A Workbook in Topology

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Preface

This is a summary of courses on General Topology, offered by me at the Department of Mathematics, University of Mumbai during the academic year 2004-2005, at the Department of Mathematics and Statistics, University of Hyderabad in Jan–April 2012 and at the School of Mathematics and Statistics, University of Hyderabad in Jan–April 2015. There are minimal number of proofs in this set of notes. This may be used as a workbook by students of topology. Because of its brevity, its emphasis on the most useful concepts in the practice of modern topology and wealth of concrete examples, the student will be adequately prepared to do mathematics which require a deeper knowledge of topology rather a nodding acquaintance to a large number of dated concepts.

Its merit, if any, lies in the choice of topics, their development and the emphasis on concrete and geometric examples and exercises. I plan to add a bit more material and a lot of pictures so that it could serve as a skeleton of a course in General Topology. Later I plan to develop this into a text-book. (So, please do not plagiarize!)

This set may be used in conjunction with many articles of mine on Topology. They are the topics which a student who wishes to specialize in Topology as practiced now need to know and they take them further into some of the topics dealt with the main text. These are appended at the end of the workbook as appendices.

Topology of Metric Spaces, 2nd edition, by S. Kumaresan is published by Narosa. The books *Topology* by Munkres and *Topology* by Armstrong are available in Indian edition. These three books may be used to fill in the details of my outline. My book is strongly recommended for pictures, geometric insights and developing a taste for topology.

I would appreciate receiving your comments and views.

Contents

1 Basic Notions

1.1 **Finite sets.** Let X be a set. We say that it is finite if $X = \emptyset$ or if there exists a bijective map of X into an initial segment $I_n := \{k \in \mathbb{N} : 1 \leq k \leq n\}$ of $\mathbb{N}$.

Using induction/well-ordering principle, one can show that if X has a bijection with I_m and I_n, then $m = n$. The unique n is called the number of elements in X. (For a proof, see my article on Finite sets.) The number of elements in the emptyset is 0.

1.2 **Countable and uncountable sets.** We say that a set X is countable if either $X = \emptyset$ or if either of the equivalent conditions are satisfied:

(a) There is a one-one map $f\colon X \to \mathbb{N}$.

(b) There exists an onto map $g\colon \mathbb{N} \to X$.

Applications: Countability of $\mathbb{N} \times \mathbb{N}$, $\mathbb{Q}^+$, $\mathbb{Q}$, countable union of countable sets, finite product of countable sets. (See my article on Countable and Uncountable sets, also Munkres.)

1.3 Uncountability of $2^{\mathbb{N}}$: Cantor's theorem: there exists no onto map from X to $P(X)$.

We prove this by contradiction. Assume $f\colon P(X) \to X$ be onto. Consider the set $S := \{x \in X : x \notin f(x)\}$. Since f is onto there exists $a \in X$ such that $f(a) \in S$. Now exactly one of the following must happen: (i) $a \in S$ or (ii) $a \notin S$. If $a \in S$, by the very definition of S, $a \notin f(a) = S$, contradiction. Similarly (ii) cannot happen. Hence we conclude that no such f exists.

1.4 Metric Spaces: In most of a first course in real analysis, we just needed the notion of a distance between two real numbers to define the concept of convergent sequences or the concept of continuous functions. Motivated by this we define a *metric* or a distance function on a (nonempty) set X as a function $d\colon X \times X \to \mathbb{R}$ which satisfies the following properties:

(a) For all $x, y \in X$, we have $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$.

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(c) For all $x, y, z \in X$, we have the traingle inequality:

$$d(x, z) \leq d(x, y) + d(y, z).$$

1.5 Metrics in $\mathbb{R}^2$: L^1 and L^∞ metrics, called the sum and max metrics:

$$\begin{aligned} d_1(x, y) &:= \sum_{k=1}^{n} |x_k - y_k| \\ d_{\max}(x, y) \equiv d_\infty(x, y) &:= \max\{|x_k - y_k| : 1 \leq k \leq n\}. \end{aligned}$$

These distances generalize to function spaces. See Item 8.

1.6 **Normed linear spaces.** A *norm* on a vector space V over $\mathbb{R}$ (or over $\mathbb{C}$) is a function $\|\ \| : V \to \mathbb{R}$ satisfying the following conditions:
(i) For $x \in V$, $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$.
(ii) For $x \in V$ and $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$ if X is vector space over $\mathbb{C}$) , we have $\|\lambda x\| = \|\lambda\| \|x\|$.
(iii) For $x, y \in V$, we have the triangle inequality $\|x + y\| \leq \|x\| + \|y\|$.

1.7 Examples of normed linear spaces:

(a) Finite dimensional normed linear spaces: On $\mathbb{R}^n$, we have the following norms:

$$\|x\|_1 := \sum_{k=1}^{n} |x_k| \quad \text{and} \quad \|x\|_\infty := \max\{|x_k| : 1 \le k \le n\}.$$

That these are norms is easily verified.

Another norm is the standard/Euclidean norm: $\|x\|_2 := \left(\sum_{k=1}^n |x_k|^2\right)^{1/2}$. We need Cauchy-Schwarz inequality to verify that this is a norm.

1.8 Function spaces.

(a) Let X be any nonempty set. Let $B(X, \mathbb{R})$ denote the real vector space of all bounded real valued functions on X. Then $\|f\|_\infty := \sup\{|f(x)| : x \in X\}$ is a norm on $B(X, \mathbb{R})$.

(b) Let $X = [0,1]$. Let $V := C(X, \mathbb{R})$ the vector space of all continuous real valued functions on X. Then $\|f\|_1 := \int_0^1 |f(t)|\, dt$ defines a norm on V.

(c) Since $C([0,1], \mathbb{R}) \subset B([0,1], \mathbb{R})$, we have another norm on V, namely, $\|f\|_\infty$.

1.9 ℓ^1, the space of sequences whose associated series are absolutely summable is defined as follows:

$$\ell^1 := \left\{ (z_n) : z_n \in \mathbb{R}; \sum_n |z_n| \text{ is convergent.} \right\}$$

Then $\|z\| = \|(z_n)\| := \sum_n |z_n|$ is a norm on ℓ^1.

1.10 **Open balls.** Let (X, d) be a metric space. Fix $a \in X$ and $r > 0$. The *open ball* $B(a, r)$ and the closed ball $B[a, r]$ centred at a and radius r are defined by

$$B(a,r) := \{x \in X : d(x,a) < r\} \;\;\&\;\; B[a,r] := \{x \in X : d(x,a) \le r\}.$$

We now look at some examples.

(a) in $\mathbb{R}$: $B(p, r) = (p - r, p + r)$.

(b) $B(0,1)$ in $\mathbb{R}^2$ with $\|\;\|_1$, $\|\;\|_2$ and $\|\;\|_\infty$. Picture!

Look at the pictures. How do we arrive at them? The "boundary" of $B(0,1)$ is identified. In the case of $\|\;\|_1$, the boundary is defined by $|x| + |y| = 1$. Hence $B(0,1)$ in this space is the 'region' enclosed by lines $x + y = 1$, $-(x + y) = 1$, $x - y = 1$ and $y - x = 1$. In the case of $\|\;\|_\infty$, the bounding lines are $x = 1$, $-x = 1$, $y = 1$ and $-y = 1$.

(c) in $\mathbb{Z}$ with the induced metric. Identify **all** open balls. Answer: Any set of $2n + 1$ consecutive integers.

(d) Relations between $B(x, r)$ and $B(y, s)$.

If $x = y$ and $r < s$, then $B(x, r) \subseteq B(x, s)$. Equality can occur. Consider the discrete metric and $r = 1/2$ and $s = 3/4$.

If d is discrete, and if $r > 1$ and $s > 1$, then for *any* x, y, we have $B(x, r) = B(y, s)$.

(e) Visualizing the open balls in $C[0,1]$ under $\|\;\|_\infty$.

(f) In an normed linear space , $B(x,r) = x + rB(0,1)$.

1.11 **Open sets in a metric space.** A subset U of a metric space is said to be *open* or d-open if for each $x \in U$, there exists $r_x > 0$ such that $B(x, r_x) \subset U$. We now look at lots of examples to build our intuition. In each of the examples, draw pictures of the sets and see whether you can enclose each of the points x in an open ball $B(x, r_x)$ contained in the given set. In most of the cases, the geometry will lead you to the 'best possible' radius r_x. This will develop your intuition to 'identify' the open sets "instantly". Pictures!

(a) in $\mathbb{R}$: various examples such as open intervals, union of open intervals and non-examples such as $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$,

(b) $\{(x,y) \in \mathbb{R}^2 : x > 0, y > 0\}$ in $\mathbb{R}^2$.

(c) $\{(x,y) \in \mathbb{R}^2 : x \geq 0, y > 0\}$ in $\mathbb{R}^2$.

(d) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 > 1\}$ in $\mathbb{R}^2$.

(e) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ in $\mathbb{R}^2$.

(f) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$.

(g) Various conic sections in $\mathbb{R}^2$.

(h) In a normed linear space V, if any vector subspace W is open, then $W = V$. Application: Is $C[0,1]$ open in $BF[0,1]$, the set of bounded functions?

(i) Is $\mathbb{R} \setminus \mathbb{Z}$ open in $\mathbb{R}$?

(j) The open ball $B(x,r)$ is open in any metric space.

Draw picture. If $y \in B(x,r)$, we need to find $s > 0$ such that $B(y,s) \subset B(x,r)$. Let us find such an s. Let $z \in B(y,s)$. We need to show $z \in B(x,r)$. That is we need an estimate for $d(z,x)$. The obvious estimate is $d(z,x) \leq d(z,y) + d(y,x) < s + d(y,x)$. If we can show $s + d(y,x) < r$, we are through. This suggest we choose $0 < s < r - d(x,y)$.

(k) $\{y \in X : d(x,y) > r\}$ is open. *Hint:* Modify the idea of the last sub-item.

(l) What are the open sets in a finite metric space?

(m) Can $\{h \in C[0,1] : f(x) < h(x) < g(x)\}$ for some $f, g \in C[0,1]$ be an open ball? Is it an open set?

(n) Is the open unit ball in $(C[0,1], \| \ \|_\infty)$ open in $(C[0,1], \| \ \|_1)$?

(o) If U is an open subset in an normed linear space , $(X, \| \ \|)$, then

   i. $x + U$ is open for any $x \in X$

   ii. $A + U$ is open for any $A \subset X$

   iii. αU is open for any nonzero scaler α.

(p) Is the set $U := \{f \in C[0,1] : f(1/2) \neq 0\}$ open in $(C[0,1], \| \ \|_\infty)$?

(q) Any open subgroup G of $\mathbb{R}$ is $\mathbb{R}$.

For, $0 \in G$ and hence $(-\varepsilon, \varepsilon) \subset G$ for some $\varepsilon > 0$. Since G is group, for all $x, y \in (-\varepsilon, \varepsilon)$ we have $x + y \in G$, that is, $(-2\varepsilon, 2\varepsilon) \subset G$. By induction, $(-n\varepsilon, n\varepsilon) \subset G$ for $n \in \mathbb{N}$.. Now let $x \in \mathbb{R}$ be nonzero. By Archimedean property, there exists $N \in \mathbb{N}$ such that $N\varepsilon > |x|$. Hence $x \in (-N\varepsilon, N\varepsilon)$. It follows that $\mathbb{R} = \cup_{n \in \mathbb{N}} (-n\varepsilon, n\varepsilon) \subset G$.

(r) A subset U of a metric space is open iff it is the union of a family of open balls. For, if $x \in U$, there exists $r_x > 0$ such that $B(x, r_x) \subset U$. We then have an indexed family $\{B(x, r_x) : x \in U\}$ of open balls. Clearly, $U = \cup_{x \in U} B(x, r_x)$.

(s) A subset $U \subset \mathbb{R}$ is open iff it is the union of a countable family of pair-wise disjoint open intervals. (See Lemma 1.2.42 on Page 23 of my book on Metric spaces.)

1.12 The class $\mathcal{T}$ of open subsets of a metric space (X, d) have the following properties:

(a) $\emptyset, X \in \mathcal{T}$.

(b) If $\{U_i : i \in I\}$ is any collection of elements in $\mathcal{T}$, then $U := \cup_{i \in I} U_i \in \mathcal{T}$.

