.

(c) $\mathrm{Int}\,(A \cap B) = \mathrm{Int}\,(A) \cap \mathrm{Int}\,(B)$.

(d) $\cup_{i \in I} \mathrm{Int}\,(A_i) \subset \mathrm{Int}\,(\cup_{i \in I} A_i)$.

107. Let X be a (metric) space and $A \subset X$. A point $x \in X$ is said to be a *boundary point* of A in X if every open set that contains x intersects both A and $X \setminus A$ non-trivially. The *boundary* of A in X is the set of boundary points of A in X. We denote it by ∂A.

108. Find the boundaries of each of the following sets:

(a) $A_1 = (a, b] \subset \mathbb{R}$ with the standard topology.

(b) $A_2 = \mathbb{R} \setminus \{0\} \subset \mathbb{R}$ with the standard topology.

(c) $A = \mathbb{Q} \subset \mathbb{R}$ with the standard topology.

(d) $\partial\emptyset = \emptyset = \partial X$ for any topological space X.

(e) The boundary of an open or closed ball in $\mathbb{R}^n$ is the sphere: $\partial B(x,r) = \partial B[x,r] = S(x,r) := \{y \in \mathbb{R}^n : d(x,y) = r\}$. Is this true in an normed linear space ? in an arbitrary metric space?

(f) In $\mathbb{R}$ with VIP topology and $\mathbb{R}$ with outcast topology, find ∂A, where $A = \{0\}, \{x\}, \mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$. ($x$ is a nonzero real number.)

(g) Let B be an open ball in $\mathbb{R}^n$. Find the boundary of B minus a finite number of points.

(h) Let $A := \{z \in \mathbb{C} : z = re^{it}, r \in [0,1], t \in (0, 2\pi)\}$. (Draw a picture.) Find the boundary of A.

109. A few more examples to sharpen our geometric intuition.

(a) Consider $A = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$. What is the boundary of A in $\mathbb{R}^2$?

(b) $A = U_1 \cup U_2 \cup U_3$ is the subset of $\mathbb{R}^2$ where $U_1 := \{x^2 + y^2 < 1, y > 0\}$, $U_2 := \{-1 \leq x \leq 1, y = 0\}$ and $U_3 := \{x^2 + y^2 = 1, y < 0\}$.

(c) $A = \{(x,y) : x^2 + y^2 = 1\}$.

110. Show that for any subset A of a topological space $(X, \mathcal{T})$, $\partial A = \overline{A} \cap \overline{X \setminus A}$. (This is the standard definition.)

111. While trying to prove the equivalence of the definition of continuity at a point (of a function between two metric spaces) with the sequential definition, we established the following.

Theorem 7. *Let X and Y be arbitrary topological spaces and $p \in X$. Let $f \colon X \to Y$ be a map.*

1. If f is continuous at p, then for every sequence (x_n) in X converging to p, we have $f(x_n) \to f(p)$.

2. Assume that X is first countable and Hausdorff. Assume further that f has the property that for every sequence (x_n) converging to p, the sequence $(f(x_n))$ converges to $f(p)$ in Y. Then f is continuous at p.

Proof. 1. Let $V \subset Y$ be open with $f(p) \in V$. By continuity of f at p, there exists an open set $U \ni p$ such that for $x \in U$, we have $f(x) \in V$. Since $x_n \to p$, for this U, there exists $N \in \mathbb{N}$ such that $k \geq N \implies x_k \in U$. Hence for $k \geq N$, we see that $f(x_k) \in V$, that is, $f(x_k) \to f(x)$.

2. We prove this by contradiction. Assume that (B_n) is a local base at p such that $B_{n+1} \subset B_n$ and $\cap_n B_n = \{p\}$. Since f is not continuous at p, there exists an open set $V \ni f(p)$ such that given any open set $U \ni p$, there exists $x \in U$ such that $f(x) \notin V$. In particular, for each $n \in \mathbb{N}$, there exists $x_n \in B_n$ such that $f(x_n) \notin V$. Clearly,

$x_n \to p$. For, let $W \ni p$ be an open set. Then there exists N such that $p \in B_N \subset W$. If $k \geq N$, then $x_k \in B_k \subset B_N \subset W$. Thus, we conclude $x_n \to p$. Now by hypothesis, $f(x_n) \to f(p)$. If we apply the definition of convergence to the set V, we find that $f(x_n) \notin V$ for any n. □

112. A subset $D \subset X$ of a topological space is dense in X if for every *nonempty* open set $U \subset X$, we have $D \cap U \neq \emptyset$, that is U intersects D non-trivially.

113. Examples of dense sets:

    (a) $\mathbb{Q}$ is dense in $\mathbb{R}$. Is $\mathbb{R} \setminus \mathbb{Q}$ dense in $\mathbb{R}$? Can you think of a countable dense subset in $\mathbb{R}^2$? in $\mathbb{R}^n$?

    (b) In $\mathbb{R}$, with the lower limit topology, the sets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense.

    (c) The set $A := \{x \in \ell_1 : x_n = 0 \text{ for all } n \geq N \text{ for some } N\}$ is dense in ℓ_1.

    (d) The set D_n of all sequences $x = (x_m) \in \ell_1$ whose terms are rational and $x_k = 0$ for $k > n$. Let $D := \cup_{n \in \mathbb{N}} D_n$. Then D is a countable dense subset of ℓ_1.

    (e) The only dense subset of a discrete space X is X itself.

    (f) In an indiscrete space, any nonempty subset is dense.

    (g) The set $\{0\}$ is dense in $\mathbb{R}$ with the VIP topology. The set $\mathbb{R} \setminus \mathbb{Q}$ is not dense.

    (h) The set $\mathbb{R} \setminus \{0\}$ is dense in $\mathbb{R}$ with the outcast topology. This space cannot have a countable dense set.

    (i) $S := \{n + m\sqrt{2} : n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. (Did you notice that $\mathbb{Z}$ and $\sqrt{2}\mathbb{Z}$ are closed and S is a sum of two closed sets? If the result is true, then S cannot be closed in $\mathbb{R}$. Why? If S is closed and dense, then $S = \mathbb{R}$, but S is countable! Hence we have an example of two closed sets in $\mathbb{R}$ whose sum is not closed.) See Lemma 2.5.7/Page 52 of my book on Metric spaces..

    (j) Is $\mathbb{Q}^2$ dense in $\mathbb{R}^2$ with the order topology?

    (k) Weierstrass approximation theorem says that the vector subspace of polynomials in the normed linear space $(C[0,1], \| \|_\infty)$ is dense. (This should be a topic for Student Seminar!)

    (l) A *dyadic rational* is a real number of the form $m/2^n$ where m is an integer and $n \in \mathbb{N}$. Let D denote the set of dyadic rationals. Then D is dense in $\mathbb{R}$. Consider an open interval of the form $(a - \varepsilon, a + \varepsilon)$. Choose n so that $1/2^n < \varepsilon$. If there is no dyadic rational in this interval, then there exists an odd integer m such that $m/2^n < a - \varepsilon$ and $(m + 2)/2^n > a + \varepsilon$. (Why?) But then

    $$2/2^n = 2^{-n}((m+2) - m) > a + \varepsilon - (a - \varepsilon) = 2\varepsilon, \text{ a contradiction.}$$

114. $D \subset X$ is dense in a space (X, d) iff every point of X is a limit point of D.

    Let D be dense in X. Let $x \in X$ and $U \ni x$ be open. Then $U \cap D \neq \emptyset$. Thus x is a limit point of D.

    Conversely, if every $x \in X$ is a limit point of D, we claim that D is dense in X. For, if not, there exists a nonempty open set U such that $U \cap D = \emptyset$. Since U is nonempty, choose $x \in U$. Then x is not a limit point of D as $U \ni x$ is open but $U \cap D = \emptyset$.

115. $D \subset X$ is dense in the space X iff its closure $\overline{D} = X$. (This is the standard definition.)

Recall (from Item 68) that the closure of any set A is the set of limit points of A. The result now follows from the last item.

116. In a metric space (X, d), a set A is dense in X iff for every $x \in X$ and $\varepsilon > 0$, there exists an $a \in A$ such that $d(x, a) < \varepsilon$. (Thus, A is dense in X, if we can "approximate" any point $x \in X$ to "any level of approximation" by an element of A. See Item 113k to understand this vague remark. Also recall that $\mathbb{Q}$ is dense in $\mathbb{R}$, which means that any real number can be approximated to any level of accuracy by a rational number.)

117. Let (X, d) be a metric space. Assume that the only dense subset is X itself. Can we say something about the topology? *Hint:* What are the maximal proper subsets of X?

118. Let A, B be two dense subsets of a space X. Is $A \cup B$ dense? Is $A \cap B$ dense?

119. If A, B are open dense subsets of a space X, is $A \cap B$ dense in X?

120. Give an example of a proper open dense subset of $\mathbb{R}$.

121. Continuation of the last item. If we write an open set $U = \dot{\cup} J_k$, as the disjoint union of open intervals (Item 12s), then we say that the "measure" or "length" of U is $\sum_k \ell(J_k)$, the sum of lengths of the intervals J_k. Given $\varepsilon > 0$, can you find an open dense subset of $\mathbb{R}$ whose length is less than or equal to ε?

122. Let D be dense in $(X, \mathcal{T}_1)$. Is D (necessarily) dense in $(X, \mathcal{T}_2)$ where $\mathcal{T}_2$ is finer (respectively, coarser) than $\mathcal{T}_1$?

123. Let X, Y be topological spaces. Assume that A is dense in X and $f\colon X \to Y$ is continuous and onto. Then $f(A)$ is dense in Y.

124. The set of matrices in $M(n, \mathbb{C})$ with distinct eigenvalues is dense. In particular, the set of all diagonalizable matrices in $M(n, \mathbb{C})$ is dense. This exercise requires a good background in Linear Algebra.

It is well-known fact in linear algebra that any $A \in M(n, \mathbb{C})$ can be brought to upper triangular form, say T, via conjugation by a unitary matrix U such that $T = UAU^{-1}$. The eigenvalues are the diagonal entries, say, d_j. We can find very small ε_j's so that $d_j + \varepsilon_j$'s are all distinct. We thus get a new upper triangular matrix, say T_1 whose entries are the same as that of T except d_j is replaced by $d_j + \varepsilon_j$. Again it is well known that T_1 is diagonalizable. The matrix $A_1 := UT_1U^{-1}$ has distinct eigenvalues and hence is diagonalizable. It is close to A if ε_j's are small. This follows from the observation $\|A\|$ on $M(n, \mathbb{C})$ comes from the inner product $(X, Y) \mapsto \operatorname{Tr} XY^*$ and the inner product is invariant under conjugation by unitary matrices. The reader is encouraged to work out the details and submit it as an assignment to the instructor. Details!

125. A topological space is *separable* if there exists a countable dense subset.

126. Examples and non-examples of separable spaces:

    (a) $\mathbb{R}^n$ is separable. Consider $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_j \in \mathbb{Q}\}$.

    (b) A discrete space X is separable iff X is countable.

(c) ℓ_1 is separable.

(d) $\mathbb{R}$ with VIP topology is separable.

(e) $\mathbb{R}$ with outcast topology is not separable.

(f) Any second countable space is separable. If $\{B_n\}$ is any countable basis, choose one element, say, $x_n \in B_n$. Then $D := \{x_n\}$ is a countable dense set.

(g) Let X be infinite with co-finite topology and let A be any infinite subset of X. Then any $x \in X$ is a limit point of A. In particular, X with co-finite topology is separable.

(h) Is $\mathbb{R}^2$ with the order topology separable? (Recall the geometric description of basic open sets in this space. See Item 19.)

127. Let X be uncountable with co-finite topology. Then X is not first countable but separable by Item 126g.

128. Let X be uncountable with co-countable topology. No countable set can have a cluster limit point and hence X is not separable.

129. Let ℓ_∞ denote the set of all bounded real sequences. It is a normed linear space with respect to the norm $\|x\|_\infty := \sup\{|x_n| : n \in \mathbb{N}\}$. The space $(\ell_\infty, \| \ \|_\infty)$ is not separable. *Hint:* Consider the uncountable subset $\{x \colon \mathbb{N} \to \{0,1\}\}$ of ℓ_∞.

130. A metric space is separable iff it is second countable.

131. $\mathbb{R}_\ell$, the space $\mathbb{R}$ with lower limit topology, is first countable but not second countable.

132. Let $f, g \colon X \to Y$ be continuous and Y be Hausdorff. Then the set $A := \{x \in X : f(x) = g(x)\}$ is closed in X.

We show that $B := X \setminus A$ is open. Let $b \in B$. Then $f(b) \neq g(b)$ and hence there exist open sets $V_1 \ni f(b)$ and $V_2 \ni g(b)$ with $V_1 \cap V_2 = \emptyset$. By continuity of f and g at b, there exist open sets $U_1 \ni b$ and $U_2 \ni b$ such that $f(U_1) \subset V_1$ and $g(U_2) \subset V_2$. Then $U_b := U_1 \cap U_2$ is an open set containing b and we have for $x \in U_b$, $f(x) \in V_1$ and $g(x) \in V_2$. Hence $f(x) \neq g(x)$ for $x \in U_b$. That is, $U_b \subset B$. Hence $B = \cup_{b \in B} U_b$ is open.

133. Let the hypothesis be as in the last item. Assume that D is dense in X and that $f(x) = g(x)$ for all $x \in D$. Then $f(x) = g(x)$ for all $x \in X$.

Let the notation be as in the last item. Then A is a closed set containing D. Since D is dense $\overline{D} = X \subset A$. That is, $A = X$.

134. Let X, Y be sets. Suppose $f \colon X \to Y$ is a bijection. Assume further that one of the sets has an extra mathematical structure such as a group, vector space, metric or a topology. Then we can transfer the structure to the other set using the bijection. We look at some specific instances.

(a) Let X be a group. Then we define $y_1 \cdot y_2$ to be $f(x_1 \cdot x_2)$ where $f(x_i) = y_i$, $i = 1, 2$. It turns out that Y is group and that $f \colon X \to Y$, by virtue of the very definition of group law on Y, is a group homomorphism (and hence an isomorphism.)

(b) Let Y be a metric space. Then we set $d(x_1, x_2) := d(f(x_1), f(x_2))$. Then the metric space (X, d) is isometric to (Y, d).

(c) Let X be a topological space. Let $\mathcal{T}_X$ be the topology on X. We then define a topology $\mathcal{T}_Y$ on Y be declaring that $V \in \mathcal{T}_Y$ iff there exists $U \in \mathcal{T}_X$ such that $V = f(U)$. Then the map $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a homeomorphism, a term not yet defined!

135. To illustrate this principle, we use the bijection $t \mapsto e^t$ from $X := \mathbb{R}$ to $Y := (0, \infty)$ to make Y into a vector space over $\mathbb{R}$. Given $y_1, y_2 \in Y$, we look at their (unique) pre-images $x_j = \log y_j$, carry out the vector addition in X, obtain $x_1 + x_2 = \log y_1 + \log y_2 = \log(y_1 y_2)$ and map it by the bijection. The result is $y_1 y_2$. Similarly, the scalar multiple of y by $\alpha \in \mathbb{R}$ is $\alpha \log y \mapsto e^{\alpha \log y} = y^\alpha$. Thus the vector addition of y_1 and y_2 is $y_1 y_2$ and the scalar multiple $\alpha \cdot y$ is y^α. The 'additive identity' is 1. Note that the map $t \mapsto e^t$ is a linear isomorphism.

136. A map $f\colon X \to Y$ between two topological spaces is a *homeomorphism* if (i) f is bijective, (ii) f is continuous and (iii) $f^{-1}\colon Y \to X$ is continuous.

This is the analogue of isomorphisms in Algebra. Note that there also one requires a bijective map f such that f and its inverse f^{-1} preserve the 'algebraic structures' such as group, ring, vector space structures. It turns out in the context of algebra, if f preserves the structure, then f^{-1} does automatically.

We say that two topological spaces X and Y are homeomorphic if there exists a homeomorphism $f\colon X \to Y$.

137. The relation of being homeomorphic is an equivalence relation among topological spaces.

138. Examples of homeomorphisms.

(a) Any $f\colon \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + b$ for a nonzero $a \in \mathbb{R}$ is a homeomorphism.

(b) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a homeomorphism.

(c) Any linear isomorphism of $\mathbb{R}^n$ is a homeomorphism.

More generally, any linear isomorphism from $(\mathbb{R}^n, \|\ \|)$ to $(\mathbb{R}^n, \|\ \|')$, where $\|\ \|$ and $\|\ \|'$ are any of the norms $\|\ \|_1$, $\|\ \|_2$ and $\|\ \|_{\max}$, is a homeomorphism.

In particular, the identity map is a homeomorphism. As a corollary, we conclude that the topologies induced by these norms are the same:

$$\mathcal{T}_{\|\ \|_1} = \mathcal{T}_{\|\ \|_2} = \mathcal{T}_{\|\ \|_{\max}}.$$

(d) Let us now look at some homeomorphisms of a normed linear space. Let $(X, \|\ \|)$ be a normed linear space. Then the maps (a) $x \mapsto \lambda x$ for $0 \neq \lambda \in \mathbb{R}$, (b) $x \mapsto x + v$, where $v \in X$ is fixed are homeomorphisms.

(e) Consider $M(n, \mathbb{R})$. Then the maps (a) $X \mapsto X^t$, (b) $X \mapsto X + A$ for fixed $A \in M(n, \mathbb{R})$ and (c) $X \mapsto AX$ for a fixed nonsingular matrix A are homeomorphisms.

(f) Any two discrete spaces are homeomorphic iff they have the same cardinality.

(g) If two metric spaces are isometric, then they are homeomorphic.

(h) In the examples of this item, the subsets are given the metric topology from the induced metric.

i. $[a, b] \simeq [0, 1]$. More generally, $[a, b] \simeq [c, d]$.

ii. $(-1, 1) \simeq \mathbb{R}$.
iii. $(0, 1] \simeq [1, \infty)$.
iv. $[0, 1) \simeq (0, 1]$.
v. Can $\mathbb{Q}$ be homeomorphic to $\mathbb{Z}$?
vi. Is $\mathbb{N} \simeq \mathbb{Z}$?

(i) A bijective continuous map need not be a homeomorphism. Examples and a non-example:

i. $\mathbb{R}$ with discrete topology and $\mathbb{R}$ with indiscrete topology.
ii. $f : [0, 2\pi) \to S^1 \subset \mathbb{C}$ given by $f(t) = e^{it}$. (A more instructive exercise.)
iii. Any bijective continuous map of a finite topological space X to itself is a homeomorphism.

(j) The spaces $(\mathbb{R}, \mathrm{VIP})$ and $(\mathbb{R}, \mathrm{Outcast})$ are not homeomorphic.
We shall see later a lot of examples of homeomorphisms.

139. **Open and closed maps.** A map $f : X \to Y$ is said to be *open* if $f(U)$ is open in Y for every U open in X. A closed map is defined similarly.

(a) A bijective continuous map is a homeomorphism iff it is an open map.
Application: The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a homeomorphism.

(b) A bijective continuous map is a homeomorphism iff it is an closed map.
Application: Item 206b. We have to wait for this.

140. We say that property of a topological space is a *topological property* if every space Y homeomorphic to X also has the property. Examples:

(a) The space being Hausdorff is a topological property.

(b) The space being first countable is a topological property.

(c) The space being second countable is a topological property.

(d) The space being separable is a topological property.

(e) Existence of a nonempty, proper subset which is both open and closed is a topological property.

(f) Let us say that a topological space X has BCP if every continuous real valued function is bounded. For example all closed and bounded intervals have this property. Is BCP a topological property? *Hint:* If $\varphi : X \to Y$ is a homeomorphism there is a "natural map" $\varphi^* : C(Y, \mathbb{R}) \to C(X, R)$ where $C(X, \mathbb{R})$ stands for the set of real valued continuous functions on X etc.
The "adjoint" map φ^* is defined by $\varphi^*(g) := g \circ \varphi$. If φ is a bijection, then φ^* is also a bijection. If φ is a homeomorphism, then $\varphi^*(g) \in C(X, \mathbb{R})$ for any $g \in C(Y, \mathbb{R})$.

(g) Two metric spaces can be homeomorphic, but one of them could be bounded while the other is not. Hence 'being bounded' is not a topological property among metric spaces.

(h) Similarly, completeness is not a topological property among the metric spaces.

We shall see later a lot of examples of topological properties.

> The study of topology is mainly understanding topological properties and using them to assert whether given two spaces are homeomorphic or not.

141. We now look at some natural questions which lead us to the generation of new topologies.

142. Given a set X and a collection $\mathcal{S}$ of subsets of X, how to we describe the open sets in the smallest topology, say, $\mathcal{T}_{\mathcal{S}}$ that contains $\mathcal{S}$? (We assume, as this is the case that occurs in practice, that for every $x \in X$, there exists $S \in \mathcal{S}$ such that $x \in S$.) We do this in two steps.

    (a) We wanted a base for some topology on X which will also contain $\mathcal{S}$. Clearly, $\mathcal{B} := \{S_1 \cap \cdots \cap S_n : S_j \in \mathcal{S}, n \in \mathbb{N}\}$ is a base for some topology and $\mathcal{S} \subset \mathcal{B}$.

    (b) The topology $\mathcal{T}_{\mathcal{B}} := \{U \subset X : \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subset U\}$ is then the smallest topology that contains $\mathcal{S}$.

    (c) Thus, we can rid of the intermediate $\mathcal{B}$ and define the topology directly in terms of $\mathcal{S}$. We say $U \in \mathcal{T}_{\mathcal{S}}$ iff for every $x \in U$, there exists $n \in \mathbb{N}$ such that we can find S_j, $1 \le j \le n$ with $x \in S_1 \cap \cdots \cap S_n \subset U$. One can again show directly that this is the smallest topology containing $\mathcal{S}$.

    (d) $\mathcal{S}$ is called a *subbase* and $\mathcal{T}_{\mathcal{S}}$ is the topology generated by $\mathcal{S}$.

143. Let us look at some concrete examples:

    (a) Consider $\mathcal{S} := \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$. The topology generated by $\mathcal{S}$ on $\mathbb{R}$ is the usual topology.

    (b) Consider $\mathbb{R}$ and $\mathcal{S} := \{\{0, x\} : x \ne 0, x \in \mathbb{R}\}$. What is the topology on $\mathbb{R}$?

    (c) Let X be a set with at least 3 elements. Let $\mathcal{S}$ be the family of two-element subsets of X. The topology generated by $\mathcal{S}$ is the discrete topology.

    (d) What is the topology on $\mathbb{R}^2$, if we take the subbase consisting of all straight lines in $\mathbb{R}^2$?

    (e) What is the topology on $\mathbb{R}^2$, if we take the subbase consisting of all straight lines parallel to the x-axis in $\mathbb{R}^2$? Which of the following sets are open in this topology? (i) the open unit disk, $B(0,1)$, (ii) the open vertical band $\{(x,y) \in \mathbb{R}^2 : 0 < x < 1\}$, (iii) the open horizontal band $\{(x,y) \in \mathbb{R}^2 : 0 < y < 1\}$, (iv) any subset which is bounded in the Euclidean metric.

    (f) What is the topology on $\mathbb{R}^2$, if we take the subbase consisting of all circles in $\mathbb{R}^2$?

    (g) What is the topology on $\mathbb{R}^2$, if we take the subbase consisting of all circles, with centre at the origin, in $\mathbb{R}^2$?

    (h) Consider $\mathcal{S} = \{X\}$ as a subbase on X. What topology do we get on X?

144. Let $f \colon X \to Y$ be *any* map between two sets. Assume that one of them is a topological space. What we wish to do is to endow the other set with an *optimal* topology in such a way that $f \colon X \to Y$ becomes a continuous map between the spaces.

(a) Let Y be a topological space. Then if we endow X with the discrete topology, then the problem is solved! But this topology has no bearing on Y and/or on the map f! So what we require is the smallest topology on X making f continuous.

(b) Let X to be a topological space. Considerations similar to the last item suggest us that we require the largest topology on Y making f continuous.

145. These problems arise in a very natural way.

(a) Let X be a subset a topological space Y. Then we have an *obvious* or *natural* map $i \colon X \to Y$, the inclusion of X in Y, that is, the restriction of the identity on Y to X.

(b) Let X be any topological space and $\sim$ an equivalence relation on X. Then as Y, we take the quotient set $X/\sim$, that is, the set of equivalence classes. Once again, we have a natural map $\pi \colon X \to Y$, where $\pi(x)$ is the equivalence class of x.

146. More general situations may also arise. Let X be a set and Y_i be topological spaces, indexed by a set I. Assume that we are given certain maps $f_i \colon X \to Y_i$ for each $i \in I$. We again ask for a single smallest topology on X making all the maps f_i continuous. Or the other way around, we have maps $f_j \colon X_j \to Y$ where X_j's topological spaces.

Typical instances of this phenomenon are:

(a) Let $\{X_j : j \in I\}$ be an indexed family of (pairwise disjoint) topological spaces. Let $X := \biguplus_{j\in I} X_j$, the disjoint union of X_j's. We have natural inclusion maps $\iota_j \colon X_j \to X$. We wish to endow X with the largest topology with respect to which all ι_j's are continuous.

(b) Let $\{X_i : i \in I\}$ be an indexed family of topological spaces. Let $X := \prod_{i\in I} X_i$. We have obvious maps $\pi_i(x) = x_i$, the i-th projection. We wish to equip X with the smallest topology such that each of the projections becomes continuous.

(c) Let E be a set and let $X := \mathcal{F}$ be a family of functions from E to $\mathbb{R}$. Consider the evaluation maps $\varepsilon_x(f) := f(x)$ for each $x \in E$. Thus, we have a family of maps $\varepsilon_x \colon X \to \mathbb{R}$ and we want the smallest topology which will make all these maps continuous.

147. Let us deal with various cases. Let X be a set and Y be a topological space and $f \colon X \to Y$ be a map. Any topology on X which makes f continuous must contain the set $\mathcal{U} := \{f^{-1}(V) : V \in \mathcal{T}_Y\}$. It turns out this collection is already a topology and hence is the smallest topology on X, as required. (We were lucky this time!)

148. Let us look at the concrete case in Item 145a. Then the topology on X is given by

$$\mathcal{T}_X := \{i^{-1}(V) : V \in \mathcal{T}_Y\} = \{V \cap A : V \in \mathcal{T}_Y\}.$$

The topology $\mathcal{T}_X$ is called the subspace topology on Y and any $U \in \mathcal{T}_X$ is said to be open *in* X. We say that $F \subset X$ is closed in X if its complement, $X \setminus F$, *in* X is open in X.

149. The following are immediate from the definition of subspace topology and are very useful in 'identifying' or visualizing open sets in subspace topology.

(a) If $\mathcal{B}$ is a basis for the topology $\mathcal{T}_Y$ on Y, then $\mathcal{B}_X := \{B \cap X : B \in \mathcal{B}\}$ is a basis for the subspace topology on X.

(b) If $\mathcal{B}_x$ is a local basis at $x \in Y$ for the topology $\mathcal{T}_Y$ on Y, then $\mathcal{B}_{x,X} := \{B \cap X : B \in \mathcal{B}_x\}$ is a local basis at $x \in X$ for the subspace topology on X.

150. Let us look at some examples to develop our intuition about subspace topology. Use the last item to identify a local basis at each point of the subset A.

(a) Consider $A = [0, 1] \subset \mathbb{R}$. Then the sets $[0, 1/2)$, $(1/2, 1]$ and $(1/2, 3/4)$ are open in in A.

(b) Consider $Y := \{(x, y) : xy = 0\} \subset \mathbb{R}^2$ be the two axes. Then the basic open sets near (0,0) are crosses (of two line segments along the x and y-axes.) At other points, just intervals around them.

(c) Let $A := \{1/n : n \in \mathbb{N}\} \cup \{0\}$. Then the basic open sets are the singletons $\{1/n\}$ for $n \in \mathbb{N}$ and $\{1/n : n \geq n_0\} \cup \{0\}$. The latter are basic opens sets near 0 in A.

(d) Let $S := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ be the unit circle in $\mathbb{R}^2$. The basic open sets in S are open arcs of the circle.

(e) Consider $A = \mathbb{Q} \subset \mathbb{R}$. Then the set $\{r \in \mathbb{Q} : -\sqrt{2} \leq r \leq \sqrt{2}\}$ is both open and closed in $\mathbb{Q}$.

(f) Let X be a metric space and $\emptyset \neq A \subset X$. Then we have two topologies on A: (i) one comes from the induced metric, call it δ_A, on A and (ii) the other is the subspace topology. They are the same.

Let $\mathcal{T}_{d_A}$ denote the metric topology on A and $\mathcal{T}_A$ denote subspace topology on A. The local base at $a \in A$ for $\mathcal{T}_A$ is $\{B_{(A,d_A)}(a, r) : r > 0\}$ and the one for $\mathcal{T}_A$ is $\{B_{(X,d)}(a, r) \cap A : r > 0\}$. But, $B_{(A,d_A)}(a, r) = B_{(X,d)}(a, r) \cap A$ for each $r > 0$. Hence the local bases are the same at each point $a \in A$ for both the topologies.

(g) Let $A := [0, 1] \times [0, 1]$. Then A has the order topology as well as the subspace topology as a subset of $\mathbb{R}^2$ with order topology. They are not the same. (Contrast this with the last item.)

Consider the set $V := \{(0, y) : 1/2 < y \leq 1\}$. Then V is is open in the subspace topology but not in the order topology on the ordered set A. Draw a picture of A and use the definitions of subspace topology and order topology. V is open in the subspace topology, since it is the intersection A with basic open set in $\mathbb{R}^2$ with an interval (in the order topology): $V = A \cap (a, b)$ where $a = (0, 1/2)$ and $b = (0, 2)$.

Let, if possible, A be open in the order topology on A. Then there exists an open interval (c, d) such that that point $p = (0, 1) \in (c, d)$. Let $c = (x_1, y_1)$ and $d = (x_2, y_2)$. Then $x_1, x_2 \geq 0$ and $y_1, y_2 \leq 1$. Now, $(x_1, y_1) < (0, 1)$ in the dictionary order. We conclude that $x_1 = 0$ and $y_1 < 1$. Similarly, $x_2 > 0$ and $y_2 \geq 0$. But an element of the form $(x_2/2, y)$ with $y \geq 0$ lies in the basic open set but not be in V.

(h) Consider $\mathbb{R}$ with VIP topology and $A = \mathbb{R}^*$. Then the subspace topology on $\mathbb{R}^*$ is the discrete topology. The subspace topology on $\mathbb{Q}$ is the VIP topology on $\mathbb{Q}$. (Do you understand this statement?)

(i) Investigate the subspace topology on $\mathbb{Q}$ considered as a subset of $\mathbb{R}$ with outcast topology.

(j) Let X be a Hausdorff space, $A \subset X$ be endowed with the subspace topology. Then A is Hausdorff.

151. Let A be nonempty and open in X. Then $U \subset A$ is open in A iff it is open in X.

152. Let $B \subset A \subset X$. Let $(X, \mathcal{T}_X)$ be a topological space. Let $\mathcal{T}_A$ denote the subspace topology on A. Let $x \in A$. Then $x \in A$ is a limit point of B **in** A iff x is a limit point of B in X.

    Let x be a limit point of B in A. Let $U \in \mathcal{T}_X$ such that $x \in U$. Since $U \cap A \in \mathcal{T}_A$ is an open set containing x, $(U \cap A) \cap B \neq \emptyset$. But, $(U \cap A) \cap B = U \cap B$. Thus, x is a limit point of B in X.

    Conversely, let $x \in A$ be a limit point of B in X. Let V be open in A with $x \in V$. We need to show that $V \cap B \neq \emptyset$. There exists $U \in \mathcal{T}_X$ such that $V = U \cap A$. Then $x \in U$ and since x is a limit point of B in X, we have $U \cap B \neq \emptyset$. Since $B = B \cap A$, it follows that $x \in (U \cap A) \cap B \neq \emptyset$, that is, $V \cap B \neq \emptyset$, or x is a limit point of B in A.

153. Let $A \subset X$. Then $F \subset A$ is closed in A iff there exists a closed set C in X such that $F = A \cap C$.

    You may use the last item to prove this. Or, we may proceed directly as follows. (This is essentially set-theoretic exercise and so the reader should try on his own.)

    Let $F = A \cap C$. We show that the complement of F in A is open. Let $U = X \setminus C$. Then U is open in X. We claim that $U \cap A = A \setminus F$. To show $(X \setminus C) \cap A \subset A \setminus F$, let $x \in (X \setminus C) \cap A$. If $x \in F$, then $x \in F = C \cap A$ and hence $x \in C$, a contradiction. For the reverse inclusion, let $x \in A \setminus F$. We need to show that $x \in X \setminus C$./ If false, then $x \in C$ and hence $x \in A \cap C = F$, that is $x \in F$, a contradiction.

    Assume that F is closed in A. Then $A \setminus F$ is open in A. Let U be open in X such that $A \setminus F = U \cap A$. Let $C := X \setminus U$. Then C is closed in X. We claim that $C \cap A = F$. To show that $C \cap A \subset F$, let $x \in C \cap A$. Suppose $x \notin F$, then $x \in (A \setminus F) = A \cap U$. Therefore, $x \in U$ or $U \notin C$, a contradiction. To prove the reverse inclusion, let $x \in F$. If $x \notin C$, then $x \in U$ and hence $x \in U \cap A = (A \setminus F)$, that is $x \notin F$, a contradiction.

    As a specific example, the set of Item 150e is open as well as closed in $\mathbb{Q}$. (Contrast this with Item 57i.)

154. We shall put to use some of the concepts we learned to gain a different perspective of the limit of a sequence.

    Consider $X = \mathbb{N} \cup \{\infty\}$, where ∞ is just a symbol representing an element not in $\mathbb{N}$. (We could have used $\star$ in place of ∞!) Let $\varphi \colon X \to Y := \{1/n : n \in \mathbb{N}\} \cup \{0\}$ be defined by $\varphi(n) = 1/n$ and $\varphi(\infty) = 0$. Then φ is a bijection. We endow Y with the subspace topology as a subset of $\mathbb{R}$. Using the bijection φ, we transfer the topology on Y to X. The local basic open set in Y at n are $\{n\}$, and at ∞ are $\{k : k \geq N\}$ for some $N \in \mathbb{N}$. (See Item 150c.)

    Now if $f \colon X \to \mathbb{R}$ is a function, when is it continuous at ∞? If we restrict f to $\mathbb{N} \subset X$, we can think of f as a real sequence, say, (a_n), in $\mathbb{R}$. Do you see any relation between the continuity of f at ∞ and the convergence of (a_n)? If we replace $\mathbb{R}$ by a topological space Z, do the results (concerning the convergence of sequences in Z) continue to be true?

We shall return to this example later when we talk of one point compactifications.

155. Let $f\colon X \to Y$ be a continuous map between two topological spaces. Let $A \subset X$ be a subset endowed with the subspace topology. Then the restriction f_A of f to A gives rise to a map $f_A\colon A \to Y$. Is it continuous?

If $V \subset Y$ is open, then $f_A^{-1}(V) = f^{-1}(V) \cap A$ is open in A, since $f^{-1}(V)$ is open in X.

156. A question 'dual' to the one in the last item: Let $f\colon X \to Y$ be continuous. Assume that $f(X) \subset B \subset Y$. We then have an induced map $g\colon X \to B$ defined by $g(x) = f(x)$ for $x \in X$. Let B be given the subspace topology. Is g continuous? The answer is 'Yes.'

Let $W \subset B$ be open **in** B. Then there exists V, an open subset of Y such that $W = V \cap B$. It is easy to check that $g^{-1}(W) = f^{-1}(V)$. Since f is continuous, $f^{-1}(V)$ is open in X and hence $g^{-1}(W)$. (We shall return to this later when we talk of universal mapping properties. See???) Reference?

Items 155–156 are most often used when we deal with subspaces *without* explicit mention.

157. At this stage we are curious about the following questions.

(a) Let X and Y be topological spaces. Assume that $\{A_i \in I\}$ is a family of subsets of X such that $\cup_{i\in I} A_i = X$. Further assume that for each i, we have a continuous function $f_i\colon A \to Y$. (Here A_i's are given the subspace topology.) Can we get 'glue' them together to get a continuous function $f\colon X \to Y$ in such a way that the restriction $f|_{A_i} = f_i$ for $i \in I$?
A necessary condition is that $f_i(x) = f_j(x)$ for each $x \in A_i \cap A_j$, $i, j \in I$. This will ensure that we get a function f from the set X to Y whose restrictions to A_i are as required.

(b) Let X and Y be topological spaces. Let $f\colon X \to Y$ be a map. Assume that $\{A_i \in I\}$ is a family of subsets of X such that $\cup_{i\in I} A_i = X$ and that $f|_{A_i}\colon A_i \to Y$ is continuous. Can we conclude f is continuous?

158. Let us investigate the situation. Let $V \subset Y$ be open. Observe that

$$f^{-1}(V) = f^{-1}(V) \cap X = \cup_{i\in I} \left(f^{-1}(V) \cap A_i\right).$$

Each term in the union, $f^{-1}(V)\cap A_i = f_i^{-1}(V)$ is open in A_i. If we can ensure that each of these open in X then $f^{-1}(V)$ is open in X. We know a sufficient condition which will ensure this, namely, we demand each A_i is open.

Let $V \subset Y$ be closed. Since each $f^{-1}(V) \cap A_i = f_i^{-1}(V)$ is closed in A_i, to ensure $f^{-1}(V)$ is closed, we may demand that each A_i is closed. But then $f^{-1}(V)$ is a union of closed sets and it is closed if we further assume that I is finite. We have thus arrived at

159. Gluing Lemma:

Lemma 8. *Let X, Y be topological spaces and let $f\colon X \to Y$ be any map. Assume that $\{A_i : i \in I\}$ is a family of subsets of X whose union is X. Assume further that $f_i := f|_{A_i}\colon A_i \to Y$ is continuous for each $i \in I$. Then*

1. *f is continuous if each A_i is open.*
2. *f is continuous if each A_i is closed and I is finite.* □

160. Let us consider the general case in Item 146. We want the smallest topology $\mathcal{T}$ that contains all sets of of the form $f_i^{-1}(V_i)$ where V_i is open in X_i and $i \in I$. That is $\mathcal{T}$ is the smallest topology containing the family of sets $\mathcal{S} := \{f_i^{-1}(V_i) : V_i \in \mathcal{T}_i; i \in I\}$, where $\mathcal{T}_i$ is the topology on X_i.

There is no reason to believe that $f_i^{-1}(V_i) \cap f_j^{-1}(V_j)$ must be again of the form $f_r^{-1}(V_r)$ for some $r \in I$. Hence $\mathcal{S}$ may not be topology on X.

161. We now want to look at the concrete case in Item 146b. As a preliminary, we review the concept of Cartesian product.

Let $\{X_i : i \in I\}$ be an indexed family of sets. Then the Cartesian product $X := \prod_{i\in I} X_i$ is defined by

$$\prod_{i\in I} X_i := \{x \colon I \to \biguplus_{i\in I} X_i : x(i) \in X_i \text{ for each } i \in I\},$$

where $\biguplus_{i\in I} X_i$ stands for the disjoint union.

(a) We usually write $x \in \prod_{i\in I} X_i$ as $x = (x_i)$, where $x_i := x(i)$. We shall call x_i as the i-th coordinate of x. Let $\pi_i \colon \prod_{j\in I} X_i \to X_j$ denote the map $\pi_j(x) = x(j) = x_j$. This is called the j-th projection of X onto the j-th factor X_j.

(b) As a convention, if $I = \{1, 2, \ldots, n\}$, we identify X with $X_1 \times \cdots \times X_n$, that is, with the set of "ordered n-tuples" $(x_1, \ldots, x_n)$. Similarly, if $I = \mathbb{N}$, we identify X with $X_1 \times X_2 \times \cdots \times X_n \times \cdots$, that is the set of ordered infinite tuples $x \mapsto (x_1, x_2, \ldots, x_n, \ldots)$.

(c) If $V_j \subset X_j$, then $\pi_j^{-1}(V_j) = \prod_{i\in I} U_i$ where $U_i = X_i$ for $i \neq j$ and $U_j = V_j$. In particular, $\pi_1^{-1}(V_1) = V_1 \times X_2$ where $X = X_1 \times X_2$ etc.

162. What we requite on $X := \prod_{i\in I} X_i$ to make the projections π_i $(i \in I)$ continuous is the smallest topology that contains

$$\mathcal{S} := \{\pi_i^{-1}(V_i) : V_i \in \mathcal{T}_i, i \in I\}.$$

This is the question we have already answered in Item 142.

163. We apply the process of Item 142.to the problem posed in Item 162. Thus we arrive at the definition of *product topology* on $\prod_{i\in I} X_i$ as follows.

As a subbase for a topology on X, we take the set

$$\mathcal{S} := \left\{ \prod_{i\in I} U_i : \text{ where } U_i = X_i \text{ for all but finitely many } i \text{ and } U_i \text{ is open in } X_i \right\}.$$

The basis for the product topology on X is finite intersections of elements from $\mathcal{S}$.

In particular, $G \subset X$ is open iff for every $x \in G$, there exists $S_1, \ldots, S_n \in \mathcal{S}$ such that $x \in S_1 \cap \cdots \cap S_n \subset G$.

Thus, $G \subset X$ is open in the product topology iff for a given $x \in G$, there exists a finite subset $F \subset I$ and open subsets $U_j \subset X_j$ for $j \in F$ such that $x \in \prod_i V_i$ where $V_i = X_i$ for $i \notin F$, $V_j = U_j$ for $j \in F$ and $x \in \prod_{i\in I} V_i \subset G$.

164. Let $\emptyset \subsetneq \emptyset \neq U_i \subsetneq X_i$, be open in X_i for $i \in I$. Then $U = \prod_{i\in I} U_i$ could never be open in X unless I is finite.

Assume I is infinite. Let $x = (x_i) \in U$. If U were open, then there exists a finite set $F \subset I$, open sets V_j for $j \in F$ such that $x \in W = \prod_i W_i \subset U$ where $W_i = X_i$ for $i \notin F$ and $W_j = V_j$ for $j \in F$. Choose an $r \in I \setminus F$. Since $V_r \neq X_r$, there exists $y_r \in X_r \setminus V_r$. Consider $z = (z_i)$ where $z_i = x_i$ for $i \neq r$ and $z_r = y_r$. Then $z \in W$ but $z \notin U$.

165. If I is finite, say, $I = \{1, 2, \dots, n\}$, then the basic open sets are of the form $U_1 \times \cdots \times U_n$ where U_i is an arbitrary open set in X_i for each $1 \leq i \leq n$.

166. **Warning:** If, at first, we defined finite products of topological spaces with basis as in the last item, we would be tempted to use the following collection as a basis for a topology on the product $\prod_{i\in I} X_i$:

$$\mathcal{B} := \left\{ \prod_{i\in I} U_i : \text{ where } U_i \text{ is an arbitrary open set in } X_i \right\}.$$

The topology given rise to by this basis is called the box topology. Evidently, this is finer than the product topology.

The product topology on X is the smallest topology which makes all the projection maps π_i continuous. We shall always use this topology on the product sets.

167. We shall see how to visualize the subbasic and basic open sets of the product topology. This will allow us to gain some geometric intuition. Pictures!

(a) Consider $X \times Y$. We visualize this the first quadrant in $\mathbb{R}^2$ where X and Y are represented by $[0, \infty)$. Then any subbasic open set if of the form $U \times Y$ or $X \times V$ where $U \subset X$ and $V \subset Y$ are open. We visualize this by a vertical strip of the from $(a, b) \times [0, \infty)$ or as a horizontal strip of the from $[0, \infty) \times (c, d)$. Hence any basic open set is represented by a rectangle of the from $(a, b) \times (c, d)$.

This can be extended to a finite product.

(b) We now consider a countable product, say $X = \prod_{n\in\mathbb{N}} X_n$. We visualize X as vertical half-lines erected at $(n, 0) \in \mathbb{R}^2$: A basic open set is therefore of the form half-lines at all points except at finitely many $n_1, \dots n_k$ and at n_j an interval of the form (a_j, b_j).

(c) Consider $X := \prod_{t\in\mathbb{R}} \mathbb{R}$. The product set can be identified with the set of functions $f\colon \mathbb{R} \to \mathbb{R}$. Each function can be represented by its graph in $\mathbb{R} \times \mathbb{R}$. Fix a finite set of points $\{t_1, \dots, t_n\}$ and a finite set of intervals (a_j, b_j), $1 \leq j \leq n$. Then the basic open set corresponding to this data is $\mathbb{R}$ at all $t \notin \{t_k : 1 \leq k \leq n\}$ and (a_k, b_k) if $t = t_k$, $1 \leq k \leq n$. Thus the elements in this basic set are functions such that $f(t_k) \in (a_k, b_k)$. We can visualize this by means of their graphs.

168. To have a feeling for the product topology, we look at the following results/questions:

(a) The product of Hausdorff spaces is Hausdorff.

Easy. if $x = (x_i)$ and $y = (y_i)$ are in $X = \prod_i X_i$ are distinct elements, then there exists $j \in I$ such that $x_j \neq y_j$. Since X_j is Hausdorff, there exist U_j and V_j open in X_j with $x_j \in U_j$, $y_j \in V_j$ and $U_j \cap V_j = \emptyset$. Consider the open set $U = \prod U_i$ and $V = \prod V_i$ where $U_i = X_i = V_i$ for $i \neq j$ and at j the disjoint open sets U_j and V_j. Then U and V separate x and y.

(b) A sequence (x_k) in the product space is convergent to an element x iff it converges coordinate-wise, that is, iff $\pi_i(x_k) \to \pi_i(x)$ for each $i \in I$.

To avoid confusion with indices, we use the Greek alphabet to denote elements of the index set I.

Let $x_k \in X = \prod_{\alpha \in I} X_\alpha$ converge to x. Fix $\beta \in I$. Let $x_{k\beta} := \pi_\beta(x_k)$ and $x_\beta := \pi_\beta(x)$. Let $U_\beta \ni x_\beta$ be open. Consider the subbasic open set $U = \prod U_\alpha$ where $U_\alpha = X_\alpha$ for $\alpha \neq \beta$ and $U_\alpha = U_\beta$ for $\alpha = \beta$. Then $x \in U$ and since $x_k \to x$, there exists $N \in \mathbb{N}$ such that $x_k \in U$ for $k \geq N$. It follows that $x_{k\beta} \in U_\beta$ for $k \geq N$. Hence $x_{k\beta} \to x_\beta$.

To prove the converse, let U be a basic open set, say, of the form $U = \prod U_\alpha$ where $U_\alpha = X_\alpha$ for $\alpha \notin F \subset I$, a finite subset and U_β are open subsets in X_β, $\beta \in F$. Since $x_{k\alpha} \to x_\alpha$, it follows that there exists N_β such that for $k \geq N_\beta$, we have $x_{k\beta} \in U_\beta$ for $\beta \in F$. Let $N = \max\{N_\beta : \beta \in F\}$. We claim that $x_k \in U$ for $k \geq N$. For, any $\alpha \notin F$, $\pi_\alpha(x_k) = x_{k\alpha} \in U_\alpha = X_\alpha$ and for $\beta \in F$, since $k \geq N$, $\pi_\beta(x_k) = x_{k\beta} \in U_\beta$.