(c) If U_k, $1 \leq k \leq n$ are in $\mathcal{T}$, then $U_1 \cap U_2 \cap \cdots \cap U_n \in \mathcal{T}$.

1.13 **Topology: Definition and Examples.** A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X which satisfies the three conditions (a)–(c) of the last item. Elements of $\mathcal{T}$ are called open sets, to be precise $\mathcal{T}$-open.

(a) Metric topology: Let (X, d) be a metric space. Then the collection of (d-) open subsets is a topology on the metric space. This topology is called the metric topology on the metric space.

(b) Discrete topology: Here $\mathcal{T} = P(X)$, the power set of X. Thus, every subset is open.

(c) The topology on a finite metric space is discrete.

(d) Indiscrete topology: U is open iff $U = \emptyset$ or $U = X$, that is, $\mathcal{T} = \{\emptyset, X\}$.

(e) Co-finite topology: U is open iff $U = \emptyset$ or $X \setminus U$ is finite, that is ,

$$\mathcal{T} = \{U \subset X : \text{ Either } U = \emptyset \text{ or } X \setminus U \text{ is finite.}\}$$

(f) Co-countable topology: U is open iff $U = \emptyset$ or $X \setminus U$ is countable, that is,

$$\mathcal{T} = \{U \subset X : \text{ Either } U = \emptyset \text{ or } X \setminus U \text{ is countable.}\}$$

(g) VIP topology: Fix $p \in X$. U is open iff $U = \emptyset$ or $p \in U$.

(h) Outcast topology: Fix $p \in X$. U is open iff $U = X$ or $p \notin U$.

(i) Outcast + co-finite topology: U is open iff either $p \notin U$ or U^c is finite.

1.14 A topology on $\mathbb{Z}$. Let $\mathcal{B}$ be the set of arithmetic progressions in $\mathbb{Z}$. Any element $B \in \mathcal{B}$ is of the form $a + \mathbb{Z}b$ for some nonzero b. Note that $\mathcal{B}$ is nothing other than the set of cosets of all additive (non-trivial) subgroups of $\mathbb{Z}$. An example: $2 + 5\mathbb{Z}$. We define a topology $\mathcal{T}$ on $\mathbb{Z}$ as follows: a subset $U \subset \mathbb{Z}$ is open iff for each $x \in U$, a coset of the from $x + \mathbb{Z}b \subset U$. Clearly, $\emptyset, \mathbb{Z} \in \mathcal{T}$. If $\{U_i\}$ is a collection of sets in $\mathcal{T}$ and $x \in \cup_i U_i$, then $x \in U_j$ for some j and hence there is a $b \neq 0$ such that $x \in x + \mathbb{Z}b \subset U_j \subset \cup_i U_i$. If $x \in U \cap V$ for some $U, V \in \mathcal{T}$, then there exist b, c such that $x + \mathbb{Z}b \subset U$ and $x + \mathbb{Z}c \subset V$. Clearly, $x + \mathbb{Z} \operatorname{lcm}(b, c) \subset U \cap V$. Hence $\mathcal{T}$ is a topology on $\mathbb{Z}$.

(1) Observe that any element of $\mathcal{B}$ can be written in the form $r + \mathbb{Z}b$ where $b > 0$ and $0 \leq r < b - 1$. Hence in view of $\mathbb{Z} = \cup_{0 \leq r < b-1} r + \mathbb{Z}b$, we see that any element of $\mathcal{B}$ and its complement are both open!

(2) Another observation is that no nonempty finite set can be open.

As an application of these observations, we now give a topological proof of Euclid's theorem on the infinitude of primes in $\mathbb{Z}$. We prove this by contradiction. Assume that $p_1, \ldots, p_n$ are the set of all primes. Now the only integers that are not divisible by any prime are ± 1. Hence

$$\mathbb{Z} \setminus \{\pm 1\} = \cup_{k=1}^{n} \mathbb{Z}p_k = \cup_k U_k, \text{ say.}$$

Let us take the complements on both sides of the above equality. The complement of left side is $\cap_k U_k^c$, a finite intersection of open sets (in view of Observation 1) and hence is open. Hence the left side, a finite set is open, a contradiction to observation 2).

1.15 We now present two interesting examples of topological spaces. We leave the detailed verifications as exercise for the readers. As we learn new concepts, we shall keep revisiting them often.

(a) Let $X = \mathbb{R}$. Let $\mathcal{T}_d$ denote the standard topology on $\mathbb{R}$. Let $\mathcal{T}$ be the collection of all subsets of the form $G := U \setminus A$ where $U \in \mathcal{T}_d$ and A is a countable subset of $\mathbb{R}$. Show that $\mathcal{T}$ is a topology on $\mathbb{R}$. In the sequel we shall always denote elements of $\mathcal{T}$ as $G = U \setminus A$ etc.

(b) We say that a subset $F \subset \mathbb{N}$ is *small* if $\sum_{k \in F} \frac{1}{k}$ is convergent. The empty set is defined to be small. The following are easy to see.

  i. Any finite subset $F \subset \mathbb{N}$ is small.
  ii. If S is small and $T \subset S$, then T is small.
  iii. If F_k is small for $1 \leq k \leq N$, then $F := \cup_{k=1}^{N} F_k$ is small.
  iv. If S is an infinite subset of $\mathbb{N}$, there exists $T \subset S$ such that T is an *infinite* small set.
  To see this, observe that for each $k \in \mathbb{N}$ there exists $n_k \in S$ such that $n_k > 2^k$.
  v. $\mathbb{N}$ is not small.

Let $X := \mathbb{N} \cup \{0\}$. We consider the following collection

$$\mathcal{T} := \{U \subset X : \text{ either } U \subset \mathbb{N} \text{ or } 0 \in U \text{ and } X \setminus U \text{ is small}\}.$$

Show that $\mathcal{T}$ is a topology on X.

1.16 Basis of a topological space and basis for a topology on a set.

Basis for a topological space. Let $(X, \mathcal{T})$ be a topological space. A subset $\mathcal{B} \subset \mathcal{T}$ of open sets is said to be a basis for $\mathcal{T}$ if every element in $\mathcal{T}$ is a union of elements from $\mathcal{B}$. In other words, $\mathcal{B}$ is a basis for $\mathcal{T}$ if for any $U \in \mathcal{T}$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. The typical example of a basis is the set of all open balls for the topology on a metric space.

1.17 Examples of bases:

(a) $\{B(x, r) : x \in X, r > 0\}$ is a basis for the metric topology on any metric space. The indexing set is $X \times (0, \infty)$.

(b) $\{B(x, 1/n) : x \in X, n \in \mathbb{N}\}$ is a basis for the metric topology on any metric space. The indexing set is $X \times \mathbb{N}$.

(c) When $X = \mathbb{R}$, we can do better than the last two bases. Consider $\mathcal{B} := \{(a, b) : a, b \in \mathbb{Q}\}$ is a basis for the standard topology on $\mathbb{R}$. Note that this basis is countable, as it is indexed by $\mathbb{Q} \times \mathbb{Q}_+$. (Why? $(a, b) = B(c, r)$ where $c = (a + b)/2 \in \mathbb{Q}$ and $r = (b - a)/2 \in \mathbb{Q}_+$.)

(d) A basis for the VIP topology is $\{p\} \cup \{\{p, q\} : q \in X, q \neq p\}$.

(e) A basis for outcast topology is $\{X\} \cup \{\{q\} : q \in X, q \neq p\}$.

(f) $\mathcal{B} := \{\{x\} : x \in X\}$ is a basis for the discrete topology on a set X.

(g) $\mathcal{B} := \{X\}$ is a basis for the indiscrete topology on a set X.

(h) Note that $(0, 1) = (0, 1/2) \cup (1/4, 1) = \cup_{n \geq n}(0, (n-1)/n)$. Hence there is no uniqueness while expressing an open set as a union of some elements from the basis.

1.18 The second notion is a basis for a topology on a set X. The question here is: given a set X and a subset $\mathcal{B} \subset P(X)$ of subsets of X, does there exist a topology $\mathcal{T}$ on X for which $\mathcal{B}$ is a basis? Suppose such a topology $\mathcal{T}$ exists. Then $X \in \mathcal{T}$ so that a first requirement is (1) $\cup_{B \in \mathcal{B}} B = X$. Also, since any $B \in \mathcal{B}$ must be in $\mathcal{T}$, $B_1 \cap B_2 \in \mathcal{T}$ for any $B_1, B_2 \in \mathcal{B}$. Hence the second condition: (2) for any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$. If these two conditions are satisfied, we define a topology $\mathcal{T}$ on X as follows:

$$\mathcal{T} := \{U \subset X : \forall x \in U, \ \exists B \in \mathcal{B} \ \text{ such that } x \in B \subset U\}.$$

It is easy to verify that $\mathcal{T}$ is a topology on X and that $\mathcal{B}$ is a basis for this topology.

1.19 Order Topology: partial and total orders, dictionary order on products, $\mathbb{C}$ is totally ordered **but** is not an ordered field. Intervals of the form (a, b) and rays of the form $(-\infty, a)$ and (b, ∞). Examples in $\mathbb{R}^2$: the rays $(-\infty, (1, 2))$, $((-1, 1), \infty)$ and the intervals $((-1, 1), (3, -2))$ and $((0, 0), (0, 10))$. Basis for order topology. What is the order topology on $\mathbb{R}$, on $\mathbb{Z}$, on $\mathbb{N}$ and on a finite totally ordered set? **Pictures!** **Details!**

(a) Let $R \subset X \times X$ be a subset with the following properties:
 (i) Given $x, y \in X$, either $x = y$ or (x, y) or $(y, x) \in R$.
 (ii) For each $x \in X$, we have $(x, x) \notin R$.
 (iii) If $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Such an R is called a simple or linear order on X. If $(x, y) \in R$, we usually denote it by $x < y$. We called (X, R) or $(X, <)$ a totally ordered set. Note that (i) is the law of trichotomy.

(b) A standard example is the standard $<$ relation on R. Another example is defined by $(x, y) \in R$ if either $x^2 < y^2$ or if $x^2 = y^2$, then $x < y$.

Question: How do you describe R as a subset of $\mathbb{R} \times \mathbb{R}$?

(c) If X and Y are totally ordered sets, the *dictionary order* on $X \times Y$ is defined as follows: $(x_1, y_1) < (x_2, y_2)$ if $x_1 < x_2$ or if $x_1 = x_2$, then $y_1 < y_2$.

(d) Let X be a totally ordered set. For any $a, b \in X$ with $a < b$, we define intervals $[a, b]$, (a, b), $[a, b)$ and $(a, b]$ as in $\mathbb{R}$. How do we define $(-\infty, b)$ or (a, ∞)?

Draw pictures of the following intervals in $\mathbb{R} \times \mathbb{R}$ with lexicographic order: (i) the ray $(-\infty, (1, 2))$, (ii) the ray $((-1, 1), \infty)$, (iii) the intervals $((-1, 1), (3, -2))$ and $((0, 0), (0, 10))$.

(e) Let $\mathcal{B}$ be the collection of all open intervals of the form (a,b) in X along with $[m,b)$ if there exists a minimum $m \in X$ and/or $(a,M]$ if there exists a maximum $M \in X$. It is easy to see that $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X, called the *order topology* on X.

(f) What is the order topology on $\mathbb{R}$, on $\mathbb{Z}$, on $\mathbb{N}$ and on a finite totally ordered set? What is the order topology on $[0,1)$?

(g) Is an open disk, say $B((0,0),1)$ open in the order topology on $\mathbb{R}^2$? Is the open interval $((0,0),(0,10))$ open in $\mathbb{R}^2$ with the standard topology?

1.20 Lower Limit Topology: Consider $\mathcal{B} := \{[a,b) : a,b \in \mathbb{R}, a < b\}$. It is easy to see that $\mathcal{B}$ satisfies both the conditions laid out in Item 1.18 on Page 10. The topology associated with this basis is known as the lower limit topology on $\mathbb{R}$, denoted by $\mathcal{T}_L$. The space $(\mathbb{R}, \mathcal{T}_L)$ is denoted by $\mathbb{R}_\ell$.

When is a subset $U \subset \mathbb{R}$ open in $\mathcal{T}_L$? If for $x \in U$, we can find $[a,b) \in \mathcal{B}$ such that $x \in [a,b) \subset U$. A picture will immediately lead you to a 'better' condition: for $x \in U$, we can find $b > x$ such that $[x,b) \subset U$. In particular, any interval $(a,b) \in \mathcal{T}_L$. Hence the lower limit topology is finer than the standard topology on $\mathbb{R}$. In fact, it is strictly finer, since $[a,b)$ is open In $\mathcal{T}_L$ but not in the standard topology.

Note that no countable sub-collection of $\{[a,b) : a,b \in \mathbb{R}, a < b\}$ will serve as a basis for the lower limit topology. For, if $\{[a_n,b_n) : n \in \mathbb{N}\}$ is one such, then choose $a \in \mathbb{R}$ such that $a \neq a_n$ for $n \in \mathbb{N}$. Then the open set $[a,a+1)$ cannot be written as a union of any such elements. Why? For, a has to be in one of them, say, $[a_k,b_k)$ Since $a_k \neq a$, it follows that $a_k < a < b_k$ so that the union will have elements from $[a_k,a)$ which are not in $[a,a+1)$.

Question: How about the collection $\mathcal{B} = \{(a,b] : a,b \in \mathbb{R}, a < b\}$? Is it a basis for some topology on $\mathbb{R}$? If so,what will you call it?

1.21 The class of all topologies on a given set is a partially ordered set: if $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, we define $\mathcal{T}_1 \leq \mathcal{T}_2$ iff $\mathcal{T}_1 \subset \mathcal{T}_2$, as subsets of $P(X)$. The indiscrete topology is the smallest element and the discrete topology is the largest element of the class of topologies on X.

The union of topologies on X need not be topology. Let $X = \{a,b,c\}$ be a three element set. Let $\mathcal{T}_1 := \{\emptyset, \{a\}, X\}$ and $\mathcal{T}_2 := \{\emptyset, \{b\}, X\}$. These are two topologies on X but their union is not a topology.

However, the intersection of a (nonempty) family of topologies on X is again a topology, as can be easily verified.

Compare this with analogous results from algebra: intersections of subgroups of a group is again a group, intersection of vector subspaces of a vector spaces a vector subspace, intersection of ideas in a ring is again an ideal and so on. Associated with this phenomenon is the concept of subgroup (a vector subspace, an ideal, or a submodule) generated by subset S in a group (in a vector space, in a ring, or in a module over a ring).

These motivate us to define the following: if $\mathcal{A}$ is an arbitrary collection of subsets of a set X, there exists a unique smallest topology on X which contains $\mathcal{A}$ and is called the topology generated by $\mathcal{A}$. We shall later see a practical way of looking at this topology. See Item 10.2 on Page 40. For the time being, let us work out two examples.

- Let X be a nonempty set with at least three elements. Let $\mathcal{S}$ be the collection of all two element subsets of X. What is the smallest topology $\mathcal{T}$ containing $\mathcal{S}$? Fix $a \in X$. We can find two distinct elements, say, $x, y \in X$ none of which is a. Then $\{a, x\}$ and $\{a, y\}$ lie in $\mathcal{S}$ and hence in $\mathcal{T}$. It follows that $\{a\} \in \mathcal{T}$. Thus,every singleton subset is in $\mathcal{T}$ and hence $\mathcal{T}$ is the discrete topology on X.

  Question: What is $\mathcal{T}$ if X has only two elements?

- Let $\mathcal{S}$ consist of single element $A \subset X$. Then $\mathcal{T} = \{\emptyset, A, X\}$.

1.22 Let X be a set and $\mathcal{T}_c$ and $\mathcal{T}_f$ be respectively co-countable and co-finite topologies on X. Then the co-countable topology is finer than the co-finite topology.

They are the same iff X is finite. If X is finite, then the two topologies are the same. To see the converse, we need a result form set theory: If X is an infinite set, then there exists a set A such that $X \setminus A$ is infinite and countable.

In Item 1.15a on Page 9, $\mathcal{T}$ is finer than $\mathcal{T}_d$. The lower limit topology $\mathcal{T}_\ell$ (Item 1.20 on Page 11) is finer than the standard topology on $\mathbb{R}$.

Note that any topology on X is finer than the indiscrete topology on X and the discrete topology on X is finer than any topology on X.

1.23 We can use bases to say something about the topologies on a set.

Theorem 1. *Let X be any set. Let $\mathcal{B}_i$ be a basis for some topology $\mathcal{T}_i$ on X, for $i = 1, 2$. Then $\mathcal{T}_1 \leq \mathcal{T}_2$ iff the following holds: if $B_1 \in \mathcal{B}_1$, then $B_1 \in \mathcal{T}_2$. In particular, $\mathcal{T}_1 = \mathcal{T}_2$ iff every $B_1 \in \mathcal{B}_1$ is in $\mathcal{T}_2$ and every $B_2 \in \mathcal{B}_2$ is in $\mathcal{T}_1$.* □

We may use this to show that the order topology $\mathcal{T}_O$ on $\mathbb{R} \times \mathbb{R}$ is (strictly) finer than the usual topology $\mathcal{T}_{\text{std}}$ on $\mathbb{R}^2$. For if a point $p = (a, b) \in B$, an open ball, then there exist $\varepsilon > 0$ such that $\{a\} \times (b - \varepsilon, b + \varepsilon) \times \subset B$. (The ε can be explicitly determined!) But the set $\{a\} \times (b - \varepsilon, b + \varepsilon)$ is a basic open set in $\mathcal{T}_O$. Further, this basic open set is not open in the standard topology. Thus, $\mathcal{T}_O$ is strictly finer than $\mathcal{T}_{\text{std}}$.

2 Continuity

2.1 Continuity: Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map and $x_0 \in X$. We say that f is continuous at x_0 if for any given open set V containing $f(x_0)$, there exists an open set U containing x_0 such that $f(U) \subset V$. This definition is an abstraction of the standard ε-δ definition of continuity, say, of functions $f\colon \mathbb{R} \to \mathbb{R}$. In this context, $V = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ and $U = (x_0 - \delta, x_0 + \delta)$. In fact, we have the following theorem:

Theorem 2. *Let $f\colon (X, d) \to (Y, d)$ be a map between metric spaces. Let $x_0 \in X$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the topologies on X and Y induced buy their respective metrics. Then $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous at x_0 iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $d(x, x_0) < \delta$, we have $d(f(x), f(x_0)) < \varepsilon$.*

Proof. Let us assume that $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous at x_0. Let $\varepsilon > 0$ be given. Then $V := B(f(x_0), \varepsilon)$ is an open set containing $f(x_0)$. Hence there exists an open set $U \ni x_0$ such that for all $x \in U$ we have $f(x) \in V$. Since U is open there exists $\delta > 0$ such that $B(x_0, \delta) \subset U$. Hence it follows $d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon$, that is, $f\colon (X, d) \to (Y, d)$ is continuous at x_0. Pictures!

The converse is similar. Let $f\colon (X, d) \to (Y, d)$ is continuous at x_0. Assume that an open $V \ni f(x_0)$ is given. Then we can find $\varepsilon > 0$ such that $B(f(x_0), \varepsilon) \subset V$. For this $\varepsilon > 0$ by the definition of continuity in metric space context, there exists $\delta > 0$ such that $d(x, x_0) < \delta \implies d(f(x), f(x_0)) < \varepsilon$. If we let $U := B(x_0, \delta)$, then U is open, $x_0 \in U$, and for $x \in U$, we have $f(x) \in B(f(x_0 < \varepsilon) \subset V$. □

2.2 Let $f\colon X \to Y$ be any map between two sets. Let $B \subset Y$. The set $f^{-1}(B) := \{x \in X : f(x) \in B\}$ is called the inverse image of B under f. The following are well-known facts: Details!

(a) If $\{B_i : i \in I\}$ is a family of subsets of Y, then
  i. $f^{-1}(\cup_{i \in I} B_i) = \cup_{i \in I} f^{-1}(B_i)$.
  ii. $f^{-1}(\cap_{i \in I} B_i) = \cap_{i \in I} f^{-1}(B_i)$.

(b) For any set $B \subset Y$, we have $X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$.

(c) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be maps. Let $W \subset Z$. Then $(g \circ f)^{-1}(W) = f^{-1}\left(g^{-1}(W)\right)$.

Thus, "the inverse images behave well under set-theoretic operations and compositions."

2.3 Let X and Y be topological spaces. Then a map $f\colon X \to Y$ is said to be continuous on X iff it is continuous at each point $x \in X$.

Let X and Y be topological spaces and $f\colon X \to Y$ be continuous at each point $x \in X$. Let $V \subset Y$ be any open subset of Y. Let $U := f^{-1}(V) = \{x \in X : f(x) \in V\}$. Let $a \in U$. By the definition of U, $f(a) \in V$. Since f is continuous at a and $V \ni f(a)$ is an open set, there exists an open set $U_a \ni a$ such that for all $x \in U_a$, we have $f(x) \in V$. This implies $U_a \subset U$. Since $a \in U$ was arbitrary, what we have shown is that for each $a \in U$, there exists an open set U_a such that $a \in U_a$ and $U_a \subset U$. In particular, $U = \cup_{a \in U} U_a$ is open. (The argument of this paragraph teaches an algorithm: in the case of a topological space if we want to show that a set U is open , we need to find an open set U_a for each $a \in U$

such that $a \in U_a$ and $U_a \subset U$. Compare this with the algorithm to show a subset of a metric space is open, Item 1.11r on Page 8.)

We have thus shown that $f^{-1}(V)$ is open in X for each open subset $V \subset Y$ of Y.

Is the converse true? That is, if $f^{-1}(V)$ is open in X for each open subset $V \subset Y$ of Y, is f continuous on X? This is easy. Let $a \in X$ and $V \ni f(a)$ be open in Y. Then $U := f^{-1}(V)$ is open by hypothesis . Clearly $a \in U$. Also, for each $x \in U$, $f(x) \in V$, that is, f is continuous at a. Since a is arbitrary, it follows that f is continuous on X.

We have thus arrived at the following result.

Theorem 3. *Let X and Y be topological spaces. Then a map $f: X \to Y$ is continuous on X iff for every open subset $V \subset Y$, the inverse image $f^{-1}(V)$ is open in X.* □

2.4 The theorem of the last item leads us to the following result:

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on the same set X. Then $\mathcal{T}_1 \leq \mathcal{T}_2$ iff the identity map $I: (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is continuous. In particular, $\mathcal{T}_1 = \mathcal{T}_2$ iff the identity maps $I: (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ and $I: (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ are continuous. (This is same as saying that the identity map is a homeomorphism, a concept to be defined in Item 6.3 on Page 36.)

2.5 We looked at the following examples:

(a) Any constant map from a topological space to another is continuous.

(b) The identity map from $(X, \mathcal{T}_X)$ to itself is continuous.

(c) If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X, then the identity map $I: (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is continuous iff $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

(d) Let $(X, \mathcal{T}_X)$ be a topological space with the property that any map $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous. Then $\mathcal{T}_X$ is discrete and conversely.

(e) Let $(Y, \mathcal{T}_Y)$ be a topological space with the property that any map $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous. Then $\mathcal{T}_Y$ is indiscrete and conversely.

(f) The identity map from X with co-countable topology to X with co-finite topology is continuous. The other way map is continuous iff X is finite.

(g) Let X be a set with at least two elements and $p \in X$. Let V (resp. O) denote the VIP topology (resp. the outcast topology) on X with respect to p. Then

   i. The identity map $I: (X, V) \to (X, O)$ is not continuous. However it is continuous at $x = 0$ and at no other point.

   ii. The identity map $I: (X, O) \to (X, V)$ is not continuous at any point.

2.6 The identity map from $\mathbb{R}$ with the lower limit topology is continuous to $\mathbb{R}$ with the usual topology.

2.7 Let $\| \ \|_k$, $k = 1, 2$, be two norms on a vector space V. Then they are equivalent iff the identity map $I: (V, \| \ \|_1) \to (V, \| \ \|_2)$ and $I: (V, \| \ \|_2) \to (V, \| \ \|_1)$ are continuous.

2.8 Let X be an uncountable set with co-countable topology $\mathcal{T}_c$. Then the only continuous functions $f: (X, \mathcal{T}_c) \to \mathbb{R}$ are constants.