Thus, the convergence in $\prod_{i \in I} X_i$ is "coordinate-wise convergence."

(c) Let $A_i \subset X_i$ and $A := \prod_{i \in I} A_i$. Then $\overline{A} = \prod_{i \in I} \overline{A_i}$. In particular, if each A_i is closed, then the product $A := \prod_{i \in I} A_i$ is closed in the product space X. Contrast this with Item 164.

The proofs of this and the next item run almost like the earlier two items. We leave them for your practice.

(d) Let D_i be dense in X_i for each i. Then $D := \prod_{i \in I} D_i$ is dense in X.

This follows from the last subitem. We encourage the reader to prove it directly.

(e) Let X_i be a discrete space for each i. When is $\prod_{i \in I} X_i$ is discrete?

(f) Let X, Y be metric spaces. We have a product metric on the product $X \times Y$ given by $\delta((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), d(y_1, y_2)\}$. Thus we have two topologies on $X \times Y$, namely, the topology induced by the metric δ and the product topology (got out of the metric topologies on X and Y). We saw that these two topologies are the same. Later, we shall see an easy proof.

Investigate whether the converses (wherever applicable) are true.

Optional: Investigate how many of them are true if we equip X with the box topology. Note that if D is dense in $(X, \mathcal{T}_2)$ and if $\mathcal{T}_1$ is another topology on X with $\mathcal{T}_1$ weaker than $\mathcal{T}_2$, then D is dense in $(X, \mathcal{T}_1)$.

169. This is a continuation the theme of Item 168.

(a) Let $f : Y \to \prod_{i \in I} X_i$ be a map from a topological space Y to the topological space $\prod_{i \in I} X_i$ with product topology. Then f is continuous iff each f_i, is continuous, where $f_i = \pi_i \circ f$ for $i \in I$,

If f is continuous, then f_i is the composition of two continuous functions and hence is continuous.

Assume that f_i are continuous. To prove that f is continuous, let U be an open subset of X. Since U is a union of basic open sets, it is enough to show that $f^{-1}(B)$ is open for any basic open set. But B is of the form $\cap_{j\in F}\pi_j^{-1}(U_j)$ where $F \subset I$ is finite, and U_j is open subset of X_j, $j \in F$. Since taking inverse images behaves well with set-theoretic operations, it suffices to show that $f^{-1}(\pi_j^{-1}(U_j))$ is open in Y for any $j \in I$ and any V_j open in X_j. But

$$f^{-1}(\pi_j^{-1}(U_j)) = (\pi_j \circ f)^{-1}(u_j) = f_j^{-1}(U_j),$$

which is open in Y by the continuity of f_j.

(b) Let $f\colon X \to Y$ be continuous. Let $\text{Graph}(f) := \{(x, f(x) : x \in X\}$ be the graph of f. Let $\text{Graph}(f) \subset X \times Y$ be endowed with the subspace topology. Then X is homeomorphic to $\text{Graph}(f)$.

Consider $\varphi\colon X \to \text{Graph}(f)$ given by $\varphi(x) = (x, f(x))$. Then φ is continuous as a map from X to $X \times Y$ by the last sub-item. By Item 156, $\varphi\colon X \to \text{Graph}(f)$ is also continuous. Clearly φ is a bijection and its inverse $\varphi^{-1}\colon \text{Graph}(f) \to X$ is given by $\varphi((x, f(x)) = f(x)$. That is, φ^{-1} is the restriction of the projection of $X \times Y$ onto its first factor. By the definition of product topology, the projection map is continuous on $X \times Y$. Its restriction is continuous on Z by Item 155.

(c) The map $x \mapsto (x, y_0)$ of X into $X \times Y$ is a homeomorphism of X with $X \times \{y_0\}$ with the subspace topology inherited from $X \times Y$.

Argue as in the last sub-item.

(d) Let X, Y be topological spaces. Let $A \subset X$ and $B \subset Y$. Let $\mathcal{T}_A$ denote the subspace topology on A induced from the topology on X etc. Let $\mathcal{T}_A \times \mathcal{T}_B$ (respectively, $\mathcal{T}_X \times \mathcal{T}_Y$) denote the product topology on $A \times B$, (respectively, the product topology on $X \times Y$). Let $\mathcal{T}_{A\times B}$ denote the subspace topology on $A \times B$ considered as a subset of $X \times Y$. Then $\mathcal{T}_A \times \mathcal{T}_B = \mathcal{T}_{A\times B}$.

Easy if you know how to set up a notation which keeps your head clear. Let $W \in \mathcal{T}_A \times \mathcal{T}_B$. Let $x = (a, b) \in W$. There exist $U_a \in \mathcal{T}_A$ and $V_b \in \mathcal{T}_B$ such that $x = (a, b) \in U_a \times V_b \subset W$. Since $U_a \in \mathcal{T}_A$, there is a $U \in \mathcal{T}_X$ such that $U_a = U \cap A$. Similarly, $V_b = V \cap B$. Hence

$$x = (a, b) \in (U \cap A) \times (V \cap B) = (U \times V) \cap (A \times B) \subset W.$$

Since $(U \times V) \cap (A \times B) \in \mathcal{T}_{A\times B}$, it follows that $W \in \mathcal{T}_{A\times B}$.

Reverse inclusion is proved similarly. Let $W \in \mathcal{T}_{A\times B}$. Then there exists $W' \in \mathcal{T}_X \times \mathcal{T}_Y$ such that $W = W' \cap (A \times B)$. Let $x = (a, b) \in W$ so that $x \in W'$. We can find $U' \in \mathcal{T}_X$ and $V' \in \mathcal{T}_Y$ such that $x = (a, b) \in U' \times V' \subset W'$. Hence it follows that

$$x = (a, b) \in (U' \cap A) \times (V' \cap B) = (U' \times V') \cap (A \times B) \subset W' \cap (A \times B).$$

Since $(U' \cap A) \times (V' \cap B)$ is a basic open set in $\mathcal{T}_A \times \mathcal{T}_B$, it follows that $W \in \mathcal{T}_A \times \mathcal{T}_B$.

(e) Let $\Delta(X)$ denote the diagonal $\{(x, x) : x \in X \times X\} \subset X \times X$. Then X is Hausdorff iff $\Delta(X)$ is closed in $X \times X$. (We leave this as a very easy exercise.)

170. Let X and Y be sets. The set Y^X of functions from X to Y can be considered as the product space $\prod_{x \in X} Y_x$ where $Y_x = Y$ for $x \in X$ via the map $\varphi(f) = (f(x)) \in \prod_{x \in X} Y_x$. Thus, what the last line of Item 168b says is that the if we use φ to transfer the topology to Y^X, then a sequence of functions (f_n) in Y^X converges to a function $f \in Y^X$ iff $f_n(x) \to f(x)$ for each $x \in X$, that is, convergence in Y^X is pointwise convergence. Because of this, product topology is known as topology of pointwise convergence.

171. Is the product of first/second countable spaces first/second countable? We show that $\prod_{t \in \mathbb{R}} \mathbb{R}$ is not first countable. Note that $\mathbb{R}$ is second countable. The key idea comes from the last item. It may be worthwhile to review Item 167c.

The product space is the set of functions from $\mathbb{R}$ to $\mathbb{R}$ and the convergence is pointwise convergence. How does local base at (the constant function) 0 look like? Fix a finite subset $F \subset \mathbb{R}$ and $k \in \mathbb{N}$. Then a typical element of the local base is of the form

$$U_{F,k} := \{f \colon \mathbb{R} \to \mathbb{R} : f(t) \in (-1/k, 1/k), t \in F\}.$$

How many such basic open sets are there? As many as in the set $\mathcal{F} \times \mathbb{N}$ where $\mathcal{F}$ is the set of finite subsets of $\mathbb{R}$, that is, the cardinality of $\mathcal{F} \times \mathbb{N}$. It is intuitively clear and we expect that this space has no countable local base at $0 \in \prod_{t \in \mathbb{R})} \mathbb{R}$.

We would like to translate these ideas into a rigorous argument. An obvious method of attack is to prove this by contradiction. Consider the set

$$E := \{f \in \mathbb{R}^{\mathbb{R}} : f(t) = 0 \text{ or } 1 \ \& \ \{t \in \mathbb{R} : f(t) = 0\} \text{ is finite .}\}$$

We claim the constant function 0 is a limit point of E. For, let $U_{F,k}$ be a basic open set containing 0. Then the function $f(t) = 1$ for $t \notin F$ and $f(t) = 0$ for $t \in F$ lies in $U_{F,k} \cap E$. If the product topology were first countable, then there exists a sequence (f_n) in E such that $f_n \to f$ in the product topology. Let $F_n := \{t : f_n(t) = 0\}$. Let $A := \cup_n F_n$. Then A is countable. Observe that for $t \notin A$, $f_n(t) = 1$ and therefore $f(t) = \lim f_n(t) = 1$ for $t \notin A$. This is a contradiction

172. Contrast Item 175b with the following. Let E be any set and let $B(E, \mathbb{R})$ denote the set of all bounded real valued functions on E. If we endow this vector space with the norm $\|f\|_\infty := \sup_{x \in E} |f(x)|$, then $f_n \to f$ in this normed linear space iff $f_n \to f$ uniformly on E. (This is Item 91b.)

173. Refer to Item 171. Each of the factors in $\mathbb{R}^{\mathbb{R}}$ is a metric space. But the topology on $\mathbb{R}^{\mathbb{R}}$ is not first countable and hence there cannot be any metric d on the product $\mathbb{R}^{\mathbb{R}}$ which will induce the product topology.

174. In most of the examples above, we looked at subsets of the product set X which are of the form $\prod_{i \in I} A_i$, where $A_i \subset X_i$. You should be aware that not all subsets of X are of this form. For example, $S := \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}$, $D := \{(x, x) \in \mathbb{R}^2 : x \in \mathbb{R} \times \mathbb{R}\}$ are not a product of subsets of $\mathbb{R}$.

For, if $D = A \times B$, then $(1, 1), (2, 2) \in D = A \times B$. Hence $1, 2 \in A$, $1, 2 \in B$ and hence $(1, 2) \in A \times B = D$!

175. It is equally important to recognize product spaces in disguise. The following are very typical of this situation.

(a) Define a topology on the set S of all real sequences such that a sequence (x_k) in S converges to $x \in S$ iff the $x_{kn} \to x_n$ as $n \to \infty$ for all k where $x_k = (x_{k1}, x_{k2}, \ldots, x_{kn}, \ldots)$. (Convergence = Coordinate-wise convergence).

(b) Let X denote the set of all real valued functions on $\mathbb{R}$. Define a topology on X such that a sequence (f_n) of functions in X converge to a function $f \in X$ iff $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$. (Convergence = point-wise convergence of functions.)

(c) Let $I = \mathbb{N}$ and $X_i = \{0, 1\}$ for $i \in \mathbb{N}$. Then the product space $X := \prod_{i \in \mathbb{N}} X_i$ "is isomorphic to" the Cantor set. We have to introduce concepts and develop some more theory to explain this satisfactorily.

176. A problem similar to Item 168a: Let X be any set and $\mathcal{F}$ be a collection of real valued functions on X with the property that for any pair of distinct points $x, y \in X$, there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$. Then the smallest topology on X which makes all the functions in $\mathcal{F}$ continuous is Hausdorff.

177. Let X be a topological space and $\sim$ is an equivalence relation on X. Let $Y := X/\sim$ be the quotient set, that is, the set of all equivalence classes. Let $\pi\colon X \to Y$ be the quotient map $\pi(x) := [x]$, the equivalence class of x. The largest topology on Y with respect to which π is continuous is called the quotient topology on Y. It is given by

$$\left\{V : \pi^{-1}(V) \text{ is open in } X\right\}.$$

178. We *studied* part of my article "Generation Topologies — A Unified View of Subspace, Product and Quotient Topologies". We also did the Universal mapping properties for Cases (i) & (ii) of the article.

179. Universal mapping properties were done in the general case and applied to concrete situations and interpreted.

(a) Universal mapping property for subspace topology.

(b) Universal mapping property for quotient topology.

(c) Universal mapping property for product topology.

180. Examples of applications of universal mapping property:

(a) The continuity of the map $[0, 2\pi]/\sim$ to S^1.

(b) The continuity of $S^n \to \mathbb{P}^n(\mathbb{R})$. (This cannot be done using UMP.)

181. **More examples of homeomorphisms.** Recall that a map $f\colon X \to Y$ between two topological spaces is a *homeomorphism* if (i) f is bijective, (ii) f is continuous and (iii) $f^{-1}\colon Y \to X$ is continuous.

(a) $B(0, 1) \simeq \mathbb{R}^n$.

(b) $S^n \setminus \{e_{n+1}\} \simeq \mathbb{R}^n$. (We investigated this in detail!)

(c) $f\colon X \to Y$ continuous. Then the graph of f with the subspace topology of $X \times Y$ is homeomorphic to X. Applications:

i. $\mathbb{R}$ is homeomorphic to the parabola $y = x^2$.

ii. $\mathbb{R}^*$ is homeomorphic to the hyperbola $xy = 1$.

(d) The product space $[-1, 1] \times S^1$ is homeomorphic to a cylinder.

(e) The annulus $\{p \in \mathbb{R}^2 : 1 \le \|p\| \le 2\}$ is homeomorphic to the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 1 \le z \le 2\}$.

(f) Let $f : X \to Y$ be a homeomorphism and let $A \subset X$. Then f induces homeomorphism between A and $f(A)$ (and between $X \setminus A$ and $f(X \setminus A)$).

This is a very useful fact. Typical ways of applying this are:

i. $[0, 1)$ is not homeomorphic to $(0, 1)$.

ii. $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

Both these results need connectedness at least in disguise, but can be proved at this stage using the intermediate value theorem.

For example, let, if possible, $\varphi : [0, 1) \to (0, 1)$ be a homeomorphism. Then we have a homeomorphism, again denoted by $\varphi : (0, 1) \to (0, 1) \setminus \{\varphi(0)\}$. Let $f : (0, 1) \setminus \{\varphi(0)\} \to \mathbb{R}$ defined $f(x) = x$ is a continuous function. Let $c, d \in (0, 1)$ be such that $c < \varphi(0) < d$. Let $\varphi(a) = c$ and $\varphi(b) = d$. Then $a, b \in (0, 1)$. The function $f \circ \varphi : (0, 1) \to \mathbb{R}$ is continuous on the interval $(0, 1)$ such that $f \circ \varphi(a) = c$, $f \circ \varphi(b) = d$. But it misses the point $\varphi(0) \in (c, d)$. This contradicts the intermediate value theorem.

(g) Homeomorphism between conic sections:

i. A circle is homeomorphic to an ellipse.

ii. A parabola is homeomorphic to a line.

iii. A (rectangular) hyperbola is homeomorphic to $\mathbb{R}^*$.

iv. A pair of intersecting lines is not homeomorphic to any of the other conic sections. More generally, a circle, a parabola, a hyperbola and a pair of intersecting lines are mutually non-homeomorphic. (We shall see a proof of this later. Meanwhile you may try to prove along this along the lines of a proof of Item 181(f)i.)

182. In any normed linear space , any two open balls are homeomorphic. Recall that $B(x, r) = x + rB(0, 1)$ and $B(y, s) = y + sB(0, 1)$.

183. In any normed linear space , any open ball is homeomorphic to the entire space. Enough to show that $B(0, 1)$ is homeomorphic to the normed linear space X. Any nonzero x is of the from $x = tu$, $0 \le t < 1$. Can we map $[0, 1)$ to $[0, \infty)$ homeomorphically? If yes, then the finite radial line tu, $0 \le t < 1$ will be mapped to the line segment emanating from 0 in the direction of u.

184. In $\mathbb{R}^n$, we have $B_\infty[0, 1] \simeq B_2[0, 1]$. For a complete proof, see Example 3.3.6 on Page 73 of my book on Metric Spaces.

185. $\mathbb{R}^m \simeq \mathbb{R}^n$ iff $m = n$. This is a highly nontrivial result and we shall not prove this is our course!

186. Another most important way of proving that a map is a homeomorphism is to use the following result which you might have seen in TYBSc.

A bijective continuous map from a compact metric space to another metric space is a closed map and hence is a homeomorphism.

We shall see a more general result later in Item 206b.

187. Let X be a topological space and $A \subset X$. We say that a collection $\{U_i : i \in I\}$ of subsets of X is an open cover of A if (i) each U_i is an open subset of X and (ii) $A \subseteq \cup_{i\in I} U_i$.

 Given an open cover $\{U_i : i \in I\}$ of A, by a subcover of A, we mean a subfamily $\{U_i : i \in J\}$ for some subset $J \subset I$ such that $\{U_i : i \in J\}$ is an open cover of A. We say that the given open cover admits a finite subcover, if J (in the notation above) is a finite set.

 For example, $\{(a, b) : a, b \in \mathbb{R}, a < b\}$ is an open cover of $\mathbb{R}$. The collection $\{(a, b) : a, b \in \mathbb{Q}, a < b\}$ is a subcover of $\mathbb{R}$.

 Let X be a space with discrete topology. The family $\{A, X \setminus A\}$ is an open cover for X where A is a nonempty proper subset of X with no proper subcover.

188. Examples of open covers:

 (a) "Non trivial" open covers of $\mathbb{R}$:

  i. $\{(-n, n) : n \in \mathbb{N}\}$.
  ii. $\{(-\infty, n) : n \in \mathbb{N}\}$.
  iii. $\{(-r, 2r) : r \in \mathbb{Q}^+\}$.

  Do they admit finite subcovers? proper subcover?

 (b) Nontrivial open covers of $(-1, 1)$.

 (c) In any metric space, $\{B(x, r_x) : x \in X\}$ is an open cover where $r_x > 0$ is pre-assigned for $x \in X$. Such a cover arises "naturally" in the following way: Let $f : X \to \mathbb{R}$ be a continuous function. Let $\varepsilon > 0$ be given. Given $x \in X$, by the continuity of f at x, there exists $r_x > 0$ such that for all y with $d(x, y) < r_x$, we have $|f(x) - f(y)| < \varepsilon$. The collection $\{B(x, r_x) : x \in X\}$ is an open cover of X.

 (d) Given a Hausdorff space with at least two elements, think of a nontrivial open cover.

 (e) Can you say something specific about any open cover of $\mathbb{R}$ with outcast topology?

 (f) Give an open cover of $\mathbb{R}$ with VIP topology which has no proper subcover. For example, $\{[a, b) : a < b\}$. Think of a non-trivial open cover which does not admit any proper subcover.

 (g) Give a non-trivial open cover of $\mathbb{R}$ with lower limit topology.

 (h) Open covers of S^n:

  i. $\mathbb{R}^{n+1} \setminus \{0\}$. (This is a trivial open cover!)
  ii. $U = \mathbb{R}^{n+1} \setminus \{N\}$ and $V := \mathbb{R}^{n+1} \setminus \{S\}$, where N, S are north and south poles of the sphere S^n respectively.
  iii. $U_i^{\pm} := \{x \in \mathbb{R}^{n+1} : x_i \lesseqgtr 0\}$, $1 \le i \le n + 1$.

 (i) Open cover for a discrete space X. Let $A \subset X$. Look at $\{A\} \cup \{\{x\} : x \notin A\}$. This is an open cover of X. What if $A = \emptyset$?

(j) Open cover for an uncountable space with co-countable topology. If $X = \mathbb{R}$ with co-countable topology and $U = \mathbb{R} \setminus \mathbb{N}$, I would think of defining $U_n := U \cup \{n\}$ or $V_n := U \cup \{k : 1 \leq k \leq n\}$. Then $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$ are open covers of $\mathbb{R}$. Do they admit any proper subcovers of $\mathbb{R}$?

What will you do for an countable set X with co-countable topology?

(k) Open cover for a set with co-finite topology.

189. A subset A of a topological space X is said to be *compact* if given any open cover $\{V_i : i \in I\}$ of A where each V_i is open in A, we can find a finite subcover. We say that X is a compact space if X is a compact subset of X.

190. Given an open cover $\{U_i : i \in I\}$ of A by means of open subsets of X, then we have a "natural" open cover $\{V_i : i \in I\}$ of a subset $A \subset X$ by means of subsets of A which are open *in* A and conversely. (Note the indices. "Naturality" does not mean that given V_i's, the U_i's are unique!)

The significance of this observation is that when dealing with compactness of a subset $K \subset X$ we may either work an open cover of K by means of open subsets in X or by sets open **in** K. See Items 193, 196, 196 where this observation is exploited.

191. Examples of compact sets.

(a) A finite subset of any space is compact. In particular, the empty set is compact.

(b) An indiscrete space is compact.

(c) A discrete space is compact iff it is finite.

(d) $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ are not compact.

(e) The intervals of the form (a, b), $[a, b)$, $(a, b]$, any infinite interval are not compact.

(f) $\mathbb{R}$ with lower limit topology $\mathcal{T}_L$ is not compact. Go through the proof. Can you make a general principle of which this is a special case?

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on the same set X. Assume that $\mathcal{T}_1 \leq \mathcal{T}_2$. Then if $(X, \mathcal{T}_1)$ is not compact, then $(X, \mathcal{T}_2)$ is not compact and $(X, \mathcal{T}_2)$ is compact, so is $(X, \mathcal{T}_1)$.

This is reminiscent of the comparison test for infinite series of positive terms.

(g) Any open ball in $\mathbb{R}^n$ (or in any normed linear space) is not compact.

(h) $\mathbb{R}^n$ is not compact.

(i) Any closed and bounded interval $[a, b] \subset \mathbb{R}$ is compact.

Let $\{U_i : i \in I\}$ be a collection of open sets in $\mathbb{R}$ such that $[a, b] \subset \cup_i U_i$. Consider the set

$$E := \{x \in [a, b] : \exists \text{ a finite set } F_x \subset I \text{ such that } [a, x] \subset \cup_{j \in F_x} U_j\}.$$

If $a \in U_i$, then there exists $\varepsilon > 0$ such that $x \in (a - \varepsilon, a + \varepsilon) \subset U_i$ and also $a + \varepsilon < b$. Then $a + \varepsilon/2 \in E$. Let $c =$ l.u.b. E. (Why does it exist?) We claim $c \in [a.b]$, $c = b$ and $c = b \in E$. If $c < b$, then $c \in U_j$. By the argument above, there exists $\varepsilon > 0$ such that $E \ni c + \varepsilon/2 < b$, a contradiction. Hence $c = b$. Repeat the same argument to conclude $b \in E$.

(j) $\mathbb{R}$ with VIP topology is not compact.

(k) $\mathbb{R}$ with outcast topology is compact.

(l) Any set with co-finite topology is compact.

(m) An uncountable set with co-countable topology is not compact.

(n) A finite union of compact sets is compact.

(o) The intersection of two compact sets need not be compact. See, however, Item 193. Consider $\mathbb{Z}$ with the discrete topology. Let $\{\pm\infty\}$ be two distinct elements not in $\mathbb{Z}$. Let $X = \mathbb{Z} \cup \{\pm\infty\}$. We say a subset $U \subset X$ is open if either (i) $U \subset \mathbb{Z}$ or if either of $\pm\infty$ lies in $U \subset X$, then both the elements lie in U and $X \setminus U$ is finite. It is easy to verify that this defines a topology on X. The sets $A := \mathbb{Z} \cup \{\infty\}$ and $B := \mathbb{Z} \cup \{-\infty\}$ are compact but their intersection $\mathbb{Z}$ is not compact. Note that neither A nor B is closed.

192. A closed subset K of a compact space X is compact.

Let $\{U_i : i \in I\}$ be an open cover of K. To exploit the compactness of X, we need an open cover of X. Clearly, if we add the open set $X \setminus K$ to the given open cover of K, we end up with an open cover, say, $\mathcal{U}$ of X. Let $\mathcal{U}_0$ be a finite subcover of X. It is possible $\mathcal{U}_0$ contains $X \setminus K$. In any case, $\mathcal{U}_0 \setminus \{X \setminus K\}$ is a finite subcover of K.