2.9 Find the set of points of continuity of all real valued functions on the following spaces.
(i) $\mathbb{R}$ with VIP topology with 0 as the VIP.
(ii) $\mathbb{R}$ with outcast topology with 0 as the outcast.
(iii) $\mathbb{N}$ with the topology $\mathcal{T} := \{\emptyset, \mathbb{N}\} \cup \{I_n : n \in \mathbb{N}\}$ where $I_n = \{1, 2, \dots, n\}$.

2.10 On any metric space X, we have lots of real valued continuous functions: $f(x) := d(x, p)$ for any fixed $p \in X$. In particular, given $p \neq q$ in X, there exists a real valued continuous function f on X such that $f(p) \neq f(q)$.

2.11 Let A be a nonempty subset of a metric space X. We defined $d_A(x) \equiv d(x, A) := \inf\{d(x, a) : a \in A\}$. Identify as much as possible d_A for the subsets below and draw their graphs. Picture!

(a) $X = \mathbb{R}$ and $A = [-1, 1]$.
(b) $X = \mathbb{R}$ and $A = \mathbb{Q}$.
(c) $X = \mathbb{R}$ and $A = \mathbb{Z}$.
(d) $X = \mathbb{R}^2$ and A is the x-axis.
(e) $X = \mathbb{R}^2$ and $A = \{(x, y) : x^2 + y^2 = 1\}$.
(f) W is a vector subspace of $\mathbb{R}^n$. *Hint:* If $\mathbb{R}^n = W \oplus W^\perp$, and if $x = w + w'$, then $d_W(x) = \|w'\| = \|x - p_W(x)\|$, where $p_W : \mathbb{R}^n \to W$ is the orthogonal projection.

2.12 We claim that for any nonempty subset A of a metric space X, the function $d_A : X \to \mathbb{R}$ is continuous.

$d_A(x) \leq d(x, a) \leq d(x, y) + d(y, a)$, for $a \in A$. Hence $d_A(x)$ is a lower bound for the set $\{d(x, y) + d(y, a) : a \in A\}$. But then $\inf\{d(x, y) + d(y, a) : a \in A\} = d(x, y) + d_A(y)$.

2.13 The function $x \mapsto \|x\|$ is continuous on an normed linear space $(V, \| \;\|)$. Note that $\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$ so that $\|x\| - \|y\| \leq \|x - y\|$. Interchanging x and y we get

$$| \, \|x\| - \|y\| \, | \leq \|x - y\| .$$

This establishes the (uniform) continuity of the norm function. Note that this has the continuity of modulus/absolute value as a special case.

2.14 The functions $\pi_j : x \mapsto x_j$, the coordinate projections are continuous on $\mathbb{R}^n$ (with respect to any of the norms $\| \;\|_i$, $i = 1, 2, \infty$):

$$|\pi_j(x) - \pi_j(a)| = |x_j - a_j| \leq \|x - a\| , \quad 1 \leq j \leq n.$$

2.15 Composite of continuous functions is continuous: Let X, Y, Z be topological spaces. Let $f : X \to Y$ be continuous at $p \in X$ and $g : Y \to Z$ be continuous at $q := f(p) \in Y$. Then $g \circ f : X \to Z$ is continuous at p.

Let $W \subset Z$ be an open set such that $(g \circ f)(p) = g(q) \in W$. Since g is continuous at q, there exists an open $V \ni q$ such that $g(V) \subset W$. Since f is continuous at p and $V \ni f(p) =$, there exists an open $U \ni x$ such that $f(U) \subset V$. Clearly, $(g \circ f)(U) \subset W$.

2.16 Let $f : X \to \mathbb{R}$ be a continuous function. Then $|f| : X \to Y$ defined by $|f|(x) := |f(x)|$ is continuous on X. For, it is the composite of two continuous functions $|f| = | \;| \circ f$, where $| \;| : \mathbb{R} \to \mathbb{R}$ is the modulus function, $| \;|(x) := |x|$.

2.17 Let X be a topological space. Let $\mathbb{R}^n$ be given the metric topology arising form the standard Euclidean metric. Let $f\colon X \to \mathbb{R}^n$. Then we can write $f(x) = (f_1(x), \dots, f_n(x))$. Note that $f_j(x) = \pi_j \circ f$ where π_j is the projection as in Item 2.14 on Page 15.

We claim that f is continuous iff each $f_j\colon X \to \mathbb{R}$, $1 \le j \le n$, is continuous. Assume that f is continuous. Since $f_j = \pi_j \circ f$, it follows from Items 2.14–2.15 on Page 15 that f_j is continuous.

Now the converse. Fix $a \in X$. Let $V \subset \mathbb{R}^n$ be open containing $f(a)$. Let $\varepsilon > 0$ be such that $B(f(a), \varepsilon) \subset V$. By continuity of f_j at a, there exists an open set $U_j \subset X$ such that $a \in U_j$ and $f_j(U_j) \subset B(f_j(a), \varepsilon/\sqrt{n})$, $1 \le j \le n$. Then $U := \cap_{j=1}^n U_j$ is an open set which contains a and is such that $f(x) \in B(f(a), \varepsilon)$ for all $x \in U$:

$$d(f(x), f(a))^2 = \sum_{j=1}^n (f_j(x) - f_j(a))^2 < n(\varepsilon^2/n) = \varepsilon^2.$$

Hence for $x \in U$, we have $f(x) \in B(f(a), \varepsilon) \subset V$, that is, f is continuous at a.

2.18 Let $f, g\colon X \to \mathbb{R}$ be continuous functions. Consider $\mathbb{R}^2$ with $\|\ \|$ being one of the three norms: $\|\ \|_1, \|\ \|_2, \|\ \|_{\max}$. Then the function $\varphi\colon X \to \mathbb{R}^2$ given by $\varphi(x) = (f(x), g(x))$ is continuous.

This is a special case of the last item.

2.19 The functions $\mathbb{R}^2 \to \mathbb{R}$ given by $\alpha\colon (x, y) \mapsto x + y$ and $\mu\colon (x, y) \mapsto xy$ are continuous.

To establish the continuity of these function we use Theorem 2 in Item 1.

Let $(a, b) \in \mathbb{R}^2$. Let $\varepsilon > 0$ be given. Assume $\delta > 0$ serves. We estimate

$$\begin{aligned} |\alpha(x,y) - \alpha(a,b)| = |(x+y) - (a+b)| &= |(x-a) + (y-b)| \\ &\le |x-a| + |y-b| \\ &\le d((x,y),(a,b)) + d((x,y),(a,b)). \end{aligned}$$

If $d((x, y), (a, b)) < \delta$, the above estimate suggests that we take $2\delta < \varepsilon$.

Let $\varepsilon > 0$ be given. Assume $\delta > 0$ serves. We may assume that $0 < \delta < 1$. If $d((x, y), (a, b)) < \delta$, then $|x - a| < \delta < 1$ and $|y - b| < \delta < 1$. Hence $|y| \le |y - b| + |b| < 1 + |b|$. We now estimate

$$\begin{aligned} |\mu(x,y) - \mu(a,b)| = |xy - ab| = |xy - ay + ay - ab| &\le |y||x-a| + |a||y-b| \\ &\le (1+|b|)|x-a| + |a||y-b| \\ &< M2\delta, \end{aligned}$$

where $M = \max\{1 + |b|, |a|\}$. If we choose $\delta < \frac{\varepsilon}{2M}$, as well as $\delta < 1$, the estimates above establish $|\mu(x, y) - \mu(a, b)| < \varepsilon$.

2.20 If f, g are continuous functions from a topological space to $\mathbb{R}$ and if $a, b \in \mathbb{R}$, then the functions $af + bg$ and fg are continuous. Hint: Use Items 15–19.

Thus the set $C(X, \mathbb{R})$ of all real valued continuous functions on a topological space is a vector space over $\mathbb{R}$. It is also a commutative ring with identity, in fact, an algebra over $\mathbb{R}$.

2.21 Given two real numbers a, b we wish to find a "formula" for $\max\{a, b\}$ and $\min\{a, b\}$. Given a, b, their mid point is $(a+b)/2$. To reach the maximum of these two, we need to move to the right for half of the distance between them, that is, we need to add $|a-b|/2$ to their mid point. Similar analysis can be done for minimum. Hence we arrive at the following formulas:

$$\max\{a, b\} = \frac{(a+b) + |a-b|}{2} \quad \text{and} \quad \min\{a, b\} = \frac{(a+b) - |a-b|}{2}.$$

2.22 If $f, g \colon X \to \mathbb{R}$ are two continuous functions on a topological space X, then $\max\{f, g\}$ and $\min\{f, g\}$ are continuous. This follows from Items 21, 16 and 20.

2.23 Any polynomial function $f \colon \mathbb{R}^n \to \mathbb{R}$ is continuous. This follows from Item 14 and 22. Examples of polynomial functions on $\mathbb{R}^2$ and $\mathbb{R}^3$ are $p(x, y) = 3x^2 + y^2 - xy^2 + 6x - 7y + 10$, $q(x, y, z) = z^{10} - 9y^2 + 17xyz^3 + 2012$ etc.

2.24 The map $\rho \colon \mathbb{R}^* \to \mathbb{R}^*$ given by $\rho(x) = 1/x$ is continuous. Look at the estimate:

$$|\rho(x) - \rho(y)| \leq \frac{|x-y|}{|xy|} \leq \frac{2|x-y|}{|x^2|},$$

if we restrict y in such a way that $|x-y| < |x|/2$.

2.25 Let $f \colon X \to \mathbb{R}$ be continuous and assume that $f(x) \neq 0$ for all $x \in X$. Then $1/f \colon X \to \mathbb{R}$ is continuous. For, $1/f$ is the composition $\rho \circ f$, where ρ is as in the last item.

2.26 Any linear map from $\mathbb{R}^n$ with any one of our three standard norms to any normed linear space is continuous. In particular, any linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$ is continuous.

More generally, any linear map $T \colon \mathbb{R}^n \to X$, where X is any normed linear space is (uniformly) continuous.

For let $\{e_i : 1 \leq i \leq n\}$ be the standard basis of $\mathbb{R}^n$. Then for any $x = (x_1, \ldots, x_n) = \sum_i x_i e_i \in \mathbb{R}^n$ we have

$$\begin{aligned} \|Tx\| = \left\| T\left(\sum_{i=1}^n x_i e_i\right) \right\| &\leq \sum_{i=1}^n |x_i| \, \|Te_i\| \\ &\leq M \sum_{i=1}^n \|x\|, \text{ where } M := \max\{\|Te_i\| : 1 \leq i \leq n\} \\ &= Mn \|x\|. \end{aligned}$$

Note that $|x_i| \leq \|x\|$ where $\| \ \|$ could be either $\| \ \|_1$, $\| \ \|_2$ or $\| \ \|_{\max}$. Hence $\|Tx - Ty\| = \|T(x-y)\| \leq Mn \|x - y\|$ so that T is Lipschitz and hence uniformly continuous.

2.27 Let $M_{m \times n}(\mathbb{R})$ denote the set of all $m \times n$ matrices with real entries. We identify it with $\mathbb{R}^{mn}$ using an obvious linear isomorphism:

$$X = (x_{ij}) \mapsto (x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{m1}, \ldots, x_{mn}).$$

We use any one of the standard norms on $M_{m \times n}(\mathbb{R})$. We let $M(n, \mathbb{R}) := M_{n \times n}(\mathbb{R})$. Then we have

(a) The 'transpose' map $X \mapsto X^T$ from $M(n, \mathbb{R})$ to itself is continuous. For, the map is $(x_{11}, x_{12}, \dots, x_{n1}, \dots, x_{nn}) \to (x_{11}, x_{21}, \dots, x_{1n}, \dots, x_{nn})$. The coordinate maps are $f_{ij}(X) = x_{ji}$ and hence are continuous. (See Item 17.)

(b) The 'trace' map $X \mapsto \mathrm{Tr}(X)$ is continuous from $M(n, \mathbb{R})$ to $\mathbb{R}$. Observe that it is a linear map.

(c) The determinant map $\det \colon M(n, \mathbb{R}) \to \mathbb{R}$, defined by $X \mapsto \det(X)$, is a "polynomial function" and hence is continuous. When $n = 2$ and the matrix is $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\det(X) = ad - bc$. For general n, recall the formula for the determinant (Laplace expansion) as an alternating sum, $\det(X) := \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) x_{1\sigma(1)} \cdots x_{n\sigma(n)}$.

2.28 One can use functions whose continuity are known to assert that certain subsets are open. This is a very useful observation.