193. Let $K \subset X$ be a compact subset of X. Is K closed in X?

That is, is $X \setminus K$ open? The only way of doing this it for each $x \in X$, to find an open set $U_x \ni x$ such that $U_x \cap K = \emptyset$. We also need to exploit the compactness of K. That is, we need to find an open cover of K (via open sets of X), which do not have x. This suggests that we may require X to be Hausdorff. (All these were arrived at by students!) We have the following result.

In a Hausdorff space a compact subset is closed and hence the intersection of compact sets is compact in a Hausdorff space.

Let X be Hausdorff and $K \subset X$ be compact. We shall show that the complement $K^c := X \setminus K$ is open. Given $p \in K^c$, we need to show that there exists an open set $U_p \ni p$ with $U_p \subset K^c$. We need to exploit Hausdorffness of X and the compactness of K. This means we need to generate an open cover of K. For any $q \in K$, we have disjoint open sets $V_q \ni q$ and $U_{pq} \ni p$. Hence $\{V_q : q \in K\}$ is an open cover of K and hence there exists a finite set $\{q_1, \ldots, q_n\}$ of K such that $K \subset \cup_{i=1}^n V_{q_i}$. Let $U_p := \cap_{i=1}^n U_{pq_i}$. Then $U_p \ni p$ is an open set and it lies in K^c. Thus, $K^c = \cup_{p \in K^c} U_p$ is open.

194. Note that the a proof in the last item establishes the following result.

Let X be a compact Hausdorff space. Let $K \subset X$ be closed and $x \notin K$. Then there exist disjoint open sets $U_x \ni x$ and $U_K \supset K$.

A space X is said to be *regular* if K is closed in X and $x \notin K$, there exist disjoint open sets $U_x \ni x$ and $U_K \supset K$.

Hence a compact Hausdorff space is regular.

Question: In the result about a compact Hausdorff space X, can we replace x by a closed set L disjoint from K?

195. Let (X, d) be a metric space. We say that $A \subset X$ is *bounded* if there exist $x_0 \in X$ and $r > 0$ such that $A \subset B(x_0, r)$. The following are easily seen results about this concept:

(a) A is bounded iff for every $x_1 \in X$, there exists $R > 0$ such that $A \subset B(x_1, R)$. Easy. Observe

$$d(a, x_1) \leq d(a, x_0) + d(x_0, x_1) < r + d(x_0, x_1).$$

Hence let $R = r + d(x_0, x_1)$.

(b) Let $(X, \| \ \|)$ be an normed linear space . Show that $A \subset X$ is bounded iff there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in A$. Easy. $A \subset B(0, M)$ for some $M > 0$ by the last subitem.

(c) Any finite set is bounded.

(d) Any open or closed ball is bounded.

(e) A is bounded iff there exists $M > 0$ such that $d(x, y) \leq M$ for all $x, y \in A$.

(f) If $A \neq \emptyset$, we set $\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}$, which is set to ∞ if the supremum does not exist. The extended real number $\operatorname{diam}(A)$ is called the diameter of A. A set A is bounded iff either $A = \emptyset$ or $\operatorname{diam}(A) < \infty$.

(g) $\operatorname{diam}(B(x, r)) \leq 2r$ and strict inequality can occur.

(h) In an normed linear space , $\operatorname{diam}(B(x, r)) = 2r$. *Hint:* Go through Item 71.

(i) Any convergent sequence in a metric space is bounded.

(j) Boundedness is not a topological property. Already seen in Item 140g.

(k) Which vector subspaces of an normed linear space are bounded subsets?

(l) The set $O(n)$ of all orthogonal matrices (that is, the set of matrices satisfying $AA^t = I = A^tA$) is a bounded subset of $M(n, \mathbb{R})$. Here $M(n, R)$ is considered as an normed linear space as in Ex. 50. Observe that $\|A\|^2 = \sum_i \left(\sum_j |a_{ij}^2|\right) = n$.

(m) The set $SL(n, \mathbb{R})$ of all $n \times n$ real matrices with determinant 1 is not bounded in $M(n, \mathbb{R})$.

(n) The set of all nilpotent matrices in $M(n, \mathbb{R})$ is not a bounded set.

(o) Let G be a subgroup of the multiplicative group $\mathbb{C}^*$ of the non-zero complex numbers. Assume that as a subset of $\mathbb{C}$ it is bounded. Then $|g| = 1$ for all $g \in G$.

196. In a metric space any compact set is bounded in X.

Let K be a compact subset of a metric space X. Fix $a \in X$. Consider $\{B(a, n) : n \in \mathbb{N}\}$ This is an open cover of X and hence the collection $\{B(a, n) : n \in \mathbb{N}\}$ has a finite subcover of K. Since $(B(a, n)$ is increasing, there exists $N \in \mathbb{N}$ such that $K \subset B(a, N)$.

Applications:

(a) $SL(n, \mathbb{R})$ is not a compact subset of $M(n, \mathbb{R})$.

(b) The set of symmetric (respectively, the skew-symmetric) matrices is not compact in $M(n, \mathbb{R})$. So is the set of matrices with trace zero.

(c) The set of nilpotent matrices in $M(n, \mathbb{R})$ is not compact.

197. In any topological space, any convergent sequence along with its limit is a compact subset.

Let $x_n \to x$. Given an open cover $\{U_i : i \in I\}$ of $\{x_n : n \in \mathbb{N}\} \cup \{x\}$, let $x \in U_j$. Then all but finitely many $x_n \in U_j$.

198. If A is a nonempty compact subset of $\mathbb{R}$, then $\sup A$ and $\inf A$ exist and they belong to A.

Let $\beta = \sup A$. Then there exists x_n such that $\beta - \frac{1}{n} < x_n \leq \beta$. Hence $x_n \to \beta$ and hence β is a limit point of A. Heine-Borel says that A is closed.

199. Assume that $f\colon X \to Y$ is continuous and that X is compact. Then $f(X)$ is compact. In particular, compactness is a topological property.

200. The product $X \times Y$ of two spaces is compact iff X and Y are compact.

To understand the proof, draw a picture of $X \times Y$ as a closed rectangle, as explained in Item 167. If $\{U_i \times V_i : i \in I\}$ is an open cover by means of basic open sets, then we have an cover of $\{x\} \times Y$, a "vertical line". Since this is compact, we have a finite subcover which turns out to be an open cover of a (super)set of the form $U_x \times Y$, $U_x \ni x$ open. (A more challenging and instructive exercise could be to carry out this in the case of an open cover of the circle $x^2 + y^2 = 1$ by means of open disks in $\mathbb{R}^2$.) These U_x's cover X and hence they have a finite subcover. Thus we end up with a finite subcover of $X \times Y$.

Let us now work out the details. WLOG, we may assume that we are given an open cover by means of basic open sets as in the last paragraph. Since the inclusion map $x \mapsto (x, y)$ is continuous (Why? See Items 169b and 169c.), $\{x\} \times Y$ is compact by Item 199. Hence there exists a finite subcover, say, $\{U_i \times V_i : i \in F_x\}$ for a finite subset $F_x \subset I$. Then $x \in U_x = \cap_{j \in F_x} U_i$ is an open set. Thus the finite subcover $\{U_i \times V_i : i \in F_x\}$ is an open cover of $U_x \times Y$. As x varies over X, we have an open cover $\{U_x : x \in X\}$ of the compact space X. Let $A \subset X$ be finite such that $\{U_x : x \in A\}$ is an open cover of X. Then the collection $\{U_i \times V_i : i \in F_x, x \in A\}$ is a finite subcover of $X \times Y$.

(Why? Let $(x, y) \in X \times Y$. Since $\{U_a : a \in A\}$ is a finite open cover of X, there exists $a \in A$ such that $x \in U_a$. Hence $(x, y) \in U_a \times Y$. Now, $\{U_j \times V_j : j \in F_a\}$ is a finite open cover of $U_a \times Y$, there exists $j \in F_a$ such that $(x, y) \in U_j \times V_j$, $j \in F_a$, $a \in A$, as claimed.)

201. A more general result known as Tykhonoff's theorem is true, which has very far-reaching applications in analysis.

Theorem 9 (Tykhonoff). *Let $\{X_i : i \in I\}$ be a family of compact spaces. The the product space $\prod_{i \in I} X_i$ with product topology is compact.* □

For a proof, see my article on Compact spaces.

202. Application of the last item: Any cube $[-R, R]^n \subset \mathbb{R}^n$ is compact. This follows from Items 191i and 200.

203. Let K be closed and bounded subset of $\mathbb{R}^n$. Let $R > 0$ be such that $\|x\| \leq R$ for $x \in K$. Then $|x_i| \leq \mathbb{R}$ for $x = (x_1, \ldots, x_n) \in K$. Thus, $K \subset [-R, R]^n$. Hence by the last result,

$[-R, R]^n$ is compact. By Item 192, K is compact. We have thus proved the sufficiency part of the following

Theorem 10 (Heine-Borel)**.** *A subset $K \subset \mathbb{R}^n$ is compact iff K is closed and bounded.*

The necessary part follows from Items 193 and 196.

204. Applications of Heine-Borel theorem.

    (a) Among the non-degenerate conics in $\mathbb{R}^2$, only circles and ellipses are compact.

    (b) The unit sphere $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is compact.

    (c) $O(n, \mathbb{R})$, the set of orthogonal matrices is compact subset of $M(n, \mathbb{R})$.

    (d) The subgroup $SL(n, \mathbb{R})$ is closed and unbounded. It is not a compact subset of $M(n, \mathbb{R})$.

    (e) The set of nilpotent matrices in $M(n, \mathbb{R})$ is closed and unbounded. It is not a compact subset of $M(n, \mathbb{R})$.

    (f) All norms on $\mathbb{R}^n$ are equivalent. Details!

    Application: Any finite dimensional vector subspace of an normed linear space is always closed. *Hints:* If two equivalent norms $\| \ \|_1$ and $\| \ \|_2$ are given on a vector space X, then $(X, \| \ \|_1)$ is complete iff $(X, \| \ \|_2)$ is complete.

205. In general, a closed and bounded subset of a metric space need not be compact. (Standard example. For another, see Item 217h.)

206. Compact sets and maps:

    (a) Assume that $f : X \to Y$ is continuous and that X is compact. Then $f(X)$ is compact. In particular, compactness is a topological property.

    (b) Let X be compact and Y be Hausdorff. Then any continuous bijection $f : X \to Y$ is a homeomorphism.

    We claim that f is a closed map. Let $C \subset X$ be a closed set. Then C is compact by Item 192. Hence $f(C)$ is a compact subset of Y by sub-item (a). Since Y is Hausdorff space, and the compact set $f(C)$ is closed in Y by Item 193.

    This is a very useful result. Some applications are given below.

    i. Typical applications arise in the theory of quotient spaces: The quotient space $[0, 2\pi]/\sim$ is homeomorphic to S^1.

    ii. Let f be any map (not assumed to be continuous) from a compact Hausdorff space X to a compact space Y. Assume that the graph of f is closed as a subset of the product space $X \times Y$. Then f is continuous.

    We have a bijection $\varphi : X \to \text{Graph}(f)$ given by $\varphi(x) = (x, f(x))$. If we show that φ is continuous, then as a component of φ, the function f must be continuous. To use Item 206b, the requirements that the domain and codomain are compact are met. We need a continuous bijection. If $\psi := \varphi^{-1}$, then $\psi(x, f(x)) = x$ is a continuous bijection from the compact space $\text{Graph}(f)$ to the compact Hausdorff X. Hence it is a homeomorphism. We conclude its inverse φ is also continuous.

This may be called a Closed Graph Theorem, in analogy with a result bearing the same name in functional analysis: Let X and Y be complete normed linear spaces, Let $T\colon X \to Y$ be a linear map whose graph is closed in $X \times Y$. Then T is continuous.

iii. Let X be a set with two distinct topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. Assume that $\mathcal{T}_1 \subset \mathcal{T}_2$ and further that $(X, \mathcal{T}_2)$ is compact Hausdorff. Then $(X, \mathcal{T}_1)$ is compact but not Hausdorff.

(c) Let X be compact and Y be a metric space. Then any continuous map $f\colon X \to Y$ is bounded.

Let $f\colon X \to Y$ be a continuous function from a compact space X to a metric space Y. Fix $q \in Y$. Consider $V_n := B(q, n)$. Then $U_n := f^{-1}(V_n)$ is open. The sequence (U_n) is increasing and $\cup_n U_n = X$. Hence $X = U_N$ for some N, that is, $f(X) \subset B(q, N)$.

Note that this also follows form Items 199 and 196.

The converse is not true, in general. See Items 31 and 191m. For metric spaces, the converse is true. For a proof, see my article on Compact Spaces. Details!

(d) Let X be compact. Then any continuous function $f\colon X \to \mathbb{R}$ attains its bounds.

Let X be a compact space and $f\colon X \to \mathbb{R}$ be continuous. By the last sub-item $f(X)$ is a bounded subset of $\mathbb{R}$. Let $M = \sup f(X)$ and $m = \inf f(X)$. If there does not exists any $\in X$ such that $f(a) = M$, then $U_n := \{x \in X : f(x) < M - \frac{1}{n}\}$ is open, $U_n \subset U_{n+1}$ and $\cup_n U_n = X$. By compactness, there exists N such that $f(X) = U_N$. But then $\sup f(X) \leq M - \frac{1}{N}$, a contradiction. Similar proof establishes the existence of $b \in X$ such that $f(b) = m$.

This can also be proved using Item 199, Heine-Borel theorem and Item 198.

Applications:

i. Let X be compact and $f\colon X \to \mathbb{R}$ be continuous. Assume that $f(x) > 0$ for all $x \in X$. Then there is a $\delta > 0$ such that $f(x) \geq \delta$ for all $x \in X$.

ii. Let K be a compact and C a closed subsets of a metric space X such that $K \cap C = \emptyset$. Then $d(K, C) > 0$.

iii. Let K be a nonempty compact subset of a normed linear space X. Then there exists $x \in K$ such that $\|y\| \leq \|x\|$ for all $y \in K$.

(e) Let X and Y be metric spaces. Assume that X is compact. Then any continuous map $f\colon X \to Y$ is uniformly continuous.

Fix $\varepsilon > 0$. For each $x \in X$, let δ_x correspond to $\varepsilon/2$ and the continuity of f at x. Then $\{B(x, \delta_x/2) : x \in X\}$ is an open cover of X. Let $\{B(x_k, \delta_k/2) : 1 \leq k \leq n\}$ be a finite subcover where $\delta_k = \delta_{x_k}$. Let $\delta := \min\{\delta_k/2 : 1 \leq k \leq n\}$.

Let $s, t \in X$ be such that $d(s, t) < \delta$. If $s \in B(x_k, \delta_k/2)$, then $d(t, x_k) \leq d(t, s) + d(s, x_k) < \delta_k$. Hence that

$$d(f(s), f(t)) \leq d(f(s), f(x_k)) + d(f(x_k), t) < \varepsilon.$$

207. Given an open cover $\{U_i : i \in I\}$ of a metric space (X, d), we say that a positive number δ is a *Lebesgue number* of the cover, if for any subset $A \subset X$ whose diameter is less than δ, there exists $i \in I$ such that $A \subset U_i$.

If δ is a Lebesgue number of the cover and $0 < \delta' \leq \delta$, then δ' is also a Lebesgue number of the given open cover.

208. In general, an open cover may not have a Lebesgue number. Let $X = (0, 1)$ with the usual metric. Let $U_n := (1/n, 1)$. Then $\{U_n : n \in \mathbb{N}\}$ is an open cover of X. Does there exist a Lebesgue number for this cover?

Theorem 11 (Lebesgue Covering Lemma)**.** *Let (X, d) be a compact metric space. Let $\{U_i : i \in I\}$ be an open cover of X. Then a Lebesgue number exists for this cover.*

We mimic the argument of Item 206e. For each $x \in X$, if $x \in U_i$, then there exists δ_x such that $B(x, \delta_x) \subset U_i$. Consider the open cover $\{B(x, \delta_x/2) : x \in X\}$ like earlier and arrive at δ, which does the job.

209. Use the last theorem to prove Item 206e. Note that the proofs of Item 206e and Lebesgue covering lemma are also similar.

210. Definition of FIP: A family of subsets $\{F_i : i \in I\}$ of a set X is said to have the *finite intersection property*, (FIP, in short), if every finite collection of members of the family has a nonempty intersection. Examples:

    (a) Let X be any set and (F_n) be a decreasing sequence of nonempty subsets of X. Then $\{F_n : n \in \mathbb{N}\}$ enjoys FIP.

    (b) Let X be noncompact. Then there exists an open cover $\{U_i : i \in I\}$ of X which does not admit a subcover. Consider the family of closed sets $\{F_i : i \in I\}$ where $F_i := X \setminus U_i$. This family of closed sets has F.I.P.

211. A topological space is compact iff every family of closed sets with FIP has a nonempty intersection.

Let X be compact. Let $\{A_i : i \in I\}$ be a family of closed sets with FIP. We are required to show that $\cap_i A_i \neq \emptyset$. Assume on the contrary that $\cap_i A_i = \emptyset$. Let $U_i := X \setminus A_i$. Then $\{U_I : i \in I\}$ is an open cover of X. Since X is compact, there exists a finite set $F \subset I$ such that $\cup_{j \in F} U_j = X$. By taking complements of this equation, we obtain $\cap_{j \in F} A_j = \emptyset$. This contradicts our hypothesis that $\{A_i : i \in I\}$ enjoys FIP.

Converse is exactly along the same lines. If X has the said property, we need to show that X is compact. Let $\{U_i : i \in I\}$ be an open cover of X. Assume that it does not admit a finite subcover. Let $A_i := X \setminus U_i$. Then $\{A_i : i \in I\}$ is a family of closed set with FIP. Hence $\cap_i A_i \neq \emptyset$ which entails $\cup_i U_i \neq X$! □

This characterization is used in the proof of Tykhonoff's theorem.

212. Cantor intersection theorem. This is an analogue of the nested interval theorem of real analysis.

Theorem 12. *Let X be any Hausdorff topological space. Let (K_n) be a decreasing sequence of nonempty compact subsets of X. Then $\cap_n K_n \neq \emptyset$.*

Assume the contrary. Let $U_n := X \setminus K_n$. Then each U_n is open. (Why?) (U_n) is an increasing sequence of open sets whose union is X. Hence $\{U_n : n \in \mathbb{N}\}$ is an open cover for K_1 and hence there exists N such that $K_1 \subset U_N$. That is, $K_1 \subset X \setminus K_N$. Since $K_N \neq \emptyset$, if we select $p \in K_N \subset K_1$, we arrive at a contradiction $p \in K_1 \subset K_N^c$.

213. A subset A of a metric space (X, d) is said to be *totally bounded* if for any given $\varepsilon > 0$, there exist a finite number of points $x_1, \ldots, x_n \in X$ such that $A \subset \cup_{k=1}^n B(x_k, \varepsilon)$.

The finite set $\{x_k : 1 \le k \le n\}$ is usually referred to as an ε-net for A.

214. Examples, non-examples and properties of totally bounded sets.

(a) Any compact subset of a metric space is totally bounded.

(b) If B is totally bounded and $A \subset B$, then A is totally bounded.

(c) If A is totally bounded, so is its closure $\overline{A}$.
If $\{x_k : 1 \le k \le n\}$ is an ε-net for A, then it is 2ε-net for $\overline{A}$.

(d) Any totally bounded subset is bounded. The converse is not true. Standard example: an infinite set with discrete metric. A slightly more demanding example: In ℓ^2, the orthonormal set $\{e_n : n \in \mathbb{N}\}$. An interesting example: $f_n(x) = x^n$, $n \in \mathbb{N}$, in $(C[0,1], \| \ \|_\infty)$.

(e) Any bounded subset of $\mathbb{R}$ is totally bounded. (This is essentially Archimedean property.) In fact, any bounded subset of $\mathbb{R}^n$ is totally bounded.
One can prove this directly. Or, if $A \subset \mathbb{R}^n$ is bounded, so is $K := \overline{A}$. Hence K is closed and bounded. By Heine-Borel, K is compact and hence totally bounded. A being a subset of K is therefore totally bounded by Item 214b.

215. Characterization of compact metric spaces.

Theorem 13. *Let X be a metric space. Then the following are equivalent.*
1. X is compact.
2. X is complete and totally bounded.
3. (Bolzano-Weierstrass property.) *Every infinite subset of X has a cluster point in X.*
4. (Sequential compactness.) *Every sequence in X has a convergent subsequence.* □

For a proof, see my article on compact spaces.

216. Applications of 2nd characterization:

(a) Arzela-Ascoli theorem as a characterization of compact subsets of $(C(X), \| \ \|_\infty)$, where X is a compact metric space. (Perhaps statement only.)

(b) A subset $A \subset \ell_1$ is compact iff A is closed, bounded and is such that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{n \ge N} |x_n| < \varepsilon$ for **all** $x \in A$.

217. Applications of (perhaps the most useful) 4th characterization.

(a) Any continuous map from a compact space to a metric space is bounded.

(b) Any continuous real valued function on a compact space attains its bounds.

(c) Let K be a nonempty compact subset of $\mathbb{R}$. Show that $\sup K, \inf K \in K$. Deduce the last item from this.
Let $\alpha := \inf K$. Then there exists $x \in K$ such that $\alpha \le x_n < \alpha + \frac{1}{n}$. Hence $x_n \to \alpha$. Since K is closed, we obtain $\alpha \in K$.
To deduce the last result, take $K = f(X)$.

(d) Let A, B be disjoint compact subsets of a metric space. Then there exist $a \in A, b \in B$ such that $d(A, B) = d(a, b)$, and hence $d(A, B) > 0$.

This result need not be true if the sets are assumed to be closed. Consider $A := \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ and $B := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0 \text{ and } xy = 1\}$. Then $A + B = \{(x, y) \in \mathbb{R}^2 : y > 0\}$.

(e) Let K be a compact subset and C a closed set in $\mathbb{R}^n$. If $K \cap C = \emptyset$, then there exist $x \in K$ and $y \in C$ such that $d(x, y) = d(K, C)$.

It is easy to see that there exists an $x \in K$ such that $d(K, C) = d(x, C)$. To get y, observe that there exists a sequence (y_n) in C such that $d(x, y_n) \to d(x, C)$. You need to apply Bolzano-Weierstrass theorem to the sequence (y_n).

(f) Let K, C be as in the last item. Then $K + C$ is closed in $\mathbb{R}^n$.

(g) Let X, Y be compact metric spaces. Then $X \times Y$ is compact.

An obvious line of attack needs a careful argument. Observe that 4th characterization applied to the sequences (x_n) and (y_n) may produce subsequences of the form (x_{2n}) and (y_{2n-1}) converging to x and y respectively. This will not help us to produce a convergent subsequence of $(x_n.y_n)$!

If $((x_n, y_n))$ is a sequence in $X \times Y$, by compactness of X, there exists a subsequence (x_{n_k}) which converges to some $x \in X$. Now consider the sequence (y_{n_k}) in the compact metric space Y. Assume a subsequence $(y_{n_{k_r}})$ converges to $y \in Y$. Then the subsequence $(x_{n_{k_r}}, y_{n_{k_r}})$ converges to (x, y) by Item 168b.

(h) Let X denote the normed linear space of all bounded real valued functions on $[0, 1]$ under the sup norm $\| \ \|_\infty$. Then the closed unit ball in X is closed and bounded but not compact.

Recall Item 91b and (x^n) in $C[0, 1]$.