(a) Since polynomial functions from $\mathbb{R}^n$ to $\mathbb{R}$ are continuous

i. The subsets $\{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$, $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\}$ and $\{(x, y) \in \mathbb{R}^2 : xy \neq 1\}$ are all open.

ii. The subset $\{(x, y) \in \mathbb{R}^2 : x^3 - 34x^2y - 28xy^2 - y^3 + 7xy - 19y + 125 \neq 0\}$ is open in $\mathbb{R}^2$.

iii. $\mathbb{R}^3 \setminus P$, where $P := \{(x, y, z) : ax + by + cz = d\}$ is a plane, is open in $\mathbb{R}^3$.

iv. The rectangle $R := (a, b) \times (c, d)$ is open in $\mathbb{R}^2$: $R = p_1^{-1}(a, b) \cap p_2^{-1}(c, d)$, where $p_1(x, y) = x$ etc.

(b) The set $\{f \in C[0, 1] : f(1/2) \neq 0\}$ in $X := (C[0, 1], \| \ \|_\infty)$ is open. *Hint:* Consider $T \colon X \to \mathbb{R}$ given by $T(f) := f(1/2)$.

(c) Let W be a vector subspace of $\mathbb{R}^n$. Then $\mathbb{R}^n \setminus W$ is open in $\mathbb{R}^n$. *Hint:* Write $\mathbb{R}^n = W \oplus W^\perp$ and let $u_1, \dots, u_k$ be an orthonormal basis of $W^\perp$. Then $x \in \mathbb{R}^n$ lies in W iff $\langle x, u_i \rangle = 0$ for all $1 \leq i \leq k$.

Alternately, consider the orthogonal projection $\pi \colon \mathbb{R}^n \to W^\perp$. Then $\mathbb{R}^n \setminus W = \pi^{-1}(W^\perp \setminus \{0\})$.

(d) $GL(n, \mathbb{R})$, the set of all invertible matrices is open in $M(n, \mathbb{R})$.

(e) The set of symmetric matrices, being a vector subspace, cannot be open in $M(n, \mathbb{R})$. *Hint:* See Item 11h.

Same holds true for the set of skew symmetric matrices.

(f) Let Y be an ordered set with the order topology. Let $f, g \colon X \to Y$ be continuous. Then the set $L := \{x \in X : f(x) < g(x)\}$ is an open set. *Hint:* Fix $a \in L$. Either there exists $\alpha \in Y$ such that $f(a) < \alpha < g(a)$ or none. In the former consider the open sets $V_1 := (-\infty, \alpha)$, $V_2 := (\alpha, \infty)$ in the former case and $V_1 = (-\infty, g(a))$ and $V_2 = (f(a), \infty)$. Consider $U := f^{-1}(V_1) \cap g^{-1}(V_2)$. Then $a \in U$. For each $x \in U$, we have $f(x) < g(x)$.

2.29 To check continuity, it suffices to show that the inverse images of basic elements in the codomain are open in the domain:

Lemma 4. *Let $(X_i, \mathcal{T}_i)$ be topological spaces $i = 1, 2$ and let $\mathcal{B}_2$ be a basis for $\mathcal{T}_2$. Then $f \colon (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ is continuous iff $f^{-1}(B_2) \in \mathcal{T}_1$ for all $B_2 \in \mathcal{B}_2$.* □

Item 6 is an immediate consequence of this.

2.30 Consider $M(n, \mathbb{R})$ the set of all $n \times n$ real matrices. Then the map Repetition: Item 27

$$\varphi \colon A \mapsto (a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{n1}, \ldots, a_{nn})$$

is a linear isomorphism of $M(n, \mathbb{R})$ onto $\mathbb{R}^{n^2}$. We use this to transfer the Euclidean norm on $\mathbb{R}^{n^2}$ to $M(n, \mathbb{R})$ as follows: $\|A\|^2 := \|\varphi(A)\|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2$.

Show that the map $M(n, \mathbb{R}) \times M(n, \mathbb{R}) \to M(n, \mathbb{R})$ given by $\mu(X, Y) = XY$, the matrix product is continuous.

2.31 Let X and Y be normed linear spaces. A linear map $T \colon X \to Y$ is continuous at $0 \in X$ iff there exists a positive constant C such that $\|Tx\| \leq C \|x\|$ for all $x \in X$. *Hint:* Use ε-δ definition of continuity at 0.

Deduce that a linear map between two normed linear space 's is continuous iff it is continuous at 0.

2.32 When do two norms $\| \ \|_j$, $j = 1, 2$ generate the same topology on a vector space X? They do iff the identity maps $I \colon (X, \| \ \|_1) \to (X, \| \ \|_2)$ and $I \colon (X, \| \ \|_2) \to (X, \| \ \|_1)$ are continuous. (Why?) By the last item, this means that we can find positive constants C_1 and C_2 such that $C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1$ for all $x \in X$. We thus arrive at the following result.

Two norms $\| \ \|_j$, $j = 1, 2$ generate the same topology on a vector space X iff positive constants C_1 and C_2 such that $C_1 \|x\|_1 \leq \|x\|_2 \leq C_2 \|x\|_1$ for all $x \in X$. We then say that the two norms $\| \ \|_1$ and $\| \ \|_2$ are *equivalent.*

2.33 In $\mathbb{R}^n$, the three norms $\| \ \|_1$, $\| \ \|_2$ and $\| \ \|_\infty$ are equivalent. This follows from Item 26. It follows also from the observation:

$$\frac{1}{n} \|x\|_1 \leq \frac{1}{\sqrt{n}} \|x\|_2 \leq \|x\|_\infty \leq \|x\|_2 \leq \|x\|_1 .$$

Later, we shall show that all norms on $\mathbb{R}^n$ induce the same topology, that is, they are all equivalent.

2.34 Closed Sets: Let $(X, \mathcal{T})$ be a topological space. A set $F \subset X$ is called a closed set (or said to be closed) in X if $X \setminus F$ is open in X. Let $\mathcal{C}$ be the class of all closed subsets in X. The following are more or less immediate:

(a) $\emptyset, X \in \mathcal{C}$.

(b) If $\{F_i : i \in I\}$ is a family of closed sets, then their intersection $\cap_{i \in I} F_i$ is again closed.

(c) If F_1 and F_2 are closed, then so is $F_1 \cup F_2$.

2.35 Examples of Closed Sets:

(a) $\emptyset$ and X are both open and closed in any topological space.

(b) $\mathbb{Z}$ is closed in $\mathbb{R}$.

(c) There exist sets which are neither open nor closed: $[0, 1)$, $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$ with usual topology,

(d) Any finite subset of a metric space is closed.

(e) Any closed ball $B[x, r]$ in a metric space is closed. Hence any closed interval $[a, b]$ is closed in $\mathbb{R}$.

(f) Any sphere $S(x, r) := \{y \in X : d(x, y) = r\}$ in a metric space is closed.

(g) The set $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ is closed in $\mathbb{R}$.

(h) The set $(-\infty, 0) \cup [1, \infty)$ is closed in $\mathbb{R}$ with lower limit topology but not closed in $\mathbb{R}$ with the usual topology.

(i) The only subsets of $\mathbb{R}$ which are both open and closed are $\emptyset$ and $\mathbb{R}$.

Let A be both open and closed in $\mathbb{R}$. Assume that A is not empty. We need to prove $A = \mathbb{R}$. Let $a \in A$. Since a is open there exists $r > 0$ such that $(a - r, a + r) \subset A$. Consider

$$E := \{c \in \mathbb{R} : c > a, (a - \varepsilon, c) \subset A\}.$$

Then $a + \varepsilon \in E$. If $\sup E = \infty$, then it follows that $(a - \varepsilon, \infty) \subset A$. Assume $\sup E = \alpha \in \mathbb{R}$. Now either $\alpha \in A$ or $\alpha \notin A$.

If $\alpha \in A$, since A is open there exists $\delta > 0$ such that $(\alpha - \delta, \alpha + \delta) \subset A$. Since $\alpha - \delta < \alpha = \sup E$, there exists $c \in E$ such that $(a - \varepsilon, c) \subset A$. Clearly, $(a - \varepsilon, \alpha + \delta) = (a - \varepsilon, c) \cup (\alpha - \delta, \alpha + \delta) \subset A$. Hence $\alpha + (\delta/2) \in E$, contradiction to $\alpha = \sup E$.

If $\alpha \notin A$, then $\alpha \in \mathbb{R} \setminus A$, an open set. Hence there exists $\delta > 0$ such that $(\alpha - \delta, \alpha + \delta) \subset \mathbb{R} \setminus A$. As earlier, there exists $c \in E$ such that $\alpha - \delta < c$. Hence the interval $(\alpha - \delta, c)$ lies in both A and its complement, a contradiction. Thus we conclude that $\sup E = \infty$ so that $(a - \varepsilon, \infty) \subset A$. Similarly, we can conclude $(-\infty, a + \varepsilon) \subset A$ and hence $A = \mathbb{R}$.

(j) The set $[0, 1)$ is neither closed nor open in $\mathbb{R}$.

(k) Any subset of a discrete space is open as well as closed.

(l) Any subset $A \subset \mathbb{R}^*$ is closed in $\mathbb{R}$ with VIP topology with 0 as the VIP.

(m) What are the sets which are both open and closed in $\mathbb{R}$ with VIP topology with 0 as the VIP?

(n) Any subset of $\mathbb{R}$ containing 0 is closed in $\mathbb{R}$ with the outcast topology with 0 as the outcast.

(o) What are the sets which are both open and closed in $\mathbb{R}$ with the outcast topology with 0 as the outcast?

(p) Any vector subspace of $\mathbb{R}^n$ is closed. So are its translates.

Let V be a vector subspace of $\mathbb{R}^n$. Let $\mathbb{R}^n = V \oplus V^\perp$ be the orthogonal decomposition. Then $x \in \mathbb{R}^n$ lies in V iff $v \cdot u \equiv \langle x, u \rangle = 0$ for all $u \in V^\perp$. The map $f_u \colon \mathbb{R}^n \to \mathbb{R}$ given by $f_u(x) := x \cdot u$ is linear and hence by Item 26, it is continuous. Hence the kernel $f_u^{-1}(0)$ is a closed subset of $\mathbb{R}^n$. Since $V = \cap_{u \in V^\perp} f_u^{-1}(0)$ is the intersection of closed sets, V is closed.

(q) The set of $n \times n$ symmetric matrices and the set of $n \times n$ skew-symmetric matrices are closed in $M(n, \mathbb{R})$.

(r) The set $GL(n, \mathbb{R})$ is not closed in $M(n, \mathbb{R})$.

(s) The set of singular matrices in $M(n, \mathbb{R})$ is closed.

(t) The set $\{f \in C[0, 1] : f(1/2) = 0\}$ in $X := (C[0, 1], \| \ \|_\infty)$ is closed.

(u) The sets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are neither closed nor open in $\mathbb{R}$.

2.36 We have the following characterization of continuity in terms of closed sets.

Theorem 5. *Let $f: X \to Y$ be a map between topological spaces. Then f is continuous iff $f^{-1}(B)$ is closed in X for every closed set $B \subset Y$.* □

2.37 As we did earlier in the case of continuity and open sets, we may use the above theorem to assert that certain subsets are closed.

(a) The set $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$, $\{(x, y) \in \mathbb{R}^2 : xy = 1\}$, $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ are closed in $\mathbb{R}^2$.

(b) The closed rectangle $R := [a, b] \times [c, d]$ is closed in $\mathbb{R}^2$.

(c) The unit n-dimensional sphere $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is closed in $\mathbb{R}^{n+1}$.

(d) The set $SL(n, \mathbb{R})$ of matrices $A \in M(n, \mathbb{R})$ with determinant 1 is closed in $M(n, \mathbb{R})$.

(e) The subset of matrices whose trace is 0 is closed in $M(n, \mathbb{R})$. (Also follows from Item 35p.)

(f) The set $O(n)$ of orthogonal matrices is closed in $M(n, \mathbb{R})$. *Hint:* The maps $M(n, \mathbb{R}) \to \mathbb{R}$ given by $A \mapsto R_i(A) \cdot R_j(A) \equiv \sum_{k=1}^{n} a_{ik}a_{jk}$ are continuous. Here $R_i(A)$ denotes the i-th row of A.