218. **Connected Spaces.** Look at

(a) $\mathbb{R}$, an interval,

(b) a circle, a parabola, an ellipse, two intersecting lines, a disk, a circle, a parabola or an ellipse along with a tangent line at one of its points in $\mathbb{R}^2$,

(c) a plane, a sphere, a ball in $\mathbb{R}^3$.

All of them seem to be in a "single piece." Consider now

(a) $\{-1, 1\}$, $\mathbb{Z}$, $(-1, 0) \cup (0, 1)$ in $\mathbb{R}$,

(b) two (distinct) parallel lines, a hyperbola, two disjoint open disks in $\mathbb{R}^2$,

(c) two distinct parallel planes, the set consisting of the unit ball $B(0, 1)$ along with the plane $x = 2$.

All of these seem to have more than one piece.

219. A topological space X is said to be *connected* if the only subsets of X which are both open and closed are $\emptyset$ and X. If there exists a subset $\emptyset \neq A \neq X$ which is both open and closed, then the space is said to be *disconnected* or not connected.

Clearly, connnectedness is a topological property.

We say that a subset A of a topological space X is connected (or a connected subset of X), if A is a connected space with the subspace topology.

220. If X is not connected, say $\emptyset \neq A \neq X$ is both open and closed, then $B := X \setminus A$ is such that $\emptyset \neq B \neq X$ and it is both open and closed. Hence, X is disconnected iff there exist .... (Complete this sentence.) Thus X has two "pieces" A and B!

One usually calls A or the pair (A, B) as a disconnection of X.

221. A topological space X is connected iff it has the following property: If U and V are nonempty open sets such that $X = U \cup V$, then $U \cap V \neq \emptyset$.

222. A subset A is connected iff the following condition is satisfied: If U and V are open subsets of X such that $U \cap A$ and $V \cap A$ are nonempty and $A \subset U \cup V$, then $U \cap V \cap A \neq \emptyset$.

223. We now give some examples. (More examples will follow once we prove a powerful characterization of connected spaces. See Items 224–225.)

   (a) $\mathbb{R}$ is connected. See Item 57i. Similar proof shows that any interval is connected.
   (b) $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are not connected. See Item 150e.
   (c) Any discrete space with more than one element is disconnected.
   (d) Any indiscrete space is connected.
   (e) Is the empty set connected?

224. The following theorem is a powerful characterization of connected spaces. The theorem remain true if we take Z to be any discrete space with at least two elements, for instance, $\mathbb{Z} \subset \mathbb{R}$ with the subspace topology.

Theorem 14. *Consider $Z := \{\pm 1\} \subset \mathbb{R}$ with subspace topology. A topological space is connected iff any continuous map $f: X \to Z$ is a constant.* □

Let X be connected. Let $f: X \to Z$ be continuous. If $f(a) = 1$ and $f(b) = -1$, let $A := f^{-1}(\{1\})$ and $B := f^{-1}(\{-1\})$. Then A and B are non-empty, open, disjoint, with $X = A \cup B$. Hence X is not connected.

Conversely, if X is not connected, let (A, B) be a disconnection of X. Define $f = 1$ on A and $f = -1$ on B. If V is an open set in $\{\pm 1\}$, then $V = \emptyset$, $V = \{\pm 1\}$, $V = \{1\}$ or $V = \{-1\}$. Their inverse images are $\emptyset$, X, A and B respectively. Hence f is a continuous function from X onto $\{\pm 1\}$.

When we use this result to deal with connectedness of subsets in a topological space, we shall make use of Items 155–156.

225. Applications of the last theorem.

   (a) Any interval is connected. Use intermediate value theorem.
   (b) A subset of $\mathbb{R}$ is connected iff it is an interval. As one can give a direct proof of this, we have the intermediate value theorem as a corollary.
   (c) Let $M(n, \mathbb{R})$ denote the set of all $n \times n$ matrices of real numbers. Then $GL(n, \mathbb{R}) := \{A \in M(n, \mathbb{R}) : \det(A) \neq 0\}$ is not connected.
   (d) $O(n, \mathbb{R}) := \{A \in GL(n, \mathbb{R}) : AA^t = I\}$ is not connected.
   (e) Let X be a topological space. Let A and B be two connected subsets of X such that $A \cap B \neq \emptyset$. Then $A \cup B$ is connected. Generalize this.

(f) Let X be a connected topological space and $g\colon X \to Y$ be a continuous map. Then $g(X)$ is connected. Applications:

   i. Any line segment in an normed linear space is connected.

   ii. The circle $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected. Similarly, the ellipse and parabola are connected.

   iii. $SO(2,\mathbb{R}) := \{A \in O(2,\mathbb{R}) : \det A = 1\}$ is connected.

   iv. $GL(n,\mathbb{R})$ is not connected.

   v. $O(n,\mathbb{R})$ is not connected.

(g) Let X be such that every pair of points of X lies in a connected subset. Then X is connected.

In fact, we can weaken the hypothesis. Let $p \in X$ be fixed. Assume that for any $x \in X$, there exists a connected subset $A_x \subset X$ such that $p, x \in A_x$. Then X is connected.

Applications:

   i. Any star-shaped subset of a normed linear space is connected. A subset E of a normed linear space X is said to be star-shaped at $p \in E$ if for any $q \in E$, the line segment $[p,q] := \{(1-t)p + tq : 0 \leq t \leq 1\} \subset E$.

   ii. A subset C of a normed linear space is said to be convex if it is star-shaped at each of its points. Hence a convex set in a normed linear space is connected.

   iii. It is easy to see that a ball $B(a,r)$ in a normed linear space is convex. If $x, y \in B(a,r)$, we have, for $t \in [0,1]$,

$$
\begin{aligned}
d((1-t)x + ty, a) &= \|(1-t)(x-a) + t(y-a)\| \\
&\leq (1-t)\|x-a\| + t\|y-a\| \\
&< (1-t)r + tr = r.
\end{aligned}
$$

   Consequently, any ball (open or closed) in a normed linear space is connected.

   iv. $\mathbb{R}^2 \setminus \{0\}$ is connected.

   v. $\mathbb{R}^2 \setminus \{(n,0) : n \in \mathbb{Z}\}$ is connected.

   vi. $\mathbb{R}$ with the lower limit topology $\mathcal{T}_L$ is connected. For given any two points, say, $a < b$, the the identity map from $(\mathbb{R}, \mathcal{T}_d)$ to $(\mathbb{R}, \mathcal{T}_L)$ is continuous and hence maps the connected set $[a,b]$ to a connected set.

(h) Let A be a connected subset of a space X. Let $A \subset B \subset \overline{A}$. Then B is connected. Let $f\colon B \to \{\pm 1\}$ be continuous. Restricted to A, f is a constant, say 1. Let $x \in B$. We show that $f(x) = 1$. Let, if possible, $f(x) = -1$. Then there exists a set $U_x \ni x$, open in B, such that $f(U_x) \subset (-3/2, -1/2)$. Since x is a limit point of A, we can find $a \in U_x \cap A$. But then $f(a) = 1 \notin (-3/2, -1/2)$.

Application:

   - Consider the set $L := \{(t,0) : t \in [0,1]\}$, $A_n := \{(1/n, y) : y \in [0,1]\}$ for $n \in \mathbb{N}$ and $A_0 := \{(0,y) : y \in [0,1]\}$. Then $E := L \cup (\cup_n A_n)$ is connected and so its closure, $E \cup A_0$ is connected. Hence the set $E \cup \{(0,1)\}$ is connected. ($X := E \cup A_0$ is known as the comb space.)

(i) Let X be the union of open disk in $\mathbb{R}^2$ along with the tangent line $x = 1$. It is connected.

(j) The open unit disk in $\mathbb{R}^2$ along with any subset of its boundary is connected. (This is geometrically 'obvious'.)

(k) Let $\{A_i : i \in I\}$ be a collection of connected subsets of a space X with the property that for all $i, j \in I$ we have $A_i \cap A_j \neq \emptyset$. Then $A := \cup_i A_i$ is connected.

Applications:

Any star-shaped subset of a normed linear space is connected. In particular, we have

- Any convex subset of a normed linear space is connected.
- Any open/closed ball in any normed linear space is connected.
- Any vector subspace in a normed linear space (in particular $\mathbb{R}^n$) is connected.
- Any coset of a vector subspace in a normed linear space (or $\mathbb{R}^n$) is connected.

(l) Let X be a topological space. Assume that $\{A_i : i \in I\}$ is a family of connected subsets of X. Let L be another connected subset such that $L \cap A_i \neq \emptyset$ for all $i \in I$. Show that $L \cup (\cup_{i \in I} A_i)$ is a connected subset of X.

(m) Let X and Y be topological spaces. Then the product space $X \times Y$ is connected iff both X and Y are connected.

Let $f : X \times Y \to \{\pm 1\}$ be a continuous function. Fix $(a, b) \in X \times Y$. We show that for any $(x, y) \in X \times Y$, we have $f(x, y) = f(a, b)$. Since $\{a\} \times Y$ is connected (why?), the restriction of f to this set is a constant. In particular, $f(a, y) = f(a, b)$. Now the subset $X \times \{y\}$ is connected and hence the restriction of f to it is a constant. In particular, $f(a, y) = f(x, y)$. Hence $f(x, y) = f(a, b)$. Picture!

Applications:

i. $\mathbb{R}^2 \setminus \{(0, 0)\}$ is connected as it is the product of $(0, \infty) \times [0, 2\pi)$.

ii. A cylinder $\{(x, y, z) : x^2 + y^2 = 1\}$ is the product of circle and $\mathbb{R}$ and hence is connected.

(n) The sphere $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is connected.

The case $n = 1$ is already seen. Assume $n > 1$.

Note that $S = S_+ \cup S_-$ where $S_\pm := \{x \in \mathbb{R}^{n+1} : \pm x_{n+1} > 0\}$, union of two closed hemi-spheres. The map $\varphi : S_- \to B[0, 1]$ given by $\varphi(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n)$ is a bijective continuous map whose inverse is $\psi : B[0, 1] \subset \mathbb{R}^n \to S_-$ given by $\psi(u) = \left(u, -\sqrt{1 - \sum u_j^2}\right)$. Hence it is homeomorphism and S_- is connected. Similarly, S_+ is connected. Now the intersection of these two hemi sphere is the equator $\{x_{n+1} = 0\}$. By Item 225e, the sphere is connected.

Alternatively, connectedness of S^n can also seen as follows. Since S^n is the image the polar coordinate map, S^n is connected. For instance, $\varphi : [-\pi/2, \pi/2] \times [0, 2\pi] \to S^2$ is given by $\varphi(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$.

A third way of seeing this is to observe that any point x other than the north pole e_{n+1} lies on a unique great circle and appeal to Item 225g.

Applications:

i. $\mathbb{R}^n \setminus \{0\}$ is connected, $n \geq 2$.

ii. A cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ is connected.

iii. An annular region $\{x \in \mathbb{R}^n : r < \|x\| < R\}$ is connected.

226. We can give a direct proof of Item 225(n)i. Draw pictures in $\mathbb{R}^2$ to understand the proof below. Let $x \in \mathbb{R}^n$ be nonzero. Let $P = \{x \in \mathbb{R}^n : x_n = 1\}$. Then P is homeomorphic to $\mathbb{R}^{n-1}$ and hence is connected (as $n > 1$). We show that any non-zero x lies on a line segment which meets P. Hence by Item 225l it will follow that the set of nonzero vectors in $\mathbb{R}^n$ $(n > 1)$ is connected.

     Let $x \in \mathbb{R}^n$ be nonzero. If $x_j \neq 0$ for some $j < n$, then the line $(x_1, \dots, x_j, \dots, t)$ passes through x, does not contain 0 and it meets P.

     If x_n is the only nonzero coordinate, the line joining x with $(1, 0, \dots, 0, 1)$ is given by $(1-t)(0, \dots, 0, x_n) + (1-t)(1, 0, \dots, 0, 1)$. It contains the given point, does not pass through origin and it meets P.

     We now prove that the sphere is connected. The continuous map $\mathbb{R}^n \setminus \to \mathbb{R}^n$ given by $x \mapsto x/\|x\|$ has the sphere as its image.

227. Connectedness can be used to settle questions on homeomorphisms:

     (a) The set of irrational numbers in $\mathbb{R}$ with subspace topology is not homeomorphic to $\mathbb{R}$.
     (b) A hyperbola cannot be homeomorphic to $\mathbb{R}$.
     (c) $\mathbb{R}$ cannot be homeomorphic to $\mathbb{R}^2$.
     (d) A pair of intersecting lines cannot be homomorphic to a parabola.
     (e) The set A of two distinct parallel lines in $\mathbb{R}^2$ is not connected. Hence a pair of intersecting lines cannot be homomorphic to A.

228. A finite metric space is connected iff is a singleton.

229. Let X be connected and $f : X \to \mathbb{R}$ be a continuous non-constant function. Show that $f(X)$ is uncountable.

230. Let X be a connected metric space with at least two elements. There X "has at least as many elements as $\mathbb{R}$." In particular, X is uncountable.

231. What are all the continuous functions from $f : \mathbb{R} \to \mathbb{R}$ that take only rational values?

232. Are there continuous functions $f : \mathbb{R} \to \mathbb{R}$ that take irrational values at rational numbers and rational values at irrational numbers?

233. Let $f : [a, b] \to \mathbb{R}$ be continuous. "Identify" the image $f([a, b])$.

234. Let f be a one-one continuous function on an interval. Then f is monotone.

235. What are all the continuous functions from a connected space to (i) a discrete space, (ii) a finite Hausdorff space?

236. Let $f : X \to Y$ be a continuous map from a connected space X **onto** a finite Hausdorff space? What can you conclude about Y?

237. Let X and Y be topological spaces and $f : X \to Y$ be a map. We say that f is *locally constant* if for each $x \in X$, there exists an open set U_x containing x with the property that f is a constant on U_x.

238. Show that any locally constant function is continuous.

239. Let $U \subset \mathbb{R}^n$ be a nonempty open set. Let $f: U \to \mathbb{R}$ be a differentiable function with derivative 0. Then f is locally constant. (It need NOT be a constant function!)

240. Let X be connected and Y be Hausdorff. Then any locally constant function $f: X \to Y$ is a constant function on X.

Fix $p \in X$. We show that $f(x) = f(p)$ for $x \in X$. Define a subset $A := \{x \in X : f(x) = f(p)\}$. Then $p \in A$. Hence p is non-empty. We shall show that A is both open and closed. Since X is connected, it will follow that $A = X$.

Let $x \in A$. Since f is locally constant, there exists an open set $U \ni x$ on which f is a constant. Hence for any $z \in U$, $f(z) = f(x) = f(p)$. That is, $U \subset A$. Hence A is open.

Let q be a limit point of A. Let $U_q \ni q$ be an open set on which f is a constant. Since q is a limit point of A, there exists $a \in A \cap U_q$. Hence $f(z) = f(a)$ for all $z \in U_q$, in particular, $f(q) = f(a) = f(p)$. Hence $q \in A$, that is, A is closed. Hence $A = X$.

This is a typical way in which connectedness hypothesis is used. If a result has connectedness as hypothesis, define a set which reflects what we want to prove and show that the set so-defined is non-empty, open and closed. Learn this proof well. For another example, refer to Item 252.

241. In Item 239, if we further assume that U is connected, then f is a constant.

242. **Path-connected Spaces.** A continuous map $\alpha: [a, b] \to X$ to a topological space X is called a *path*. Since any two intervals are homeomorphic, it is a standard practice to assume that $a = 0$ and $b = 1$. The point $p := \alpha(0)$ is called the initial point and $q := \alpha(1)$ is called the terminal point of the path α. We also say that p is path connected to q by the path α.

Note that if p is connected to q by a path to $\alpha: [0, 1] \to X$ with $\alpha(0) = p$ and $\alpha(1) = q$, then the *reverse* path $\tilde{\alpha}: [0, 1] \to X$ defined by $\tilde{\alpha}(t) := \alpha(1 - t)$ is a path from q to p.

Thus if p is connected to q by a path iff q is connected to p by a path. Because of this we simply say that p and q are path-connected, without specifying which is the initial point etc.

243. It is important not to identify the path α with its image $\alpha([0, 1])$ in X. (It is called the trace of α. Mnemonic: the trains could be different but the tracks may be the same.) The paths $\alpha, \beta: [0, 1] \to \mathbb{R}^2$ given by $\alpha(t) = (t, 0)$ and $\beta(t) = (t^3, 0)$ have the same trace.

244. Two point p and q may be connected by more than one path. Think of at least 3 different paths connecting $(-1, 0)$ to $(0, 1)$ in $\mathbb{R}^2$.

245. If x and y are path-connected and y and z are path-connected in a space, then x and z are path connected.

This is an application of gluing lemma. Assume that $\alpha: [0, 1] \to X$ connects x to y and $\beta: [0, 1] \to X$ connects y to z. Then the map $\gamma: [0, 1] \to X$ defined as

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{if } 0 \leq t \leq 1/2 \\ \beta(2t - 1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Since $\alpha(1) = y = \beta(0)$, we can apply gluing lemma to conclude that γ is path connecting x to z.

246. We say that a topological space X is *path-connected* if any two points of X are connected by a path.

247. X is path connected iff there exists $p \in X$ such that any point $x \in X$ is path connected to p.

248. Any path connected space is connected.

Let X be path connected. Let $f\colon X \to \{\pm 1\}$ be continuous. Let $p, q \in X$ and $\alpha\colon [0,1] \to X$ be a path joining p to q. Now $f \circ \alpha\colon [0,1] \to \{\pm 1\}$ is continuous and hence is a constant. In particular, $f(p) = f(\alpha(0)) = f(\alpha(1)) = f(q)$, Hence f is a constant.

Or, observe that any two points lie on the trace of a path, which is connected. Hence, by Item 225g, X is connected.

249. The converse is not true. Two examples:

(a) Comb space: Let $L := \{(x, 0) : 0 \le x \le 1\}$ and $A_n := \{(1/n, y) : 0 \le y \le 1\}$, for $n \in \mathbb{N}$. Let $P = \{(0, 1)\}$. Then $L \cup (\cup_{n\in\mathbb{N}} A_n)$ is connected and its closure contains $X := L \cup (\cup_{n\in\mathbb{N}} A_n) \cup \{P\}$. Hence X is connected. It is not path connected. If possible, let γ be path joining P to $Q = (1, 0) \in X$. Choose an open disk $B(P, r)$ which does not meet the x-axis. Let $[0, \delta)$ be such that $\gamma(t) \in B(P, r)$ for $t \in [0, \delta)$. Let $\gamma = (\gamma_1, \gamma_2)$. Then $\gamma_1([0, \delta))$ is a connected subset of $B(P, r) \cap X$. It follows that $\gamma_1(t) = 0$ for $t \in [0, \delta)$. Hence $\gamma_1(\delta) = 0$. We Let $t_0 := \sup\{t \in [0, 1] : \gamma_1(t) = 0\}$. Then $\gamma_1(t_0) = 0$. We claim that $t_0 = 1$. If not, repeat argument using the continuity of γ at t_0. We then get there exists $s > t_0$ such that $\gamma_1(s) = 0$. Thus we conclude that $\gamma(t) = P$ for $t \in [0, 1]$.

(b) Topologist's sine curve. Consider

$$X := \{(x, \sin(1/x)) : x > 0\} \cup \{(x, 0) : -1 \le x \le 0\} = A \cup B \text{ (say.)}$$

Clearly each of A and B is connected. Also, the point $(0, 0)$ is a limit point of the set A and hence $A_1 = A \cup \{(0, 0)\} \subset \overline{A}$ is connected. Since B and A_1 have a point in common their union X is connected.

We claim that X is not path-connected. In fact, we show that there is no path connecting $(1/\pi, 0)$ with $(0, 0)$. Let $\gamma\colon [0, 1] \to X$ be path such that $\gamma(0) = (1/\pi, 0)$ and $\gamma(1) = (0, 0)$. Then $\pi_1 \circ \gamma$ must take all values that lie between 0 and $1/\pi$. In particular, there exist $t_n \in [0, 1]$ such that $\pi_1 \circ \gamma(t_n) = \frac{1}{(2n+\frac{1}{2})\pi}$. Then, $\gamma(t_n) \to (0, 1)$ as $n \to \infty$. By Bolzano-Weierstrass, there exists a convergent subsequence, (t_{n_k}). Let t_0 be the limit of this subsequence. Then $\pi_1 \circ \gamma(t_{n_k}) \to 0$. Thus, $\gamma(t_0)$ must be $(0, y)$ for some y. Since $\gamma(t_0) = (0, y) \in X$, it follows that $y = 0$. But, $\pi_2 \circ \gamma(t_0) = \lim \pi_2 \circ \gamma(t_{n_k}) = 1$. This contradiction shows that there is no such path γ. □

250. The continuous image of a path connected space is path connected.

Let $f\colon X \to Y$be continuous with X path connected. Let $Y = f(X)$. Given $y_j = f(x_j) \in Y$, $j = 1, 2$. Let γ be a path connecting x_1 to x_2. Then $f \circ \gamma$ is a path connecting y_1 to y_2.

An application. The proof in Item 226 showed that $\mathbb{R}^n \setminus \{0\}$, $n \geq 2$, is path connected. Hence S^n, being a continuous image of $\mathbb{R}^n \setminus \{0\}$ is also path connected.

251. The product space of path connected spaces is path connected.

Let $x = (x_i), y = (y_i) \in \prod_i X_i$. Let γ_i be a path connecting x_i to y_i in the space X_i. Define $\gamma(t) := (\gamma_i(t))$. Then γ is a path connecting x to y .

An application. The third proof of the connectedness of S^n in Item 225n established its path connectedness. Hence $\mathbb{R}^n \setminus \{0\} = (0, \infty) \times S^{n-1}$ is path connected.

252. Any open subset of a normed linear space is connected iff it is path connected.

Let A be a connected open subset of a normed linear space X. Fix $p \in A$. It suffices to show that there is a path connecting p to any $q \in A$. (See Item 247.) Let

$$E := \{x \in A : x \text{ is path-connected to } p\}.$$

From here onwards, the proof is exactly similar to the one in Item 240.

Clearly, $p \in E$ and hence $E \neq \emptyset$. Let $x \in E$. Then there exists an open ball $B(x, r) \subset A$, since A is open. Now any $z \in B(x, r)$ is connected to x via the line segment $t \mapsto (1 - t)z + tx$. Since $x \in E$, x is path connected to p. Hence by Item 245, z is path-connected to p and hence $B(x, r) \subset E$. We conclude that E is open.

Let $q \in A$ be a limit point of E **in** A. As earlier, there exists $B(q, r) \subset A$. Since q is a limit point of E, there exists $z \in B(q, r) \cap E$. Now, q is path connected by the line segment $(1 - t)q + tz$ to z which in turn is path connected to p, as $z \in E$. Hence q is path connected to p, or $q \in E$. Hence E is closed.

253. **Connected Components.** In a topological space X, the relation $x \sim y$ if there exists a connected set A with $x, y \in A$ is an equivalence relation. The equivalence classes are called the *connected components* or components of X. The following are immediate:

(a) If C is a component, then C is a closed connected set.

(b) Any component C is a maximal connected set in the sense that if A is connected and $C \subset A$, then $C = A$.

(c) If C is a component, $x \in C$ and if A is a connected set with $x \in A$, then $A \subset C$.

254. Examples of components:

(a) The only component of a connected space X is X.

(b) The components of a discrete space are the singleton sets.

(c) The components of $\mathbb{Q}$ are the singleton sets. (Note that the topology on $\mathbb{Q}$ is not discrete topology. We gave two proofs of this. One is direct use of subspace topology and another used existence of non trivial convergent sequences.) Details!

(d) What are the components of $\mathbb{R}$ with VIP topology? with outcast topology?