Or, use the fact that the map $F: A \mapsto (A, A^T)$ composed with $(A, B) \to AB$ is continuous. Then $O(n, \mathbb{R})$ is the inverse image $F^{-1}(I)$.

(g) The set of singular matrices in $M(n, \mathbb{R})$ is closed.

(h) The set of nilpotent matrices in $M(n, \mathbb{R})$ is closed.

(i) Keep the notation of Item 28f. Then the set $\{x \in X : f(x) \leq g(x)\}$ is closed. .

2.38 Let A be a subset of a topological space. The characteristic function χ_A of A is defined by

$$\chi_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

What can you conclude about A if the function χ_A is continuous on X?

2.39 Go back to Item 21. If you understand the principle in work in it, you would have foreseen what follows. For any set A of a topological space $(X, \mathcal{T})$, the smallest closed set containing A exists. It is denoted by $\overline{A}$ and called the closure of A in X. (Compare this with the existence of the smallest topology containing a family $\{A_i : i \in I\}$ of subsets of a set X.) Note that $A \subset \overline{A}$.

2.40 Examples of closures:

(a) The closure of $(a, b) \subset \mathbb{R}$ is $[a, b]$.

(b) The closure of $\mathbb{Q}$ in $\mathbb{R}$ is $\mathbb{R}$.

(c) The closure of an open ball $B(x, r)$ in $\mathbb{R}^n$ is the closed ball $B[x, r]$. In a general metric space, this need not be true. Consider $B(x, 1)$ and $B[x, 1]$ in a discrete metric space with at least two points.

(d) Let $\mathbb{R}$ be given the VIP topology with 0 as the VIP. Then the closure of $A = \{0\}$ is $\mathbb{R}$. The closure of $\mathbb{R} \setminus \mathbb{Q}$ is itself. The closure of $\{a\}$ is itself if $a \neq 0$.

(e) Investigate the case of $\mathbb{R}$ with outcast topology.

2.41 Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then $x \notin \overline{A}$ iff there exists an open set $U \ni x$ with $U \cap A = \emptyset$. Hence, $x \in \overline{A}$ iff for every open set $U \ni x$, we have $U \cap A \neq \emptyset$. This suggests the following definition.

3 Limit and Cluster Points

3.1 $x \in X$ is said to be a *limit point* of A if for every open set $U \ni x$, we have $U \cap A \neq \emptyset$.

This is NOT the standard definition and hence should not be confused with the notion of cluster or an accumulation point which we shall see below. We shall follow our nomenclature only.

3.2 Consider the lower limit topology $\mathcal{T}_L$ on $\mathbb{R}$. Let $A = [a, b)$. Is b in the closure of A? That is, is b a limit point of $[a, b)$?

3.3 Consider $\mathbb{R}^2$ with order topology. Let $Q := \{(x, y) \in \mathbb{R}^2 : x > 0 \ \& \ y > 0\}$ be the first quadrant. What is $\overline{Q}$? Points of the other three quadrants are not in the closure. Any point $(a, 0)$ with $a > 0$ is in $\overline{Q}$ while $(0, 0)$ is not.

3.4 Every point of A is a limit point of A.

3.5 $x \in \overline{A}$ iff x is a limit point of A. (This is true because of our definition of a limit point. See Item 12.)

For, let $x \in \overline{A}$ and $U \ni x$ be open. If $U \cap A = \emptyset$, then $A \subset X \setminus U$, a closed set and hence $\overline{A} \subset X \setminus U$. But $x \in \overline{A}$ and $x \notin X \setminus U$, a contradiction. Hence x is a limit point of A.

Conversely, if x is a limit point of A and $x \notin \overline{A}$, then $x \in U := X \setminus \overline{A}$, an open set. But $U \cap A \subset U \cap \overline{A} = \emptyset$. Hence x is a not a limit point of A, a contradiction.

3.6 Let (X, d) be a metric space, $A \subset X$. Then $x \in X$ is a limit point of A iff there exists a sequence (a_n) in A such that $a_n \to x$.

3.7 With the notation as in the last item, $x \in \overline{A}$ or x is a limit point of A iff $d_A(x) = 0$.

3.8 In any normed linear space $(X, \| \ \|)$, the closure of an open ball $B(p, r)$ is $B[p, r]$. Thus, $q \in X$ is a limit point of $B(p, r)$ iff $d(p, q) \leq r$. In particular, $\overline{B(p, r)} = B[p, r]$.

If $q \in B[p, r]$, consider the line segment $(1 - t)p + tq$, $0 \leq t \leq 1$. Draw picture. All points with $0 \leq t < 1$ are in $B(p, r)$. From this line segment, you can find a sequence $p_k \in B(p, r)$ which converges to q. Or, consider $B(q, \varepsilon)$ for $\varepsilon > 0$. Then for any $0 < t < 1$, we have

$$d(q, (1 - t)p + tq) = \| (1 - t)(q - p) \| = (1 - t)r < \varepsilon,$$

if t is near to 1. Thus, any open set containing q contains points of $B(p, r)$ other than q,

3.9 The set theoretic results about the closure operation:

(a) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(c) $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. Strict containment can occur.

(d) $\cup_{i \in I} \overline{A_i} \subset \overline{\cup_{i \in I} A}$. Strict containment can occur.

(a) follows from the fact that any closed set that contains B will contain A. In particular, the smallest closed set $\overline{B}$ that contains B will contain A. Hence $\overline{A}$, the smallest closed set containing A, will be contained in $\overline{B}$.

(b). Since LHS is the smallest closed set containing $A \cup B$, and since $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$, it follows that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. Let $x \in \overline{A} \cup \overline{B}$. Assume WLOG that $x \in \overline{B}$. Then for any open set $U \ni x$, we have $\emptyset \neq U \cap B \subset U \cap (A \cup B)$. That is, x is a limit point $A \cup B$ and hence $x \in \overline{A \cup B}$.

(c) Since $\overline{A} \cap \overline{B}$ is a closed set containing $A \cap B$, it follows that $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. An instance of the strict containment, consider $A = \mathbb{Q}$ and $B = \mathbb{R} \setminus \mathbb{Q}$ in $\mathbb{R}$.

(d) Consider $\mathbb{Q} = \cup_{x \in \mathbb{Q}} \{x\}$.

3.10 $x \in X$ is a *cluster* or an *accumulation point* of A iff for every open set $U \ni x$, the set $(U \setminus \{x\}) \cap A \neq \emptyset$, that is, any open set $U \in x$ contains a point of A other than x.

3.11 Intuitively, A accumulates or clusters around x. (They are like celebrities of A!) Obviously, any cluster point of A is a limit point of A, but not conversely. The notion of a cluster point is much stronger and more stringent than that of a limit point.

3.12 Let $(X, \mathcal{T})$ be any topological space and $A \subset X$. Then $\overline{A}$ is the union of A and the cluster points of A. (Compare and contrast this with Item 5.)

3.13 Every point of $A = \mathbb{Z} \subset \mathbb{R}$ is a limit point of A but there exists no cluster point of A in $\mathbb{R}$.

Since $\overline{\mathbb{Z}} = \mathbb{Z}$, this examples also shows that 'limit point' cannot be replaced by 'cluster point' in Item 5.

3.14 Consider $\mathbb{R}$ with VIP topology with 0 as the VIP. Then any nonzero real number is a cluster point of $A = \{0\}$. Zero is obviously a limit point of A but not a cluster point of A.

3.15 The last example also shows that the following can occur. x may be a cluster point of A, but there may exist open sets $U \ni x$ with $U \cap A$ is finite!

3.16 Any point in any ball (open or closed) in an normed linear space is a cluster point of the ball. The idea in Item 8 proves this.

3.17 Analyze the situation in a metric space. In a metric space, if x is a cluster point of A, then every open set $U \ni x$ will contain infinitely many points of A. The proof suggested the following definition.

3.18 A topological space X is said to be *Hausdorff* iff for every pair $x, y \in X$ of distinct points, there exist open set U, V such that $x \in U$ and $y \in V$ and $U \cap V = \emptyset$. That is, any two distinct points can be "separated by open sets."

We also say that a topology $\mathcal{T}$ on a set X is Hausdorff if the space $(X, \mathcal{T})$ is Hausdorff.

3.19 Let $(X, \mathcal{T})$ be a Hausdorff (topological) space and $A \subset X$. Then $x \in X$ is a cluster point of A iff for every open set $U \ni x$, the set $U \cap A$ is infinite.

We prove this by contradiction. Let x be a cluster point of A and assume that there exists an open set U such that $x \in U$ and $U \cap A$ is finite. Let $(U \setminus \{x\}) \cap A = \{a_1, \dots, a_n\}$. Since X is Hausdorff, for each j, $1 \leq j \leq n$, the exists an open set $U_j \ni x$ and $V_j \ni a_j$ such that $U_j \cap V_j = \emptyset$, $1 \leq j \leq n$. Then $U = \cap_{j=1}^{n} U_j$ is an open set such that $U \cap A$ is at most $\{x\}$.

3.20 A finite set in a Hausdorff space cannot have a cluster point. (Hausdorff condition is required. Look at $\mathbb{R}$ with VIP topology with zero as the VIP and $A = \{0\}$.) If a subset A of a Hausdorff space X has a cluster point, then A is infinite.

But there exists an infinite set in a Hausdorff space which has no cluster point. Look at $\mathbb{Z}$ in $\mathbb{R}$.

3.21 Let us now look at some examples of Hausdorff spaces.

(a) Any metric space is Hausdorff. For if $x_1, x_2 \in (X, d)$ are distinct, then $d(x_1, x_2) > 0$. Let $r = d(x_1, x_2)/2$. Then $B(x_j, r)$ is an open set containing x_j and $B(x_1, r) \cap B(x_2, r) = \emptyset$. For, x is a common point, then

$$d(x_1, x_2) \leq d(x_1, x) + d(x, x_2) < r + r < d(x_1, x_2),$$

a contradiction. In particular, $\mathbb{R}^n$ with the standard metric and normed linear spaces are Hausdorff.

(b) Any discrete topology is Hausdorff.

(c) The indiscrete topology on a set X with at least two elements is not Hausdorff.

(d) $(\mathbb{R}, \mathcal{T}_V)$ with 0 VIP is not Hausdorff.

(e) If we have $\mathcal{T}_1 \leq \mathcal{T}_2$ and $\mathcal{T}_1$ is Hausdorff, so is $\mathcal{T}_2$. As special cases, we have the following.

  i. The order topology on $\mathbb{R}^2$ with dictionary order is Hausdorff.

  ii. The lower limit topology on $\mathbb{R}$ is Hausdorff.

(f) Let $f \colon X \to Y$ be a 1-1 continuous function. If Y is Hausdorff, so is X. Let x_1, x_2 be distinct elements of X. Then $f(x_1)$ and $f(x_2)$ are distinct elements of Y and hence there exist disjoint open sets $V_j \ni f(x_j)$. Consider $U_j := f^{-1}(V_j)$, $j = 1, 2$.

3.22 We now give an example of a Hausdorff space in which two disjoint closed sets cannot be separated by open sets.

Let $X = \mathbb{R}$. For any fixed $p \in \mathbb{R}$ and $m \in \mathbb{N}$, let $B_{p,m} := \{p + km : k \in \mathbb{Z}_+\}$. Let $\mathcal{T}$ be the set of all subsets $U \subset \mathbb{R}$ such that for any $p \in U$, there exists $m \in \mathbb{N}$ such that $B_{p,m} \subset U$. Then it is easy to check that $\mathcal{T}$ is a topology on $\mathbb{R}$.

We claim that it is Hausdorff. Consider $p \neq q$ in $\mathbb{R}$. If $p - q$ is not an integer, then $B_{p,m} \cap B_{q,m} = \emptyset$ for any $m \in \mathbb{N}$. For, otherwise, if z is a common element, then $z = p + km = q + kn$. It follows that $p - q = k(n - m)$, an integer — a contradiction.

If $p - q = m \in \mathbb{Z}$,say, then the basic open sets $B_{p,2m}$ abd $B_{q,2m}$ separate p and q. (Verify this.)

Fix $p \in \mathbb{R}$ and $m \in \mathbb{N}$. We claim that each element of $\{p - km : k \in \mathbb{N}\}$ is a cluster point of $B_{p,m}$. Let $q = p - km$. Consider a basic open set $B_{q,n} \ni q$. The element $q + mkn \in B_{q,n}$. Since

$$q + kmn = p - km + kmn = p + mk(n - 1) \in B_{p,m},$$

the claim follows.