255. If $f : X \to Y$ is a homeomorphism, then f induces a natural bijective correspondence between the components of X and those of Y: If C is a component of X, then $f(C)$ is a component of Y. Application: The pair of intersecting lines is not homeomorphic to $\mathbb{R}$. (If they are, remove the point of intersection from the pair of lines and its image from $\mathbb{R}$. Count the components.)

256. Path components are defined in an obvious way. If C_x (resp. P_x) is the component (resp. path-component) containing $x \in X$, then $P_x \subseteq C_x$.

257. Going through the proof in Item 252, we are led to the concept of locally path connected spaces. First of all a definition.

258. Let X be a topological space and $x \in X$. A subset U is called a *neighbourhood* of x in X if there exists an open set G such that $x \in G \subset U$. Example: $[0, 1)$ is a neighbourhood of any $x \in (0, 1)$ but not of $x = 0$.

259. A set in a topological space is open iff it is a neighbourhood of each of its points.

260.

Locally P spaces

General Philosophy: Let P be a topological property. We say that a space X is *locally P* (or enjoys P locally) if for each $x \in X$ and an open set $U \ni x$, there exists a neighbourhood N of x where N has the property P and $N \subset U$.

261. Let X be a topological space. Then X is said to be *locally path connected* if for each $x \in X$ and an open set $U \ni x$, there exists a path connected neighbourhood N of x such that $N \subset U$.

Now you can similarly define *locally connected* and *locally compact* spaces.

Do you see the need for introducing the notion of neighbourhoods? If we replace a neighbourhood by an open set in the locally P spaces, what will happen if we wanted a Hausdorff space to be locally compact?

262. The proof of Item 252 yields the following result: An open set in a locally path connected space is connected iff it is path-connected.

263. An important remark: In general X may have property P but it may not be locally P. For instance, the complete comb space is connected but not locally connected. (Look for a connected neighbourhood of the point $(0, 1)$.) Similarly, there exists a compact space (Item 274c) which is not locally compact. (Do NOT get confused with the 'bad' definition of Munkres and hence his "note" that any compact space is locally compact!)

Similarly, the space X may be locally P, but X may not enjoy P. For instance, consider $\mathbb{R}$ with discrete topology. Then it is locally connected, locally path-connected and locally compact. But it is not connected, not path connected and not compact.

264. A space X is locally connected iff the components of any open subset (with subspace topology) are open in X. In particular, the components of X are open. Details!

265. The components in a locally path connected space are open.

266. Let U be an open subset of a locally path connected space. Then U is connected iff it is path-connected.

267. In a locally path connected space, the components and path components are the same.

268. Can we define locally $\mathbb{R}^n$ or locally Euclidean spaces?

We say that a space X is *locally Euclidean* or locally $\mathbb{R}^n$ if for each $x \in X$, there exists a neighbourhood $U_x \ni x$ which is homeomorphic to a neighbourhood in $\mathbb{R}^n$. (Note that n is fixed.)

269. Can we define locally Hausdorff spaces? Is it necessarily Hausdorff?

Consider $X = \mathbb{R}^* \cup \{\theta_1, \theta_2\}$ where θ_j are two elements not in $\mathbb{R}^*$. (We shall think of them as "two zeros" or "the zero with split personality!") As a local basis for $x \in \mathbb{R}^*$, we take $\{(x - 1/k, x + 1/k) : k \in \mathbb{N}\}$. At θ_j, we take $\{(-1/k, 0) \cup \{\theta_j\} \cup (0, 1/k) : k \in \mathbb{N}\}$. Then we get a topological space which is locally Euclidean and hence it is locally Hausdorff. However, it is not Hausdorff.

270. Locally Compact Spaces:

271. The following are descendants of Item 193.

(a) Let K be a compact subset of a Hausdorff space X and $x \notin K$. Then there exist disjoint open sets U and V such that $x \in U$ and $K \subset V$. (This is Item 194.)

(b) Let A and B be disjoint compact subsets of a Hausdorff space. Then there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

(c) Let X be a compact Hausdorff space. Let A and B be disjoint closed subsets of X. Then there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

272. A space X is said to be *normal* if given two disjoint closed sets A and B, there exist disjoint open sets $U \supset A$ and $V \supset B$.

Last item shows that a compact Hausdorff space is normal.

273. Another example of a normal space is any metric space.

To appreciate this, look at $A = \{(x, y) \in \mathbb{R}^2\}$, the set of axes which are asymptotes of the rectangular hyperbola $B := \{(x, y) \in \mathbb{R}^2 : xy = 1\}$.

We now prove the result. If $a \in A$, then A is a not a limit point of B. Hence $d(a, B) > 0$, by Item There exists $r_a > 0$ such that $B(a, r_a) \cap B = \emptyset$. $U := \cup_{a \in A} B(a, r_a)$. We can do similarly for B.

274. Examples of locally compact spaces:

(a) $\mathbb{R}$, $\mathbb{R}^n$ are locally compact.

(b) $\mathbb{Q}$ is not locally compact.

(c) A compact space need not be locally compact. Example: Consider $\mathbb{Q}$ with the usual topology, adjoin an extra element, say ∞. The neighbourhoods of $x \in \mathbb{Q}$ are either the neighbourhoods of x in $\mathbb{Q}$ or ∞ added to the standard neighbourhoods. The neighbourhoods of ∞ are complements in $\mathbb{Q}$ of a finite subset of F along with ∞.

(d) An normed linear space is locally compact iff it is finite dimensional. (One way is easy; the proof of the other is omitted.)

(e) A locally compact metric space need not be complete. A trivial example is $(0, 1)$!

Theorem 15. *The following are equivalent for a Hausdorff space:*

1. *X is locally compact.*

2. *For every $x \in X$ and a neighbourhood U of x, there exists an open set V such that $x \in V$, $\overline{V}$ is compact and $\overline{V} \subset U$.*

3. *Each $x \in X$ has a compact neighbourhood.*

Proof. (1) $\implies$ (2): Let $x \in X$ and K be a compact neighbourhood of x. Then there exists an open set $V \subset K$ with $x \in V$. Since X is Hausdorff, K is closed. Hence $\overline{V} \subset K$. Hence $\overline{V}$ being a closed subset of a compact set K, is compact.

(2) $\implies$ (3): Take $\overline{V}$ of (2).

(3) $\implies$ (1): □ Details!

Since locally compact spaces such as $\mathbb{R}^n$ arise quite often, whenever we say X is locally compact, we shall assume that X is Hausdorff also.

275. Local compactness is a topological property. In fact, more is true: Let $f\colon X \to Y$ be a continuous open map of a locally compact space X onto Y. Then Y is locally compact.

Let $y \in Y$ and $V \ni y$ be open. Let $x \in X$ be such that $f(x) = y$. Then $U := f^{-1}(V)$ is an open set with $x \in U$. Since X is locally compact, there exists an open set $W \ni x$ with $\overline{W}$ compact and $W \subset U$. Since f is open $f(W) \ni y$ is open, since $\overline{W}$ is compact, $f(\overline{W})$ is compact. Thus, $f(\overline{W})$ is a compact neighbourhood of y.

276. A closed (respectively open) subspace of a locally compact space is locally compact.

277. A Hausdorff topological space X is called an *n-dimensional topological manifold* if for each $p \in X$, we can find an open set $U_p \ni p$ such that U_p is homeomorphic to an open subset of $\mathbb{R}^n$ for n fixed. Thus, a manifold is a Hausdorff space which is locally Euclidean.

Typical examples are (i) open subset of $\mathbb{R}^n$ and (ii) $S^n \subset \mathbb{R}^{n+1}$. A non-example is a pair of intersecting lines in $\mathbb{R}^2$. Modern topology deals mostly with manifolds.

278. Given $X = (0, 1] \subset \mathbb{R}$, by adding just the point 0, we can make it to be compact. Note that $(0, 1]$ is dense in $[0, 1]$.

Similarly, the subspace topology on the set $\{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$ is discrete. If we add the point 0 to it, then the resulting space is compact in which the original set is dense.

Can we so something similar to any locally compact, non-compact Hausdorff space X?

That is, can we add a new point, which is denoted by ∞ to X and obtain a compact Hausdorff space? Let us work backwards. Assume $X_\infty := X \cup \{\infty\}$ is compact Hausdorff. We would like to retain open subset of X in tact. So we need to provide a local base at ∞. If $U \ni \infty$ is an open set , then $X_\infty \setminus U$ is a closed subset of the compact space X_∞ and hence is compact. But, it is in fact a subset of X. This suggests a way of defining a local base at ∞, namely, a subset $U \ni \infty$ is open if $X_\infty \setminus U$ is a compact subset of X.

279. **One point compactification.** Given a locally compact noncompact Hausdorff space X, let $X_\infty := X \cup \{\infty\}$ where $\infty \notin X$. Let $\mathcal{T}$ denote the topology on X. Consider

$$\mathcal{T}_\infty := \mathcal{T} \cup \{V \subset X_\infty : X_\infty \setminus V \text{ is a compact subset of } X.\}.$$

Then Details!

(i) $\mathcal{T}_\infty$ is a Hausdorff topology on X_∞.
(ii) The subspace topology on X is $\mathcal{T}$.
(iii) $(X_\infty, \mathcal{T}_\infty)$ is compact.
(iv) X is dense in X_∞.

280. Let X be noncompact, locally compact Hausdorff space. Let Y be a compact Hausdorff space. Assume that there exists $q \in Y$ and a homeomorphism $f\colon X \to Y \setminus \{q\}$. Then the one point compactification X_∞ of X is homeomorphic to Y. Details!

281. Examples:

   (a) $\mathbb{R}^n \cup \{\infty\} = S^n$.

   (b) Let $x\colon \mathbb{N} \to X$ be a sequence in X. Then $x_n \to x_\infty$ iff the function $x\colon \mathbb{N}_\infty \to X$ defined by $x(n) = x_n$ and $x(\infty) = x_\infty$ is continuous at ∞.

   Application: Use this to give another solution of Item 168b.

   (c) Let X be a discrete space. What is its one point compactification?

282. Functions vanishing at infinity: Let X be a locally compact Hausdorff space. A continuous function $f\colon X \to \mathbb{R}$ is said to *vanish at infinity* if for any given $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ for $x \notin K$. (We can also define continuous function vanishing at ∞ for functions taking values in a normed linear space in an obvious way.)

   A continuous function $f\colon X \to \mathbb{R}$ vanishes at infinity iff it extends to a continuous function $f_\infty\colon X_\infty \to \mathbb{R}$ with $f_\infty(\infty) = 0$.

   (a) Let $f\colon X \to \mathbb{R}$ be given. Its *support* is by definition the **closure** of the set $\{x \in X : f(x) \neq 0\}$, that is,

   $$\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}.$$

   We say that f has compact support if the support of f is compact. Evidently, any continuous function with compact support vanishes at infinity.

   (b) What are the entire functions $f\colon \mathbb{C} \to \mathbb{C}$ which vanish at infinity?

283. A closely related concept is proper maps between (locally compact Hausdorff) spaces. See ???? Details!

   This concept is so important that **any** proof of Fundamental theorem of algebra has to either directly or indirectly use the fact that the any non-constant polynomial with complex coefficients when considered as a map from $\mathbb{C}$ to $\mathbb{C}$ is proper.

284. A subset $A \subset X$ of a topological space is said to be *nowhere dense* in X, if given any nonempty open set U, we can find a nonempty open subset $V \subset U$ such that $A \cap V = \emptyset$.

   This definition is equivalent to the standard one found in all text-books: A is nowhere dense in X iff the interior of the closure of A in X is empty: $\operatorname{Int}(\overline{A}) = \emptyset$.

285. Prototype examples of nowhere dense sets:

   (a) Let V be any proper vector subspace of $\mathbb{R}^n$. More generally, any proper vector subspace of a normed linear space .

   (b) The set of zeros of any polynomial map $\mathbb{R}^n \to \mathbb{R}$.

286. **Baire Category theorem.** We shall give the formulation of Baire category theorem in a form which will be more useful than the one which uses the notion of category.

Theorem 16. *Let (X, d) be a complete metric space.*

(1) *Let U_n be open dense subsets of X, for $n \in \mathbb{N}$. Then $\cap_n U_n$ is non-empty.*
(2) *X cannot be a countable union of nowhere dense closed subsets F_n.*

We first observe that both the statements are equivalent. For, G is open and dense iff its complement $F := X \setminus G$ is closed and nowhere dense. Hence any one of them follows from the other by taking complements. So, we confine ourselves to proving the first. In fact, we shall show that $\cap_n U_n$ is dense in X.

The basic idea is to get into a situation like nested interval theorem. Since we need to exploit completeness, we need to produce a Cauchy sequence whose limit is likely to be in the intersection of U_n's. If we have a nested sequence of open balls, say, $(B(x_n, r_n)$ such that $B(x_n, r_n) \subset B(x_{n-1}, r_{n-1})$, we get a sequence (x_n). If we sih to show it is Cauchy, the only obvious estimate available (for $n > m$) is

$$f(x_n, x_m) \leq d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \leq \sum_{k=m}^{n} r_k.$$

Thus we are lead to make the sequence (r_n) of radii as the terms of a convergent sequence. The standard way of doing this is to demand $0 < r_n < 2^{-n}$.

We can also replace (U_n) by a nested sequence (V_n) of open dense sets. Define $V_1 := U_1$. Having defined V_n, define $V_{n+1} = U_{n+1} \cap V_n$. Clearly, V_1 is open dense. Assume that we have shown V_n is open dense. Let U be any nonempty open set. We need to show that $U \cap V_{n+1} \neq \emptyset$. Observe that

$$U \cap V_{n+1} = (U \cap V_n) \cap U_{n+1}.$$

Since by induction $U \cap V_n$ is nonempty open, it must have nonempty intersection with the dense set U_{n+1}. Thus we have produce a nested sequence (V_n) of open dense sets with $\cap_n V_n = \cap_n U_n$. We now show that $\cap_n V_n$ is dense. Let $B(p, r)$ be an open ball. Since V_1 is dense, there exists $x_1 \in V_1 \cap B(p, r)$. Since the intersection is open, there exists a positive $r_1 < 1/2$ such that $B[x_1, r_1] \subset V_1 \cap B(p, r)$. Repeating the same argument with $B(x_1, r_1) \cap V_2$, we find x_2 and $0 < r_2 < 2^{-2}$ such that

$$B[x_2, r_2] \subset B(x_1, r_1) \cap V_2 \subset V_1 \cap V_2 \cap B(p, r).$$

By induction we get sequences (x_n) and (r_n) such that

$$B[x_n, r_n] \subset B(x_{n-1}, r_{n-1}) \cap V_n \cap \cdots \cap V_1 \cap V_2 \cap B(p, r).$$

Clearly (x_n) is Cauchy. Since X is complete, (x_n) converges, say, to $x \in X$. Observe that $\{x_k : k \geq n\} \subset B[x_n, r_n]$. Hence x is a limit point of the closed ball $B[x_n, r_n]$ so that $x \in B[x_n, r_n]$. Since this is true for all n, we obtain $x \in B(p, r) \cap (\cap_n V_n)$. The theorem is proved.

Let $U := \cap_n U_n$. We have to prove that U is dense in X. Let $x \in X$ and $r > 0$ be given. We need to show that $B(x, r) \cap U \neq \emptyset$. Since U_1 is dense and $B(x, r)$ is open there exists $x_1 \in B(x, r) \cap U_1$. Since $B(x, r) \cap U_1$ is open, there exists r_1 such that $0 < r_1 < 1/2$ and $B[x_1, r_1] \subset B(x, r) \cap U_1$. We repeat this argument for the open set $B(x_1, r_1)$ and the dense set U_2 to get $x_2 \in B(x_1, r_1) \cap U_2$. Again, we can find r_2 such that $0 < r_2 < 2^{-2}$

and $B[x_2, r_2] \subset B(x_1, r_1) \cap U_2$. Proceeding this way, we get for each $n \in \mathbb{N}$, x_n and r_n with the properties

$$B[x_n, r_n] \subset B(x_{n-1}, r_{n-1}) \cap U_n \text{ and } 0 < r_n < 2^{-n}.$$

Clearly, the sequence (x_n) is Cauchy: if $m \leq n$,

$$d(x_m, x_n) \leq d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \leq \sum_{k=m}^{n} 2^{-k}.$$

Since $\sum_k 2^{-k}$ is convergent, it follows that (x_n) is Cauchy.

Since X is complete, there exists $x_0 \in X$ such that $x_n \to x_0$. Since x_0 is the limit of the sequence $(x_n)_{n \geq k}$ in the closed set $B[x_k, r_k]$, we deduce that $x_0 \in B[x_k, r_k] \subset B(x_{k-1}, r_{k-1}) \cap U_k$ for all k. In particular, $x_0 \in B(x, r) \cap U_k$ for all $k \in \mathbb{N}$.

287. A most often used corollary of Baire's theorem is the following: If a complete metric space X can be written as a countable union of closed sets F_n, then at least one F_n will have a nonempty interior.

288. Applications:

   (a) $\mathbb{R}^n$ cannot written as the union of a countable family of its proper vector subspaces. In particular, $\mathbb{R}^2$ is not the union of a countable family of lines through the origin.

   (b) No infinite dimensional complete normed linear space can be countable dimensional. (Algebraic sense!)

   (c) There can exist no metric d on $\mathbb{Q}$ such that d induces the usual topology on $\mathbb{Q}$ and $(\mathbb{Q}, d)$ is complete.

   (d) Let (X, d) be complete and $f_n \colon X \to \mathbb{R}$ be a sequence of continuous functions. Assume that $f_n \to f$ pointwise on X. Then the set $A := \{x \in X : f \text{ is continuous at } x\}$ is dense in X.

   Proof. Our proof is a beautiful application of both versions of Baire's theorem. Fix $\varepsilon > 0$. Define, for each $k \in \mathbb{N}$,

$$E_k(\varepsilon) := \{x \in X : |f_n(x) - f_m(x)| \leq \varepsilon, \text{ for all } m, n \geq k\}.$$

   Then we claim that $E_k(\varepsilon)$ is closed for each k.

   Reason: Fix $m, n \geq k$. Since $|f_n - f_m|$ is continuous, the set

$$E_k^{m,n}(\varepsilon) := \{x \in X : |f_n(x) - f_m(x)| \leq \varepsilon\}$$

   is a closed subset of X. Now, since $E_k(\varepsilon) = \cap_{m,n \geq k} E_k^{m,n}(\varepsilon)$, the claim follows.

   It is easy to show that $X = \cup_k E_k(\varepsilon)$.

   Reason: Let $x_0 \in X$. Since $f_n(x_0) \to f(x_0)$, the sequence $(f_n(x_0))$ is Cauchy. Hence for the given $\varepsilon > 0$, there exists k_0 such that for $m, n \geq k_0$, we have $|f_m(x_0) - f_n(x_0)| \leq \varepsilon$. Hence we conclude that $x_0 \in E_{k_0}(\varepsilon)$.

Since $X$is a complete metric space, at least one of $E_k(\varepsilon)$ should have nonempty interior. Let $U_\varepsilon := \cup_k \text{Int}\,(E_k(\varepsilon))$. Then U_ε is a nonempty open subset of X.

Let $U_n := U_{1/n}$. We claim that each U_n is dense in X.

Reason: It is enough if we show that every closed ball $B := B[x,r]$ meets U_n non-trivially. (Why?)

Reason: To show a set A is dense in a metric space, it suffices to show that $A \cap B(x,r) \neq \emptyset$ for any $x \in X$and $r > 0$. Assume that $A \cap B[z,\rho] \neq \emptyset$ for any $z \in X$ and $\rho > 0$. Then given any $B(x,r)$, we may take $z = x$ and $\rho = r/2$. Then $\emptyset \neq A \cap B[x,\rho] \subset A \cap B(x,r)$.

Observe that the closed set (and hence a complete metric space) B is the union of a countable family of closed sets: $B = \cup_n (B \cap E_k(1/n))$. By Baire, at least one of them has nonempty interior, say, $\text{Int}\,(B \cap E_k(1/n)) \neq \emptyset$. Since $\text{Int}\,(B \cap E_k(1/n)) \subset B \cap \text{Int}\, E_k(1/n)$, it follows that $B[x,r] \cap U_n \neq \emptyset$ and hence the claim is proved.

Let $D := \cap_n U_n$. By Baire, D is dense in X. We claim that every $x \in D$ is a point of continuity of f.

Reason: Fix $p \in D$. Let $\varepsilon > 0$ be given. Choose $N \gg 0$ such that $1/N < \varepsilon$. Since $p \in D$, $p \in U_N$ and hence there exists $k \in \mathbb{N}$ such that $p \in \text{Int}\,(E_k(1/N)$. By continuity of f_k at p, there exists an open neighbourhood V of p contained in $\text{Int}\, E_k(1/N)$ such that

$$|f_k(x) - f_k(p)| < \varepsilon, \text{ for all } x \in V. \qquad (1)$$

For $x \in V$, since $V \subset E_k(1/N)$, by the definition of $E_k(\varepsilon)$'s, we have

$$|f_m(x) - f_k(x)| \leq 1/N, \text{ for all } m \geq k. \qquad (2)$$

Letting $m \to \infty$ in the above equation, we obtain

$$|f(x) - f_k(x)| \leq 1/N, \text{ for all } x \in V. \qquad (3)$$

We are now ready for the kill. We claim that $|f(x) - f(p)| < 3\varepsilon$ for $x \in V$.

$$\begin{aligned} |f(x) - f(p)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(p)| + |f_k(p) - f(p)| \\ &\leq 1/N + \varepsilon + 1/N \\ &< 3\varepsilon. \end{aligned}$$

This shows that f is continuous at every point of D.

□

289. An amusing exercise: Let (x_n) be any sequence of real numbers. Show that the set $\{x \in \mathbb{R} : x \neq x_n, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$. Hence conclude that $\mathbb{R}$ is uncountable. Details!

290. **Baire category theorem for locally compact spaces.** Let X be a locally compact Hausdorff space. Let (U_n) be a sequence of open dense sets in X. Then $\cap_n U_n$ is dense in X.

Let G be a nonempty open set in X. We need to prove that there exists $x \in G$ such that $x \in U_n$ for all n. The strategy is to mimic the proof in the case of metric spaces replacing open balls by the existence of open sets V such that $\overline{V}$ is compact and $x \in V \subset \overline{V} \subset U$ for any given open set U and $x \in U$ and then invoking Cantor intersection theorem for a decreasing sequence of compact sets.

Since G is a nonempty open set and U_1 is dense, there exists $x_1 \in G \cap U_1$. Since $G \cap U_1$ is open, $x \in G \cap U_1$ and X is locally compact hausdorff space, there exists an open set V_1 such that $x \in V_1$, $\overline{V}_1$ is compact and $\overline{V}_1 \subset G \cap U_1$. Assume, by way of induction, that we have chosen x_i, $V_i \ni x_i$, $\overline{V}_i$ is compact and that $x_i \in V_i \subset \overline{V}_i \subset V_{i-1} \cap U_i$, for $1 \leq i \leq n$.

Now given a nonempty open set V_n, since $V_n \cap U_{n+1}$ is nonempty, there exists $x_{n+1} \in V_n \cap U_{n+1}$. Since X is locally compact and hausdorff, there exists an open set $V_{n+1} \ni x_{n+1}$ such that $\overline{V}_{n+1}$ is compact and $x_{n+1} \in V_{n+1} \subset \overline{V}_{n+1} \subset V_n \cap U_{n+1}$. Let $K_n := \overline{V}_n$. Thus we have a decreasing sequence (K_n) of nonempty compact subsets. Hence by Cantor intersection theorem, there exists $x \in \cap_n K_n$. Since $x \in K_n = \overline{V}_n \subset U_n$, it follows that $x \in \cap U_n$. Also, $x \in K_1 \subset U$. □

291. Locally closed sets: A subset A of a topological space is *locally closed* if for every $a \in A$, there exists an open set U_a in X such that $a \in U_a$ and $U_a \cap A$ is closed **in** U_a.

    (a) A characterization of locally closed sets: $A \subset X$ is locally closed iff there exist an open set U and a closed set C such that $A = U \cap C$.

    (b) The characterizations gives us easy examples of locally closed sets: $[0, 1)$ is neither closed nor open in $\mathbb{R}$ but is locally closed in $\mathbb{R}$.

292. Separation axioms. They deal with separating various kinds of disjoint objects by means of disjoint open sets that contain the given objects. The prominent ones are given below.

    (a) Hausdorff spaces: Given two distinct points $x \neq y$, if we can find open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

    (b) Regular spaces: Given a point x and a closed set F with $x \notin F$, there exist open sets U and V such that $x \in U$ and $V \subset V$ with $U \cap V = \emptyset$.

    (c) Normal spaces: Given two disjoint closed sets A, B, there exist open sets U, V such that $A \subset U$ and $B \subset V$ with $U \cap V = \emptyset$.

    (d) Completely regular spaces: Given two disjoint (nonempty) closed sets, we can find disjoint a continuous function $f : X \to \mathbb{R}$ such that $f = 0$ on A and $f = 1$ on B.

    (e) Clearly, a completely regular space is regular. How about completely Hausdorff and completely normal spaces? These could be the spaces the objects under question are separated by means of continuous real valued functions. Make precise definitions.

        These spaces will be useful for analysts since they assure that there is an 'abundant' supply of real valued continuous functions on the given space!

293. Some standard examples and facts concerning the above concepts:

    (a) Examples of regular spaces.

        i. Any metric space is regular.
        Let A be a closed subset of a metric space X and $x \notin A$. Let $U := X \setminus A$. Then U is open and $x \in U$. hence there exists $r > 0$ such that $B(x, 3r) \subset U$. Then the open sets $B(x, r)$ and $X \setminus B[x, 2r]$ are open set which separate x and A.

ii. Any locally compact Hausdorff space is regular. Let A be a closed subset of a locally compact space X and $x \notin A$. Then $x \in X \setminus A$ and hence there exists an open set U such that $\overline{U}$ is compact and $x \in U \subset \overline{U} \subset X \setminus A$. The open sets U and $X \setminus \overline{A}$ separate x and A.

(b) Examples of normal spaces.

i. Any metric space is normal.
We give two proofs of this.
Let A and B disjoint closed subsets of a metric space X. For each $a \in A$, since $a \notin B$, a is not a limit point of B. hence there exists $r_a > 0$ such that $B(a, 2r_a) \cap B = \emptyset$. Similar analysis holds for each $b \in B$. Now consider $U := \cup_{a \in A} B(a, r_a)$ and $V := \cup_{b \in B} B(b, r_b)$. Then U and V are open sets containing A and B respectively. If $x \in U \cap V$. then $x \in B(a, r_a) \cap B(b, r_b)$ for some $a \in A$ and $b \in B$. We observe

$$d(a, b) \leq d(a, x) + d(x, b) < r_a + r_b \leq 2 \max\{r_a, r_b\}.$$

Thus, $a \in B(b, 2r_b)$ if $r_b \geq r_a$ or $b \in B(a, 2r_a)$ if $r_a \geq r_b$. This contradicts our choice of r_a etc. Hence $U \cap V = \emptyset$.
The second is based on Urysohn's lemma for metric spaces. See Item 295.

ii. Any compact Hausdorff space is normal.
We adapt the argument which showed that in a Hausdorff space, compact sets are closed. Let A and B be disjoint closed subsets of a compact Hausdorff space X. Fix $x \in A$. For each $b \in B$, there exist open sets $U_b \ni a$ and $V_b \ni b$ such that $U_b \cap V_b = \emptyset$. Since B is compact, the open cover $\{V_b : b \in B\}$ admits a finite subcover, say, $B \subset V := \cup_{b \in F} V_b$ for a finite subset $F \subset B$. Consider $U_a := \cap_{b \in F} U_b$. Then U_a, being a finite intersection of open sets, is open and $a \in U_a$. Clearly, $U_a \cap V = \emptyset$. Note that this argument shows that a compact Hausdorff space is regular.
Given $a \in A$, by the last paragraph, there exist open sets $U_a \ni a$ and $V_a \supset B$ such that $U_a \cap V_a = \emptyset$. Now the open cover $\{U_a : a \in A\}$ of A admits a finite subcover, say, $\{U_a : a \in G\}$ for a finite subset $G \subset A$. Let $U := \cup_{a \in G} U_a$ and $V := \cap_{a \in G} V_a$. It is easy to see that U and V separate A and B.

(c) A normal space in which all singleton sets are closed is regular.

294. The most important result about normal spaces is the Urysohn's lemma.

Theorem 17 (Urysohn's Lemma)**.** *Let A, B be disjoint non-empty closed subsets of a normal space. Then there exists a continuous function $f \colon X \to [0, 1]$ such that $f = 0$ on A and $f = 1$ on B.* □

295. We prove Urysohn's lemma in the case of a metric space. Look at

$$f(x) := \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Note that f makes sense, as the denominator is nonzero. For, $d(x, A) + d(x, B) = 0$ implies that each of the non-negative terms is zero. That is, $d(x, A) = 0$ and $d(x, B) = 0$. Hence x is a limit point of the closed sets A and B (Item ???) and hence $x \in A$ and $x \in B$, a contradiction. By Item ??, f is continuous. Clearly, $f(x) = 0$ iff $x \in A$ and $f(x) = 1$ iff $x \in B$. (This is stronger than what is required!)

296. Note that Urysohn's lemma says that a space is normal iff it is completely normal.

297. A key fact needed for Urysohn's lemma for normal spaces is the following observation.

Lemma 18. *A space X is a normal space iff for each closed set F and an open set V containing A there exists an open set U such that $F \subset U \subset \overline{U} \subset V$.*

Let X be normal and F, V as above. Then F and $X \setminus V$ are disjoint closed sets. By normality of X there exist open sets U and W such that $F \subset U$ and $X \setminus V \subset W$ and $U \cap W = \emptyset$. Since $U \subset X \setminus W$ and $X \setminus W$ is closed, we see that $\overline{U} \subset X \setminus W \subset V$. Thus U is as required.

To see the converse, let A and B disjoint closed subsets of X. Let $V_1 := X \setminus B$. Then $V_1 \supset A$ is an open subset. Hence by hypothesis, there exists U_1 such that $A \subset U_1 \subset \overline{U_1} \subset V_1$. Let $V_2 = X \setminus \overline{U_1}$. Then $V_2 \supset B$ is an open set. Let U_2 be an open set such that $B \subset U_2 \subset \overline{U}_2 \subset X \setminus U_1$. We claim that $U_1 \cap U_2 = \emptyset$. For if $x \in U_1 \cap U_2$, then $x \in U_1$ and $x \in U_2 \subset (X \setminus \overline{U_1}$ and hence $x \notin \overline{U_1}$. Since $U_1 \subset \overline{U}_1$, this is a contradiction. □

298. A clean and neat proof of Urysohn's lemma is in Munkres.

A key step in the proof of Urysohn's lemma is the construction of a sequence (U_n) of open set index by dyadic rations in [0,1].

Lemma 19. *Let X be a normal space. If A and B are closed subsets of X, for each dyadic rational $r = k2^{-n} \in (0,1]$, there is an open set U_r with the following properties: (i) $A \subset U_r \subset X \setminus B$, (ii) $\overline{U}_r \subset U_s$ for $r < s$.*

Let $U_1 := X \setminus B$. By the last lemma, there exist disjoint open sets V and W such that $A \subset V$ and $B \subset W$. Let $U_{1/2} = V$. Then, since $X \setminus W$ is closed, we have

$$A \subset U_{1/2} \subset \overline{U}_{1/2} \subset X \setminus W \subset X \setminus B = U_1.$$

Applying the same lemma once again to the open set $U_{1/2}$ containing A and to the open set U_1 containing $\overline{U}_{1/2}$, we get open sets $U_{1/4}$ and $U_{3/4}$ such that

$$A \subset U_{1/4} \subset \overline{U}_{1/4} \subset U_{1/2} \subset \overline{U}_{1/2} \subset AU_{3/4} \subset \overline{U}_{3/4} \subset V.$$

Continuing this manner, we construct, for each dyadic rational $r \in (0,1)$, an open set U_r with the following properties:
(i) $\overline{U}_r \subset U_s$, $0 < r < s \leq 1$.
(ii) $A \subset U_r$, $0 < r \leq 1$. (iii) $U_r \subset U_1$, $0 < r \leq 1$.

More formally, we proceed as follows. We select U_r for $r = k2^{-n}$ by induction on n. Assume that we have chosen U_r for $r = k2^{-n}$, $0 < k < 2^n$, $1 \leq n \leq N-1$. To find U_r for $r = (2j+1)2^{-N}$, $0 \leq j < 2^{N-1}$, observe that $\overline{U}_{j2^{1-N}}$ and $X \setminus U_{(j+1)2^{1-N}}$ are disjoint closed sets. So once again appealing to the last lemma, we can choose an open set U_r such that

$$\overline{U}_{j2^{1-N}} \subset U_r \subset \overline{U}_r \subset U_{(j+1)2^{1-N}}.$$

These U_r's are as desired. □

299. We are now ready to prove

Theorem 20. Urysohn's Lemma. *A space X is a normal space iff the following is true: For any two disjoint closed subsets A and B of X there exists a continuous function $f\colon X \to [0,1]$ such that $f = 0$ on A and $f = 1$ on B.*

Let U_r's be as in the lemma of the last item. We define the function f so that the sets ∂U_r are the level sets of f for the value r. We achieve this by defining

$$f(x) = \begin{cases} 0, & x \in U_r \text{ for all } r \\ \sup\{r : x \notin U_r\}, & \text{otherwise.} \end{cases}$$

Clearly, $0 \le f \le 1$, $f = 0$ on A and $f = 1$ on B. We need only establish the continuity of f.

Let $x \in X$ be such that $0 < f(x) < 1$. Let $\varepsilon > 0$. Choose dyadic rationals r and s in $(0,1)$ such that $f(x) - \varepsilon < r < f(x) < s < f(x) + \varepsilon$. Then $x \notin U_t$ for dyadic rationals $t \in (r, f(x))$. By (i), $x \notin \overline{U}_r$. On the other hand $x \in U_s$. Hence $W = U_s \setminus \overline{U}_r$ is an open neighbourhood of x. If $y \in W$, then from the definition of f we see that $r \le f(y) \le s$. In particular, $|f(y) - f(x)| < \varepsilon$ for $y \in W$. Thus f is continuous at x. The cases when $f(x) = 0$ or 1 are easier and left to the reader.

300. We now prove Tietze extension theorem, an important tool for analysts and topologists.

301. The standard proof of Tietze extension theorem runs as follows.

  (a) Let $f_0 = f$ and $M_0 = \sup\{|f_0(y)| : y \in Y\}$. Define

$$\begin{aligned} A_0 &:= \{y \in Y : f_0(y) \ge M_0/3\} \\ B_0 &:= \{y \in Y : f_0(y) \le -M_0/3\}. \end{aligned}$$

  Then A_0 and B_0 are disjoint closed subsets of Y. Let $g_0\colon X \to [-M_0/3, M_0/3]$ be a continuous function such that $g_0 = M_0/3$ on A_0 and $g_0 = -M_0/3$ on B_0.

  (b) Let $f_1 := f_0 - g_0$ on Y. Let $M_1 := \sup\{|f_1(y)| : y \in Y\}$. Observe that $M_1 \le (2M_0)/3$. Apply the construction of the last subitem to f_1 to obtain a continuous function g_1 on X. What properties does g_1 have?

  (c) Repeating the constructions of the two subitems above, we get a sequence (g_n) of continuous functions $g_n\colon X \to \mathbb{R}$. The bounds M_n on g_n ensure the applicability of Weierstrass M-test to conclude that the series $\sum_n g_n$ is uniformly convergent on X. Let $g := \sum_n g_n$. Then g is an extension as required.

302. The argument outlined in the last item can be abstracted as follows.

Lemma 21. *Let $\mathbf{X}$ and $\mathbf{Y}$ be complete normed linear spaces. Let $T\colon \mathbf{X} \to \mathbf{Y}$ be a continuous linear map. Assume that for $y_0 \in \mathbf{Y}$ there exist constants M and $r \in (0,1)$ such that there exists $x \in \mathbf{X}$ such that $\|x\| \le M\|y_0\|$ and $\|y_0 - Tx\| \le r\|y\|$. Then there exists $z \in \mathbf{X}$ such that $Tz = y_0$ with $\|z\| \le M/(1-r)$.*

Let $y \in \mathbf{Y}$ be given. We may assume without loss of generality that $\|y\| = 1$. Given $y \in \mathbf{Y}$ let $z_1 = x$ as given in the lemma. For $y_0 = y - Tz_1$, we can find a $z_2 \in \mathbf{X}$

such that $\|z_2\| \leq M\|y - Tz_1\| \leq Mr$ and $\|y - Tz_1 - Tz_2\| \leq r\|y - Tz_1\| \leq r^2$. Proceeding by induction, we get a sequence (z_n) in $\mathbf{X}$ such that (i) $\|z_n\| \leq Mr^{n-1}$ and (ii) $\|y - \sum_{i=1}^n Tz_i\| \leq r^n$. The series $\sum_{n=1}^\infty z_n$ converges to an element $z \in \mathbf{X}$. We have $Tz = y_0$. □

303. We now prove Tietze theorem in the following form.

Theorem 22 (Tietze Extension Theorem)**.** *Let X be a normal space and Y a closed subset of X. Let $f \in \mathbf{Y} := C_b(Y, \mathbb{R})$. Then there exists a $g \in \mathbf{X} := C_b(X, \mathbb{R})$ such that $g(y) = f(y)$ for all $y \in Y$ and $\sup\{g(x) : x \in X\} = \sup\{f(y) : y \in Y\}$.*

Let $T \colon \mathbf{X} \to \mathbf{Y}$ denote the restriction map $g \mapsto g_{|_Y}$. We show that T satisfies the hypothesis of the previous lemma. Without loss of generality, assume that $|f(y)| \leq 1$ for all $y \in Y$. Let $A := f^{-1}([-1, -1/3])$ and $B := f^{-1}([1/3, 1])$. Then A and B are closed in Y and hence in X. By Urysohn's lemma, there exists a $g \in \mathbf{X}$ such that $|g(x)| \leq 1/3$ for $x \in X$ and $g = -1/3$ on A and $g = 1/3$ on B. One easily checks that $\|Tg - f\|_{\mathbf{X}} \leq 1/3$. If we take $M = 1/3$ and $r = 2/3$, then T satisfies the previous lemma. Note that the assertion about the equality of the norms is also obtained. □

304. Exercises.

(a) Let X be a normal space and F a closed subset. Assume that $f \colon F \to (-R, R)$ be a continuous function. Then f can be extended to a continuous function from X to $(-R, R)$. *Hint:* You may need Urysohn's lemma.

(b) Let X be a normal space and F a closed subset. Assume that $f \colon F \to \mathbb{R}$ be a continuous function. Then f can be extended to a continuous function from X to $\mathbb{R}$. *Hint:* $\mathbb{R}$ is homeomorphic to $(-1, 1)$.

(c) Assuming Tietze extension theorem, prove Urysohn's lemma.
Consider $f \colon A \cup B \to \mathbb{R}$ where $f = 0$ on A and 1 on B.

(d) A topological space is normal iff every continuous function from a closed subset to [0,1] extends to a continuous function from X to [0,1].
By the last item, Urysohn's lemma is valid for X. Use Item 296.

(e) Let A be a closed subset of a normal space X. Let $f \colon A \to S^n$ be continuous. Show that there exists an open set $U \supset A$ (U depends on f) and an extension g of f to U.

(f) Show that with the notation of Exer. 304e that f may not extend to all of X. *Hint:* What happens (i) if $n = 0$ and X is connected or (ii) if $X := B[0, 1] \subset \mathbb{R}^{n+1}$, $A := S^n$ and f is the identity?

305. **An example for practice.**

Consider $X = \mathbb{R}$ with the topology $\mathcal{T}$ consisting of sets of the form $U \setminus A$ where U is open in the standard topology $\mathcal{T}_d$ and $A \subset \mathbb{R}$ is any countable subset.

(a) A subset F is closed in $\mathcal{T}$ if $F = E \cup B$ where E is closed in $\mathcal{T}_d$ and B is a countable subset.

(b) If $E = U \setminus A$ is open in $\mathcal{T}$, show that the closure of E in $\mathcal{T}$ is the closure of E in $\mathcal{T}_d$.

(c) Is $\mathbb{Q}$ dense in $(X, \mathcal{T})$?
(d) What are compact subsets in $\mathcal{T}$?
(e) Show that any open cover of $(X, \mathcal{T})$ admits a countable subcover.
(f) Show that $(X, \mathcal{T})$ is not first countable.
(g) Show that any countable subset is closed in $\mathcal{T}$ and hence $(X, \mathcal{T})$ is not separable.
(h) Show that $(X, \mathcal{T})$ is connected but not path-connected.

306. **Quotient spaces.** In the next few items (307–318), we shall deal with quotient spaces. We refer the reader to our article on on "Quotient Spaces."

The best way to develop intuition on quotient topology is to start with a lot of examples and use paper models. Let us look at a few of them.

(a) Take a piece of string, which represents say the interval $[0, 1]$. Glue the ends. We get a 'loop'. Thus, we expect that if we identify the endpoints of the interval the resulting 'topological space' is homeomorphic to a circle.
(b) Take a rectangular piece of paper. Glue a pair of opposite sides, say, the horizontal sides. We get a cylinder. Thus we expect to get a cylinder if we identify the opposite sides of a square or a rectangle.
(c) If we glue the pair of horizontal sides and again the pair of vertical sides, we get a 'cycle tube'. We thus expect to get a 'torus' (circle $\times$ circle) if we identify pairs of opposite sides of square.
(d) Take circular mat/coaster made of cloth. If we sew the entire rim/boundary of the mat, we seem to get the surface of a drop of a liquid. We therefore expect to get a space homeomorphic to a (2-dimensional sphere) if we identify/collapse all points on the boundary of the closed unit disk in $\mathbb{R}^2$.
(e) Consider $\mathbb{R}^2$. If we collapse all points on a vertical line parallel y-axis and if we do this for all such vertical lines, we seem to get a line. This suggests that if we identify vertical lines parallel to y-axis with their point of intersection on the x-axis, the resulting space is homeomorphic to $\mathbb{R}$.
(f) Let us take a circle, say of radius 1 with centre at the origin. If we identify pairs of diametrically opposite points, we end up with closed semi-circle whose end points are to be glued. We seem to end up with a circle again.

Below, we develop a theory with which we shall be able to establish all the above examples rigorously.

307. We recalled concept of quotient topology. Let X be a set and $\sim$ be an equivalence relation on X. Let $X/\sim$ be the quotient set or the set of equivalence classes of $\sim$. Let $\pi\colon X \to X/\sim$ be the *quotient map* defined by $\pi(x) = [x]$, the equivalence class of x. The quotient topology on $X/\sim$ is the set of $V \subset X/\sim$ such that $\pi^{-1}(V)$ is open in X.

308. Let X be a topological space and $\sim$ an equivalence relation on X. Then the quotient topology on $X/\sim$ is the largest topology for which the natural quotient map $\pi\colon X \to X/\sim$ is continuous.

309. The theorem below, though easy, is the 'only' result needed to check the continuity of maps from quotient spaces to others.

Theorem 23 (Universal Mapping Property)**.** *Let $\pi\colon X \to X/\sim$ be a quotient map. A map $f\colon X/\sim \to Y$ is continuous iff $f \circ \pi$ is continuous.* $\square$

If f is continuous, then $f \circ \pi$ is continuous. To see the converse, let V be open in Y. Then $(f \circ \pi)^{-1}(V)$ is open in X. That is, $\pi^{-1}(f^{-1}(V))$ is open in X. By the definition of quotient topology, $f^{-1}(V)$ is open in $X/\sim$. $\square$

310. The next theorem tells us how to generate quotient spaces.

Theorem 24. *Let $f\colon X \to Y$ be continuous. Let $\sim$ be the equivalence relation on X defined by $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. Then there exists a continuous function $g\colon X/\sim \to Y$ such that $f = g \circ \pi$.* $\square$

It is trivial to see that that $\sim$ is an equivalence relation. If we set $g([x]) := f(x)$, then g is well-defined and we have $f = g \circ \pi$. In view of universal mapping property, g is continuous. $\square$

Note that g is one-one. For, if $g([x_1]) = g([x_2])$, then $f(x_1) = f(x_2)$ by the definition of g. Hence $x_1 \mathrm{Re}\, x_2$ or $[x_1] = [x_2]$.

311. Keep the notation of the last item. If f is onto, then g is onto. Thus, if $f\colon X \to Y$ is continuous and onto, we have a continuous bijection $\tilde{f} \equiv g\colon X/\sim \to Y$. Can we think of conditions under which $\tilde{f}$ becomes a homeomorphism?

312. The next result gives us a recipe to identify the quotient spaces. If we have some guess that the quotient space $X/\sim$ is homeomorphic to Y, we try to find a surjective continuous map $f\colon X \to Y$ such that the equivalence relation defined by f is $\sim$ and such that f is either open or closed.

Theorem 25. *Let $f\colon X \to Y$ be an open (or closed) continuous surjective map. Let $\sim$ be the equivalence relation defined by f: $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. Then Y is homeomorphic to the quotient space $X/\sim$ via the map $\tilde{f}\colon X/\sim \to Y$ defined by $\tilde{f}(x) = f(x)$.*

The proof is essentially an exercise in set theory. We have already seen that $\tilde{f}$ is a bijective continuous map. We show that if $W \subset X/\sim$ is open (resp. closed) we show that $\tilde{f}(W)$ is open (resp. closed) in Y.

If you draw a picture for this set-up, you will be led to conclude that $\tilde{f}(W) = f(\pi^{-1}(W))$ for *any* set $W \subset X/\sim$. Assume that this is true. Now the proof is clear. If W is open/closed, by continuity of π, the set $\pi^{-1}(W)$ is open/closed in X. If f is open/closed, it follows that the set $f(\pi^{-1}(W))$ is open/closed. Since $\tilde{f}(W) = f(\pi^{-1}(W))$, we conclude that $\tilde{f}(W)$ is open/closed. Thus $\tilde{f}$ is bijective, continuous and open/closed and hence is a homeomorphism. This completes the proof.

Let us attend to the claim: $\tilde{f}(W) = f(\pi^{-1}(W))$ for *any* set $W \subset X/\sim$. If $y \in \tilde{f}(W)$, then $y = \tilde{f}([x])$ for some $[x] \in W$. Hence $y = f(x)$, by the definition of $\tilde{f}$. Since $\pi(x) = [x] \in W$, clearly $x \in \pi^{-1}(W)$. It follows that $y \in f(\pi^{-1}(W))$. The reverse inclusion is similar. Let $y \in f(\pi^{-1}(W))$. Then $y = f(x)$ for some $x \in \pi^{-1}(W)$. Hence $\pi(x) = [x] \in W$. Hence $y = f(x) = \tilde{f}([x])$ for some $[x] \in W$. $\square$

313. Theorem 25 is the analogue of the first fundamental theorems of homomorphisms in algebra. Let us look at group theory. If H is a normal subgroup of a group G and if we have a guess that the quotient group G/H is isomorphic to K, to proves this what all we need to do is this: find a surjective group homomorphism $f\colon G \to K$ whose kernel is H. The induced map $\tilde{f}\colon G/H \to K$ is a bijective homomorphism and hence is an isomorphism. As we have already see, in topology we need the inverse of the induced bijection $\tilde{f}$ to be continuous.