Consider now the two disjoint sets $F_1 := \{1\}$ and $F_2 := \{x \in \mathbb{R} : x \leq 0\}$. F_1 is closed since the space is Hausdorff. F_2 is also closed, since its complement is open. For, note that for any $p > 0$ and $m \in \mathbb{N}$, $U_{p,m} \subset (0, \infty)$.

We claim that they cannot be separated by open sets. Assume the contrary. Let $U_1 \supset F_1$ and $U_2 \supset F_2$ be open sets separating them. Then there exists a basic open set $B_{1,m} \subset U_1$. Now $1-2m$ is a cluster point of $B_{1,m}$. But no point of F_2 can be cluster point of U_1 since $F_2 \subset U_2$ and $U_2 \cap F_1 \subset U_2 \cap U_1 = \emptyset$.

Thus we have an example of a Hausdorff space in which two distinct points can be separated by open sets but not any two disjoint closed sets.

3.23 This examples is from Munkres. Consider $\mathbb{R}$ with the topology $\mathcal{T}_K$ whose basic open sets are open intervals (a, b) and open intervals $(a, b) \setminus K$ where $K := \{1/n : n \in \mathbb{N}\}$. Then $\{0\}$ and K are disjoint closed subsets which cannot be separated by open sets.

3.24 We say that a sequence (x_n) in a topological space $(X, \mathcal{T})$ *converges* to a point $x \in X$, if for every open set $U \ni x$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. The point x is called the limit of the sequence and (x_n) is said to be *convergent*.

3.25 If $(X, \mathcal{T})$ is a Hausdorff (topological) space, then any convergent sequence has a unique limit.

This need not be true in a general space. For instance, if we consider $\mathbb{R}$ with indiscrete topology, any sequence is convergent to any point of $\mathbb{R}$!

3.26 Consider the sequence $(1/n)$ in $\mathbb{R}$ with co-finite topology. Then $1/n \to x$, for any $x \in \mathbb{R}$! (Co-finite topology on $\mathbb{R}$ is not the discrete topology.)

3.27 In any Hausdorff space, any finite set is closed. Details!

This need not be true in an arbitrary topological space. For instance, consider the indiscrete topology on $\mathbb{R}$. Or, the set $\{x, 0\}$, $x \neq 0$, in $\mathbb{R}$ with VIP topology with VIP=0.

Hence conclude: The topology of any finite Hausdorff is discrete. (See also Item 11l.)

3.28 Examples of Convergent sequences:

(a) The only convergent sequences in any discrete space are eventually constant sequences.

For, let $x_n \to x$. Then $\{x\} \ni x$ is open so there exists $N \in \mathbb{N}$ such that $\geq N \implies x_k \in \{x\}$. Thus $x_k = x$ for $k \geq N$.

(b) In the normed linear space $(B(X, \mathbb{R}), \| \|_\infty)$, a sequence (f_n) converges to $f \in B(X, \mathbb{R})$ iff f_n converges to f uniformly on X.

Assume that $f_n \to f$ uniformly on X. Let $\varepsilon > 0$ be given. Choose N such that for all $k \geq N$, and $x \in X$, we have $|f(x) - f_k(x)| < \varepsilon/2$. Hence $\sup_{x \in X} |f(x) - f_k(x)| \leq \varepsilon/2 < \varepsilon$. That is, $\| f_k - f \|_\infty < \varepsilon$ for $k \geq N$ and hence f_k converge to f in the norm. Other way implication is easier.

(c) A sequence (x_k) in $\mathbb{R}^n$ converges to $x \in \mathbb{R}^n$ iff $x_{kj} \to x_j$ as $k \to \infty$ for $1 \leq j \leq n$.

3.29 Let $f \colon X \to Y$ be continuous. Let (x_n) be a sequence in X such that $x_n \to x$. Then $f(x_n) \to f(x)$ in Y.

Let $V \subset Y$ be an open set containing $f(x)$. Since f is continuous at x, there exists an open set $U \ni x$ such that $f(z) \in V$ fo rany $z \in U$. Since $x_n \to x$ there exists $N \in \mathbb{N}$ such that for all $k \geq N$, we ave $x_k \in U$ and hence $f(x_k) \in V$.

3.30 We analyzed the proof of Item 6 and arrived at the following conclusion:

> Let $(X, \mathcal{T})$ be a space with the following property: For every $x \in X$, there exists a countable collection of open sets $\{U_{n,x} : n \in \mathbb{N}\}$ such that
>
> (a) For every open set $U \ni x$, there exists n such that $x \in U_{n,x} \subset U$
>
> (b) $\cap_n U_{n,x} = \{x\}$.
>
> Then, $x \in X$ is a limit point of $A \subset X$ iff there exists a sequence (a_n) in A such that $a_n \to x$.

4 First and Second Countable Spaces

4.1 The foregoing item led us to the following concepts.

4.2 Let $(X, \mathcal{T})$ be a topological space and $p \in X$. Then by a *local base* at p, we mean a family $\{U_{p,i} : i \in I\}$ of open sets containing p with the property that if U is an open set containing p, then there exists $i \in I$ such that $x \in U_{p,i} \subset U$.

A typical example to keep in mind: $\{B(p, r) : r > 0\}$ is a local base at p in a metric space X.

4.3 A space is said to be *first countable* if there exists a countable local base at every point $p \in X$.

4.4 Observe that if $(X, \mathcal{T})$ is first countable, then we may assume that a local base $\{U_{p,n} : n \in \mathbb{N}\}$ at p is decreasing sequence. For, if $\{V_{p,n}\}$ is a local base at p, consider $U_{p,n} := V_{p,1} \cap \cdots \cap V_{p,n}$.

4.5 We look at some examples:

(a) In $\mathbb{R}$ with standard topology, $\{(p - \frac{1}{n}, p + \frac{1}{n}) : n \in \mathbb{N}\}$ is a local base at p. Hence $\mathbb{R}$ is first countable. More generally, $\{B(p, 1/n) : n \in \mathbb{N}\}$ is a local base at p in any metric space. Hence any metric is first countable.

(b) If $\mathbb{R}$ is endowed with the discrete topology, then a local base at x can be taken as $\{x\}$. Hence $\mathbb{R}$ with discrete topology is first countable.

(c) Consider $\mathbb{R}$ with VIP topology. (Convention: VIP is always 0.) Then the set $\{p, 0\}$ is a local base at any $p \in \mathbb{R}$. (If $p = 0$, then the set $\{p, 0\} = \{0\}$!) Hence $\mathbb{R}$ with VIP topology is first countable.

(d) Any indiscrete topology is first countable.

(e) $\mathbb{R}$ with the lower limit topology $\mathcal{T}_L$ is first countable. At any $a \in \mathbb{R}$, consider $\{[a, a + \frac{1}{n}) : n \in \mathbb{N}\}$.

4.6 Let $(X, \mathcal{T})$ be a Hausdorff, first countable space. Let $\{U_{p,n} : n \in \mathbb{N}\}$ be a countable local base. Then $\cap_n U_{n,p} = \{p\}$. (We do not need the full power of Hausdorff condition. We could have achieved the same result with less stringent hypothesis, but we shall not worry about this!)

4.7 In view of Item 30 and Item 6, we have the following.

Theorem 6. *Let $(X, \mathcal{T})$ be first countable and Hausdorff. Then x is a limit point of A iff there exists a sequence (a_n) in such that $a_n \to x$.*

4.8 We say that a topological space $(X, \mathcal{T})$ is *second countable* if there exists a countable basis for $\mathcal{T}$.

4.9 Clearly, any second countable space is first countable.

4.10 Examples and non-examples:

(a) $\mathbb{R}$ with the standard topology is second countable. (See Item 17c.)

(b) A discrete space X is second countable iff the set X is countable.

(c) $\mathbb{R}$ with VIP topology is first countable but not second countable. Why? Consider the basis $\{\{x,0\} : x \in \mathbb{R}\}$. If $\{B_n : n \in \mathbb{N}\}$ is a countable basis, then for $x \in \{x,0\}$ there will be $n(x) \in \mathbb{N}$ such that $x \in B_{n(x)} \subset \{x,0\}$. Since $B_{n(x)}$ will always contain $\{0,x\}$, it follows that $B_{n(x)} = \{0,x\}$. But the family $\{\{x,0\} : x \in \mathbb{R}\}$ is uncountable where as $\{B_n : n \in \mathbb{N}\}$ is countable.

(d) The outcast topology on $\mathbb{R}$ is first countable but not second countable. For the second countability part, argue with $\{\{x,0\} : x \neq 0\}$ and consider $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{N}$.

(e) The lower limit topology on $\mathbb{R}$ is not 2nd countable. Assume the contrary and let $\{B_n : n \in \mathbb{N}\}$ be a countable basis. For each basic open set $[a, a+1)$ there exists $n(a) \in \mathbb{N}$ such that $a \in B_{n(a)} \subset [a, a+1)$. Observe that $\inf B_{n(a)} = a = \inf[a, a+1)$. Hence the map $f\colon \mathbb{R} \to \mathbb{N}$ given by $f(a) = n(a)$ is one-one. That is, R is countable, an absurdity.

Thus $(R, \mathcal{T}_L)$ is a first countable, separable space which is not second countable.

(f) Any indiscrete space is second countable.

4.11 Think over this: What will be the counter part (in terms of open sets) of the smallest closed set containing A? It is the largest open set contained in A. It is called the *interior* of A and is denoted by $\mathrm{Int}\,(A)$.

4.12 Examples of interior of a set:

(a) The interior of an open set is itself.

(b) The interior of $[a,b] \subset \mathbb{R}$ is (a,b).

(c) The interior of $\mathbb{Q} \subset \mathbb{R}$ is the empty set. What is $\mathrm{Int}\,(\mathbb{R} \setminus \mathbb{Q})$?

(d) The interior of a proper vector subspace of $\mathbb{R}^n$ is empty. Does this generalize to any normed linear space ?

(e) The interior of a closed ball $B[p,r]$ in any normed linear space is the open ball $B(p,r)$. In a general metric space, such a result is not true.

(f) Let $(X, \mathcal{T})$ be a discrete space. Then $\mathrm{Int}\,(A) = A$ for any $A \subset X$.

(g) Let $(X, \mathcal{T})$ be an indiscrete space. Then $\mathrm{Int}\,(A) = \emptyset$ for any $A \subset X$, $A \neq X$.

(h) Consider $\mathbb{R}$ with the VIP topology (VIP is 0). The interior of $\mathbb{R}^*$ is the empty set. What is $\mathrm{Int}\,(\mathbb{Q})$ and $\mathrm{Int}\,(\mathbb{R} \setminus \mathbb{Q})$ in this topology? More generally, if $0 \in A$, then $\mathrm{Int}\,(A) = A$ and if $0 \notin A$, then $\mathrm{Int}\,(A) = \emptyset$.

(i) Consider $\mathbb{R}$ with the outcast topology (outcast is 0). The interior of any set A is $A \setminus \{0\}$.

4.13 A is open iff $A = \mathrm{Int}\,(A)$.

4.14 Set theoretic results about the interior operation:

(a) If $A \subset B$, then $\mathrm{Int}\,(A) \subset \mathrm{Int}\,(B)$.

(b) $\mathrm{Int}\,(A) \cup \mathrm{Int}\,(B) \subset \mathrm{Int}\,(A \cup B)$.

(c) $\mathrm{Int}\,(A \cap B) = \mathrm{Int}\,(A) \cap \mathrm{Int}\,(B)$.

(d) $\cup_{i \in I} \mathrm{Int}\,(A_i) \subset \mathrm{Int}\,(\cup_{i \in I} A_i)$.

4.15 Let X be a (metric) space and $A \subset X$. A point $x \in X$ is said to be a *boundary point* of A in X if every open set that contains x intersects both A and $X \setminus A$ non-trivially. The *boundary* of A in X is the set of boundary points of A in X. We denote it by ∂A.

4.16 Find the boundaries of each of the following sets:

(a) $A_1 = (a, b] \subset \mathbb{R}$ with the standard topology.

(b) $A_2 = \mathbb{R} \setminus \{0\} \subset \mathbb{R}$ with the standard topology.

(c) $A = \mathbb{Q} \subset \mathbb{R}$ with the standard topology.