314. Illustrations of the use of Theorem 25.

(a) The quotient space obtained from $[0,1]$ got by identifying the end points 0 and 1 is S^1.

Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Consider the map $f\colon [0,1] \to S^1$ given by $f(t) := e^{2\pi i t}$. The induced map $\tilde{f}\colon [0,1]/\sim\, \to S^1$ is a bijective continuous from a compact space to a Hausdorff space and hence is a homeomorphism.

(b) The quotient space got by identifying two of the opposite sides of a rectangle is homeomorphic to a finite closed cylinder.

Let the square be $X = [0,1] \times [0,1]$. Let the cylinder be $Y := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 0 \le z \le 1\}$. Consider the map $f\colon X \to Y$ defined by $f(u,v) := (\cos 2\pi u, \sin 2\pi u, v)$.

(c) The quotient space of S^1 obtained by identifying the diametrically opposite points is again S^1!

Note that the diametrically opposite points are $\pm z$, $z \in S^1$. An obvious map f such that $f(z) = f(-z)$ is $f(z) = z^2$. Consider $f\colon S^1 \to S^1$ given by $f(z) = z^2$.

(d) The quotient space of the unit square identifying the corresponding points of the horizontal sides as well as the points on the vertical sides is homeomorphic to $S^1 \times S^1$, a torus (a vada or a cycle tube).

(e) For any space X and a subset A of X, the space X/A stands for the quotient space of X with respect to the equivalence: $x_1 \sim x_2$ iff $x_1 = x_2$ or $x_1, x_2 \in A$. Thus X/A is the space obtained from X by collapsing A to a single point.

Example: $D^n := B[0,1] \subset \mathbb{R}^n$. Then $D^n/S^{n-1} \simeq S^n$.

(f) Let A be a closed subset of a compact Hausdorff space X. The quotient space obtained from X by identifying A to a single point is homeomorphic to the one-point compactification of $X \setminus A$.

Let $Y = (X \setminus A) \cup \{\infty\}$ be the one-point compactification of $X \setminus A$. The obvious map is $f\colon X \to Y$ is given by $f(x) = x$ if $x \notin A$ and $f(a) = \infty$ if $a \in A$. It is easy to see that f is continuous and does the job.

(g) The last item may be used to prove that $D^n/S^{n-1} \simeq S^n$.

(h) If $X = S^1 \times [0,1]$ is the cylinder and $A = S^1 \times \{0\}$ is the bottom circle, then $X/A \simeq D^2$.

Consider $f\colon X \to D^2$ defined by $f(x,y,z) = (zx, zy)$.

How did one think of such a map? We visualize the cylinder as a stack of circles on the plane $z = r$ where $r \in [0,1]$ and the disk as the union of concentric circles of radius $r \in [0,1]$. So, we map the circle $(\cos t, \sin t, r)$ to $(r\cos t, r\sin t)$.

(i) 1-dimensional real projective space. Consider $X = \mathbb{R}^2 \setminus \{(0,0)\}$. Define an equivalence relation by setting $(x_1, y_1) \sim (x_2, y_2)$ iff there exists (necessarily) nonzero $t \in \mathbb{R}$ such that $t(x_1, y_1) = (x_2, y_2)$. The quotient set can be thought of as the set of lines passing though the origin (minus the origin, if you wish). The circle $S^1 \subset \mathbb{R}^2$ meets each equivalence class at two points which are antipodes (diametrically opposite to each other). Hence we expect the quotient space to be the same as in Item 314c. The map $f: \mathbb{R}^2 \setminus \{(0,0)\} \to S^1$ defined by $f(r\cos t, r\sin t) = (\cos 2t, \sin 2t)$ will be as required.

315. We say that an equivalence relation $\sim$ on X is open if whenever $U \subset X$ is open in X so is its *saturation* $[U] := \{x' \in X : x' \sim x \text{ for some } x \in U\}$.

Proposition 26. *An equivalence relation $\sim$ on X is open iff the quotient map $\pi: X \to X/\sim$ is open.* □

316. Hausdorffness of quotient spaces. The following result is the most useful (sufficient) condition on $\sim$ that ensures the quotient space is Hausdorff.

Theorem 27. *Let $\sim$ be an open equivalence relation on X. Assume that the relation $R := \{(x,y) \in X \times X : x \sim y\}$ is closed as a subset of $X \times X$. Then $X/\sim$ is Hausdorff.* □

317. Projective spaces over $\mathbb{R}$. Let $X := \mathbb{R}^{n+1} \setminus \{0\}$. The relation on X defined by $x \sim y$ iff $x = ty$ for some nonzero $t \in \mathbb{R}$ is a equivalence. The quotient $X/\sim$ is known as the n-dimensional projective space over the reals. It is denoted by $\mathbb{P}^n(\mathbb{R})$. The following are some of the properties of $\mathbb{P}^n(\mathbb{R})$.

(a) $\mathbb{P}^n(\mathbb{R})$ is a compact Hausdorff space.

(b) $\mathbb{P}^n(\mathbb{R})$ is homeomorphic to the quotient of S^n with respect to the relation on S^n: $x \sim y$ iff $x = \pm y$.
In this we had to deal with the continuity of a map *into* the quotient space. Go through the proof again. It shows the typical way in which the continuity of a map $f: Y \to X/\sim$ into a quotient space is dealt with. (Universal mapping property cannot deal with this situation.) The trick was to write f as the composite of a continuous map $g: Y \to X$ followed by the quotient map $\pi: X \to X/\sim$.

(c) The one dimensional projective space is homeomorphic to S^1.

318. Two very popular and important examples of quotient spaces.

(a) Möbius Strip. On the unit square X we define the equivalence relation as follows:

$$(x,y) \sim (x',y') \iff (x,y) = (x',y') \text{ or } \{x,x'\} = \{0,1\} \text{ and } y = 1 - y'.$$

Thus two points of opposite vertical sides are identified *cross-wise*. The quotient space is known as the Möbius strip.

(b) Klein's bottle. Let X be the unit square. Define an equivalence relation on X whose nontrivial relations are given by

$$(0,y) \sim (1,y) \quad \text{and} \quad (x,0) \sim (1-x,1).$$

The quotient space is called the Klein's bottle.

319. Let X be a topological space. A *loop* in X is a path $\alpha\colon [0,1] \to X$ with $\alpha(0) = \alpha(1)$. We say that α s a loop based at $\alpha(0)$.

Recall that if $\alpha, \beta\colon [0,1] \to X$ are paths such that $\alpha(1) = \beta(0)$, then their *join* $\alpha * \beta$ is defined by

$$\alpha * \beta(t) := \begin{cases} \alpha(2t) & \text{for } 0 \le t \le 1/2 \\ \beta(2t-1) & \text{for } 1/2 \le t \le 1. \end{cases}$$

Then, $\alpha * \beta$ is continuous (by gluing lemma) and we say that it is got by concatenation.

Standard Notation in homotopy theory: Let $I = [0,1]$.

320. Let X, Y be topological spaces. Let $f, g\colon X \to Y$ be continuous maps. We say that they are homotopic if there exists a continuous map $F\colon X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. We say that $f_t(x) := F(x,t)$ for $t \in I$ and $x \in X$.

The map F is called a homotopy from f to g and we write $f \overset{F}{\simeq} g$.

If $f(a) = g(a)$ for all $a \in A \subset X$ and if the homotopy F is such that $F(a,t) = f(a)$ for all $t \in I$ and $a \in A$, we say that f is homotopic to to g relative to A. We denote this by $f \overset{F}{\simeq} g$ rel A.

If α and β are paths in X with the same initial and terminal points, then saying that α is homotopic to β relative to $\{0,1\}$ is the same as saying that all the intermediate paths $\alpha_t(s) := F(s,t)$ have the same initial and terminal points, that is, they satisfy $F(0,t) = \alpha(0)$ and $F(1,t) = \alpha(1)$.

321. Examples:

(a) Let $C \subset \mathbb{R}^n$ be convex. Let $f, g\colon X \to C$ be continuous maps. Then the map $F(x,t) := (1-t)f(x) + tg(x)$ is a homotopy from f to g. If f and g agree on a set $A \subset X$, then F is a homotopy relative to A.

(b) Let $f, g\colon X \to S^n$ be continuous maps such that $f(x) \neq -g(x)$ for $x \in X$. Then the map

$$F(x,t) := \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$

is a homotopy from f to g.

(c) The map $f\colon S^1 := \{z \in \mathbb{C} : |z| = 1\} \to S^1$ defined by $f(z) = -z$ is homotopic to the identity map $g(z) = z$.

(d) Let $f\colon X \to S^n$ be a continuous map which is not onto. Then it is null-homotopic, that is, homotopic to a constant map.

(e) Consider $X := \{p \in \mathbb{R}^2 : 1 \le \|p\| \le 2\}$. Let α be 'the inner circle' and β be the ellipse lying in X and circumscribing α. Assume that they both start and end at $(0,1)$. They are homotopic in X. (Note that X is not convex.)

322. The relation of homotopy between the continuous maps from a space X to another space Y is an equivalence relation.

For, if $f \overset{F}{\simeq} g$ and $g \overset{G}{\simeq} h$, then

$$H(x,t) := \begin{cases} F(x,2t) & 0 \le t \le 1/2 \\ G(x,2t-1) & 1/2 \le t \le 1, \end{cases}$$

is a homotopy from f to h.

323. The relation of homotopy between the continuous maps from a space X to another space Y relative to a subset $A \subset X$ is an equivalence relation among maps that agree on A.

324. Homotopy behaves well with respect to composition of maps.

   (a) Let $f, g\colon X \to Y$ be homotopic relative to a set $A \subset X$ via the homotopy F. Let $h\colon Y \to Z$ be a map. Then $h \circ f \overset{h\circ F}{\simeq} h \circ g$ relative to A.

   (b) Let $f\colon X \to Y$ be a map. Assume that $g, h\colon Y \to Z$ are homotopic relative to $B \subset Y$ via a homotopy G. Then $g \circ f \overset{F}{\simeq} h \circ f$ relative to $f^{-1}(B)$, where $F(x,t) := G(f(x), t)$.

325. Fix a base point $p \in X$. Let α be a loop at p. The equivalence class $\langle\alpha\rangle$ of all loops based at p homotopic to α relative to $\{0,1\}$ is called a *homotopy class*. The collection of homotopy classes of loops at p is denoted by $\pi_1(X,p)$.

326. Construction of the fundamental group. We make $\pi_1(X,p)$ into a group as follows. For $\langle\alpha\rangle, \langle\beta\rangle \in \pi_1(X,p)$, we let $\langle\alpha\rangle * \langle\beta\rangle := \langle\alpha * \beta\rangle$.

   (a) The above multiplication is well-defined.
   For, $\alpha \overset{F}{\simeq}$ and $\beta \overset{G}{\simeq} \beta'$, then $\alpha*\beta \overset{H}{\simeq} \alpha*\beta'$ where $H(s,t) := \begin{cases} F(2s,t) & 0 \le s \le 1/2 \\ G(2s-1,t) & 1/2 \le s \le 1. \end{cases}$

   (b) The multiplication is associative.
   First of all, we compute

$$((\alpha * \beta) * \gamma)(s) = \begin{cases} \alpha(4s) & 0 \le s \le 1/4 \\ \beta(4s-1) & 1/4 \le s \le 1/2 \\ \gamma(2s-1) & 1/2 \le s \le 1 \end{cases}$$

$$(\alpha * (\beta * \gamma))(s) = \begin{cases} \alpha(2s) & 0 \le s \le 1/2 \\ \beta(4s-2) & 1/2 \le s \le 3/4 \\ \gamma(4s-3) & 3/4 \le s \le 1 \end{cases}.$$

   Define $f\colon I \to I$ by setting

$$f(s) := \begin{cases} 2s & 0 \le s \le 1/4 \\ s + \dfrac{1}{4} & 1/4 \le s \le 1/2 \\ (s+1)/2 & 1/2 \le s \le 1 \end{cases}.$$

   Since $f(0) = 0$ and $f(1) = 1$, we see that $f \simeq 1_I$, that is, f is homotopic to the identity map 1_I of I relative to $\{0,1\}$. We have

$$\begin{aligned} (\alpha * \beta) * \gamma &= (\alpha * (\beta * \gamma)) \circ f \\ &\simeq (\alpha * (\beta * \gamma)) \circ 1_I \\ &= \alpha * (\beta * \gamma). \end{aligned}$$

(c) Existence of the identity. Let $e = e_p$ denote the constant loop at p: $e(t) = p$ for $0 \le t \le 1$. Then $\langle e \rangle$ serves as the identity for the multiplication. Again, proceeding as earlier, we have

$$\begin{aligned} e * \alpha(s) &= \begin{cases} e(2s) & 0 \le s \le 1/2 \\ \alpha(2s-1) & 1/2 \le s \le 1 \end{cases} \\ e * \alpha &= \alpha \circ f \\ \text{where } f(s) &= \begin{cases} 0 & 0 \le s \le 1/2 \\ 2s-1 & 1/2 \le s \le 1. \end{cases} \end{aligned}$$

Thus we have

$$e * \alpha = \alpha \circ f \simeq \alpha \circ 1_I \text{ rel } I = \alpha.$$

Similarly, one shows that $\alpha * e \simeq \alpha$.

(d) Existence of inverse. The inverse of $\langle \alpha \rangle$ is $\langle \alpha^{-1} \rangle$, where α^{-1} is the reverse path defined by $\alpha^{-1}(s) := \alpha(1-s)$.

i. The inverse s well-defined. If $\alpha \overset{F}{\simeq} \beta$ relative to $\{0,1\}$, then $\alpha^{-1} \overset{G}{\simeq} \beta^{-1}$ relative to $\{0,1\}$ where $G(s,t) := F(1-s,t)$.

ii. We show that $\alpha * \alpha^{-1} = \alpha \circ f$ where

$$f(s) = \begin{cases} 2s & 0 \le s \le 1/2 \\ 2-2s & 1/2 \le s \le 1. \end{cases}$$

Now, $f \simeq g$ relative to $\{0,1\}$ where $g(s) = 0$ for $0 \le s \le 1$. Hence,

$$\alpha * \alpha^{-1} = \alpha \circ f \simeq \alpha \circ g \text{ rel } \{0,1\} = e.$$

One similarly, shows that $\alpha^{-1} \circ \alpha \simeq e$.

(e) Explicit homotopies can also be given. (Of what use?)

i. Existence of identity.

- $\alpha * e \simeq \alpha$ via

$$H(s,t) := \begin{cases} \alpha\left(\frac{2t}{s+1}\right) & s \ge 2t-1 \\ p & s \le 2t-1. \end{cases}$$

- $e * \alpha \simeq \alpha$ via

$$H(s,t) = \begin{cases} p & s \ge 2t \\ \alpha\left(\frac{2t-s}{2-s}\right) & s \le 2t \end{cases}$$

ii. Existence of inverse. $\alpha * \alpha^{-1} \simeq e$ via

$$H(s,t) = \begin{cases} \alpha(2t) & s \ge 2t \\ \alpha(s) & s \le 2t \text{ and } s \le 2-2t \\ \alpha(2-2t) & s \ge 2-2t \end{cases}$$

iii. Associativity. $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$ via

$$H(s,t) = \begin{cases} \alpha(\frac{4t}{s+1}) & 4t-1 \le s \\ \beta(4t-s-1) & 4t-2 \le s \le 4t-1 \\ \gamma(\frac{4t-2s}{2-s} - 1) & s \le 4t-2. \end{cases}$$

I have not verified these, simply copied from a book!

327. Let α, β be two paths such that $\alpha(1) = \beta(0)$. Then proceeding as in the last item, we show the following, as the same homotpies work as they take care of the end points!

   (a) If $\alpha' \simeq \alpha$ relative to $\{0,1\}$ and If $\beta' \simeq \beta$ relative to $\{0,1\}$, then $\alpha * \beta \simeq \alpha' * \beta'$ relative to $\{0,1\}$.

   (b) If α, β, γ are paths such that $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ make sense, then

   $$\alpha * (\beta * \gamma) \simeq (\alpha * \beta) * g \text{ relative to } \{0,1\}.$$

   (c) We have $\alpha \circ \alpha^{-1} \simeq e_{\alpha(0)}$ relative to $\{0,1\}$ and $\alpha^{-1} \circ \alpha \simeq e_{\alpha(1)}$ relative to $\{0,1\}$.

328. If X is path connected, then $\pi_1(X,p)$ is isomorphic to $\pi_1(X,q)$ for $p, q \in X$. This isomorphism depends on the choice of path joining p and q.

329. Let $p \colon E \to B$ be a continuous map. An open subset $U \subset B$ is said to be *evenly covered* by p if $p^{-1}(U)$ is the union $\cup_i V_i$ of disjoint open subsets V_i of E such that the restriction p_i of p to V_i is a homeomorphism of V_i onto U.

   We say that p is a *covering map* if (i) p is onto and (ii) each $b \in B$ has an open neighbourhood U_b which is evenly covered by p.

   The set $p^{-1}(b)$ is called the *fibre* over b.

   The sets V_i are called *sheets* of $p^{-1}(U)$.

   E is called the total space and B, the base of the covering map p.

330. Properties of a covering map.

   (a) Any covering map is open.

   (b) Each of the fibres $p^{-1}(b)$ is discrete.

   (c) Each $b \in B$ has an open neighbourhood U such that $p^{-1}(U)$ is homeomorphic to $p^{-1}(b) \times U$.

331. Examples.

   (a) The exponential map $p \colon \mathbb{R} \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$ is a covering.

   (b) The quotient map $\pi \colon S^n \to \mathbb{P}^n(\mathbb{R})$ is a covering.

   (c) Products of covering maps is again a covering map. (precise statement?)

   (d) Consider the exponential map $\exp \colon \mathbb{C} \to \mathbb{C}^*$. The open set $U := \mathbb{C}^*$ is not evenly covered by exp.

   In fact, an open set $U \subset \mathbb{C}^*$ is evenly covered by the exponential map iff there exists a continuous logarithm L on U, that is, a continuous map $L \colon U \to \mathbb{C}$ such that $\exp(L(z)) = z$ for all $z \in U$.

   Note however that $\exp \colon \mathbb{C} \to \mathbb{C}^*$ is a covering map.

332. Let $p\colon E \to B$ a covering map. Let $f\colon X \to B$ be continuous map. Then a map $g\colon X \to E$ such that $p \circ g = f$ is called a *lift* of f. One has the following commutative diagram. (Figure?)

333. Uniqueness of lifts.

Theorem 28. *Let $p\colon E \to B$ be a covering map and X a connected space. Let $f\colon X \to B$ be a map. If $g, h\colon X \to E$ are lifts of f such that $g(x) = h(x)$ for some $x \in X$, then $g = h$.* □

334. Path lifting lemma.

Theorem 29. *Let $p\colon E \to B$ be a covering map. Let $c\colon I \to B$ be a path. Let $e_0 \in E$ be such that $p(e_0) = c(0)$. then there exists a unique path $\gamma\colon I \to E$ such that $\gamma(0) = e_0$ and $p\colon \gamma = c$.* □

335. A Version of homotopy lifting lemma:

Theorem 30. *Let $p\colon E \to B$ be a covering map. Let $F\colon I \times I \to B$ be a continuous map. Let $e_0 \in p^{-1}(F(0,0))$. Then there exists a unique lift $G\colon I \times I \to E$ of F such that $G(0,0) = e_0$.* □

336. Let (E, e) and (B, b) be topological spaces with base points e and b respectively. Let $p\colon E \to B$ be a covering map. If c is a loop at b and γ is its lift through e, we cannot conclude that γ is a loop at e but $p(\gamma(1)) = b$, that is, $\gamma(1) \in p^{-1}(b)$. Example: Consider the spaces $(\mathbb{R}, 0)$ and $(S^1, 1)$. A lift of $c(t) = e^{2\pi i t}$ is $\gamma(t) = t$ in $\mathbb{R}$.

337. Let c_0 and c_1 be homotopic loops at b with F as a a homotopy. We thus get a lift $G\colon I \times I \to E$ of F such that $G(0,0) = e$ and $p(G(s,t)) = c_t(s)$, for $(s,t) \in I \times I$. Let $\gamma_t(s) := G(s,t)$. Then all these paths start at e and have the same end point $\gamma_0(1)$.

As a corollary, if $\langle c \rangle \in \pi_1(B, b)$ and γ is a lift of c through e, then

$$\pi_1(B, b) \to \pi^{-1}(b) \text{ defined by } \varphi\colon\ \langle c \rangle \mapsto \gamma(1) \tag{4}$$

is well-defined.

338. Simply connected space. We say a path-connected topological space X is *simply connected* if $\pi_1(X, x)$ is trivial for some (and hence for any) $x \in X$. Examples:

   (a) Any convex subset of $\mathbb{R}^n$ is simply connected.
   (b) The parabola $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$ is not convex but simply connected.
   (c) We shall show below (Item 341) that S^n for $n \geq 2$ is simply connected.

339. Let $p\colon (E, e) \to (B, b)$ be a covering map. Assume that E is simply connected. Then the map defined in (4) is a bijection of $\pi_1(B, b)$ with $\pi^{-1}(b)$.

As a corollary (under the above hypothesis), for any $q \in \pi^{-1}(b)$, if we let γ_y be a path joining e to y, then given a loop c at p, we have a unique $q \in \pi^{-1}(b)$ such that c is homotopic to $p \circ \gamma_y$.

340. Applications.

(a) Fundamental group of $\mathbb{P}^n(\mathbb{R})$ ($n \geq 2$). For $n \geq 2$, $\pi_1(\mathbb{P}^n(\mathbb{R}), [e_1])$ is isomorphic to $\mathbb{Z}_2$.

(b) Fundamental group of S^1 is isomorphic to $\mathbb{Z}$. The following are the main steps.

i. Given $\langle c \rangle \in \pi_1(S^1, 1)$, $\varphi(\langle c \rangle) \in \mathbb{Z}$. We call the integer the index of c.

ii. The map $\langle c \rangle \mapsto \varphi(\langle c \rangle)$ is a group homomorphism of $\pi_1(S^1, 1)$ to $\mathbb{Z}$.

341. Let X be a space, U, V be simply connected open subsets of X such that (i) $X = U \cup V$ and (ii) $U \cap V$ is path connected. Then X is simply connected.

Application. S^n is simply connected for $n \geq 2$.

342. Applications of the index of loops in S^1.

(a) No retraction theorem. There is no continuous map $f\colon B^2 \to S^1$ such that $f(z) = z$ for $z \in S^1$. □

(b) Brouwer fixed point theorem. Any continuous map of B^2 to itself has a fixed point. □

(c) Borsuk-Ulam theorem. Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map. Then there exist antipodal points $\pm v \in S^2$ such that $f(v) = f(-v)$. □

This has a physical interpretation.

(d) Ham-Sandwich theorem. Let A, B, C be bounded connected open subsets of $\mathbb{R}^3$. Then there exists a plane in $\mathbb{R}^3$ that divides each of the sets into two subsets of equal volume. □

Proof of this relied on some intuitively obvious facts on volumes.

(e) Fundamental theorem of algebra. □

For proofs, you may refer to my relevant articles in *Expository Articles.*

To add as appendices:

1. Finite sets
2. Cardinality
3. Subspace Topology
4. Quotient Topology
5. Generating Topologies
6. Tykonoff's theorem
7. Compact Spaces
8. Connected Spaces
9. Existence of Continuous Functions
10. Proper maps

11. Covering spaces
12. Topological groups
13. Discrete Subgroups of $\mathbb{R}^n$.

Consider $X = \mathbb{R}$ with the smallest topology $\mathcal{T}$ containing sets of the form $U \setminus A$ where U is open in the standard topology $\mathcal{T}_d$ and $A \subset \mathbb{R}$ is any countable subset.

Then $(X, \mathcal{T})$ is Hausdorff. It is not first countable. For,