(d) $\partial\emptyset = \emptyset = \partial X$ for any topological space X.

(e) The boundary of an open or closed ball in $\mathbb{R}^n$ is the sphere: $\partial B(x, r) = \partial B[x, r] = S(x, r) := \{y \in \mathbb{R}^n : d(x, y) = r\}$. Is this true in an normed linear space ? in an arbitrary metric space?

(f) In $\mathbb{R}$ with VIP topology and $\mathbb{R}$ with outcast topology, find ∂A, where $A = \{0\}, \{x\}, \mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$. ($x$ is a nonzero real number.)

(g) Let B be an open ball in $\mathbb{R}^n$. Find the boundary of B minus a finite number of points.

(h) Let $A := \{z \in \mathbb{C} : z = re^{it}, r \in [0, 1], t \in (0, 2\pi)\}$. (Draw a picture.) Find the boundary of A.

4.17 A few more examples to sharpen our geometric intuition.

(a) Consider $A = \mathbb{R} \times \{0\} \subset \mathbb{R}^2$. What is the boundary of A in $\mathbb{R}^2$?

(b) $A = U_1 \cup U_2 \cup U_3$ is the subset of $\mathbb{R}^2$ where $U_1 := \{x^2 + y^2 < 1, y > 0\}$, $U_2 := \{-1 \leq x \leq 1, y = 0\}$ and $U_3 := \{x^2 + y^2 = 1, y < 0\}$.

(c) $A = \{(x, y) : x^2 + y^2 = 1\}$.

4.18 Show that for any subset A of a topological space $(X, \mathcal{T})$, $\partial A = \overline{A} \cap \overline{X \setminus A}$. (This is the standard definition.)

4.19 While trying to prove the equivalence of the definition of continuity at a point (of a function between two metric spaces) with the sequential definition, we established the following.

Theorem 7. *Let X and Y be arbitrary topological spaces and $p \in X$. Let $f : X \to Y$ be a map.*

1. If f is continuous at p, then for every sequence (x_n) in X converging to p, we have $f(x_n) \to f(p)$.

2. Assume that X is first countable and Hausdorff. Assume further that f has the property that for every sequence (x_n) converging to p, the sequence $(f(x_n))$ converges to $f(p)$ in Y. Then f is continuous at p.

Proof. 1. Let $V \subset Y$ be open with $f(p) \in V$. By continuity of f at p, there exists an open set $U \ni p$ such that for $x \in U$, we have $f(x) \in V$. Since $x_n \to p$, for this U, there exists $N \in \mathbb{N}$ such that $k \geq N \implies x_k \in U$. Hence for $k \geq N$, we see that $f(x_k) \in V$, that is, $f(x_k) \to f(x)$.

2. We prove this by contradiction. Assume that (B_n) is a local base at p such that $B_{n+1} \subset B_n$ and $\cap_n B_n = \{p\}$. Since f is not continuous at p, there exists an open set $V \ni f(p)$ such that given any open set $U \ni p$, there exists $x \in U$ such that $f(x) \notin V$. In particular, for each $n \in \mathbb{N}$, there exists $x_n \in B_n$ such that $f(x_n) \notin V$. Clearly, $x_n \to p$. For, let $W \ni p$ be an open set. Then there exists N such that $p \in B_N \subset W$. If $k \geq N$, then $x_k \in B_k \subset B_N \subset W$. Thus, we conclude $x_n \to p$. Now by hypothesis, $f(x_n) \to f(p)$. If we apply the definition of convergence to the set V, we find that $f(x_n) \notin V$ for any n. □

5 Dense Subsets in a Topological Space

5.1 A subset $D \subset X$ of a topological space is dense in X if for every *nonempty* open set $U \subset X$, we have $D \cap U \neq \emptyset$, that is U intersects D non-trivially.

5.2 Examples of dense sets:

(a) $\mathbb{Q}$ is dense in $\mathbb{R}$. Is $\mathbb{R} \setminus \mathbb{Q}$ dense in $\mathbb{R}$? Can you think of a countable dense subset in $\mathbb{R}^2$? in $\mathbb{R}^n$?

(b) In $\mathbb{R}$, with the lower limit topology, the sets $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense.

(c) The set $A := \{x \in \ell_1 : x_n = 0 \text{ for all } n \geq N \text{ for some } N\}$ is dense in ℓ_1.

(d) The set D_n of all sequences $x = (x_m) \in \ell_1$ whose terms are rational and $x_k = 0$ for $k > n$. Let $D := \cup_{n \in \mathbb{N}} D_n$. Then D is a countable dense subset of ℓ_1.

(e) The only dense subset of a discrete space X is X itself.

(f) In an indiscrete space, any nonempty subset is dense.

(g) The set $\{0\}$ is dense in $\mathbb{R}$ with the VIP topology. The set $\mathbb{R} \setminus \mathbb{Q}$ is not dense.

(h) The set $\mathbb{R} \setminus \{0\}$ is dense in $\mathbb{R}$ with the outcast topology. This space cannot have a countable dense set.

(i) $S := \{n + m\sqrt{2} : n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. (Did you notice that $\mathbb{Z}$ and $\sqrt{2}\mathbb{Z}$ are closed and S is a sum of two closed sets? If the result is true, then S cannot be closed in $\mathbb{R}$. Why? If S is closed and dense, then $S = \mathbb{R}$, but S is countable! Hence we have an example of two closed sets in $\mathbb{R}$ whose sum is not closed.) See Lemma 2.5.7/Page 52 of my book on Metric spaces..

(j) Is $\mathbb{Q}^2$ dense in $\mathbb{R}^2$ with the order topology?

(k) Weierstrass approximation theorem says that the vector subspace of polynomials in the normed linear space $(C[0, 1], \| \ \|_\infty)$ is dense. (This should be a topic for Student Seminar!)

(l) A *dyadic rational* is a real number of the form $m/2^n$ where m is an integer and $n \in \mathbb{N}$. Let D denote the set of dyadic rationals. Then D is dense in $\mathbb{R}$. Consider an open interval of the form $(a - \varepsilon, a + \varepsilon)$. Choose n so that $1/2^n < \varepsilon$. If there is no dyadic rational in this interval, then there exists an odd integer m such that $m/2^n < a - \varepsilon$ and $(m + 2)/2^n > a + \varepsilon$. (Why?) But then

$$2/2^n = 2^{-n}((m + 2) - m) > a + \varepsilon - (a - \varepsilon) = 2\varepsilon, \text{ a contradiction.}$$

5.3 $D \subset X$ is dense in a space (X, d) iff every point of X is a limit point of D.

Let D be dense in X. Let $x \in X$ and $U \ni x$ be open. Then $U \cap D \neq \emptyset$. Thus x is a limit point of D.

Conversely, if every $x \in X$ is a limit point of D, we claim that D is dense in X. For, if not, there exists a nonempty open set U such that $U \cap D = \emptyset$. Since U is nonempty, choose $x \in U$. Then x is not a limit point of D as $U \ni x$ is open but $U \cap D = \emptyset$.

5.4 $D \subset X$ is dense in the space X iff its closure $\overline{D} = X$. (This is the standard definition.)

Recall (from Item 5) that the closure of any set A is the set of limit points of A. The result now follows from the last item.

5.5 In a metric space (X, d), a set A is dense in X iff for every $x \in X$ and $\varepsilon > 0$, there exists an $a \in A$ such that $d(x, a) < \varepsilon$. (Thus, A is dense in X, if we can "approximate" any point $x \in X$ to "any level of approximation" by an element of A. See Item 2k to understand this vague remark. Also recall that $\mathbb{Q}$ is dense in $\mathbb{R}$, which means that any real number can be approximated to any level of accuracy by a rational number.)

5.6 Let (X, d) be a metric space. Assume that the only dense subset is X itself. Can we say something about the topology? *Hint:* What are the maximal proper subsets of X?

5.7 Let A, B be two dense subsets of a space X. Is $A \cup B$ dense? Is $A \cap B$ dense?

5.8 If A, B are open dense subsets of a space X, is $A \cap B$ dense in X?

5.9 Give an example of a proper open dense subset of $\mathbb{R}$.

5.10 Continuation of the last item. If we write an open set $U = \dot{\cup} J_k$, as the disjoint union of open intervals (Item 11s), then we say that the "measure" or "length" of U is $\sum_k \ell(J_k)$, the sum of lengths of the intervals J_k. Given $\varepsilon > 0$, can you find an open dense subset of $\mathbb{R}$ whose length is less than or equal to ε?

5.11 Let D be dense in $(X, \mathcal{T}_1)$. Is D (necessarily) dense in $(X, \mathcal{T}_2)$ where $\mathcal{T}_2$ is finer (respectively, coarser) than $\mathcal{T}_1$?

5.12 Let X, Y be topological spaces. Assume that A is dense in X and $f \colon X \to Y$ is continuous and *onto*. Then $f(A)$ is dense in Y.

5.13 The set of matrices in $M(n, \mathbb{C})$ with distinct eigenvalues is dense. In particular, the set of all diagonalizable matrices in $M(n, \mathbb{C})$ is dense. This exercise requires a good background in Linear Algebra.

It is well-known fact in linear algebra that any $A \in M(n, \mathbb{C})$ can be brought to upper triangular form, say T, via conjugation by a unitary matrix U such that $T = UAU^{-1}$. The eigenvalues are the diagonal entries, say, d_j. We can find very small ε_j's so that $d_j + \varepsilon_j$'s are all distinct. We thus get a new upper triangular matrix, say T_1 whose entries are the same as that of T except d_j is replaced by $d_j + \varepsilon_j$. Again it is well known that T_1 is diagonalizable. The matrix $A_1 := UT_1U^{-1}$ has distinct eigenvalues and hence is diagonalizable. It is close to A if ε_j's are small. This follows from the observation $\|A\|$ on $M(n, \mathbb{C})$ comes from the inner product $(X, Y) \mapsto \operatorname{Tr} XY^*$ and the inner product is invariant under conjugation by unitary matrices. The reader is encouraged to work out the details and submit it as an assignment to the instructor. Details!

5.14 A topological space is *separable* if there exists a countable dense subset.

5.15 Examples and non-examples of separable spaces:

(a) $\mathbb{R}^n$ is separable. Consider $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_j \in \mathbb{Q}\}$.

(b) A discrete space X is separable iff X is countable.

(c) ℓ_1 is separable.

(d) $\mathbb{R}$ with VIP topology is separable.

(e) $\mathbb{R}$ with outcast topology is not separable.

(f) Any second countable space is separable. If $\{B_n\}$ is any countable basis, choose one element, say, $x_n \in B_n$. Then $D := \{x_n\}$ is a countable dense set.

(g) Let X be infinite with co-finite topology and let A be any infinite subset of X. Then any $x \in X$ is a limit point of A. In particular, X with co-finite topology is separable.

(h) Is $\mathbb{R}^2$ with the order topology separable? (Recall the geometric description of basic open sets in this space. See Item 19.)

No. For, consider the uncountable collection of pair-wise disjoint basic open sets of the form $\{B_x := \{x\} \times \mathbb{R} : x \in \mathbb{R}\}$. If D is a countable dense set, then for each $x \in \mathbb{R}$, there exists $y(x) \in D \cap B_x$. The map $f : \mathbb{R} \to D$ given by $x \mapsto y(x)$ is one-one and hence $\mathbb{R}$ is countable.

5.16 Let X be uncountable with co-finite topology. Then X is not first countable but separable by Item 15g.

5.17 Let X be uncountable with co-countable topology. No countable set can have a cluster limit point and hence X is not separable.

5.18 Let ℓ_∞ denote the set of all bounded real sequences. It is a normed linear space with respect to the norm $\|x\|_\infty := \sup\{|x_n| : n \in \mathbb{N}\}$. The space $(\ell_\infty, \| \ \|_\infty)$ is not separable. *Hint:* Consider the uncountable subset $\{x : \mathbb{N} \to \{0, 1\}\}$ of ℓ_∞.

5.19 A metric space is separable iff it is second countable.

Let X be a separable metric space with $D := \{a_n : n \in \mathbb{N}\}$ as a countable dense set. Then the collection $\{B(a_n, 1/k) : n, k \in \mathbb{N}\}$ is a countable basis for the metric topology. Let U be an open set and $x \in U$. Then there exists $r > 0$ such that $B(x, r) \subset U$. Choose k such that $1/k < r$. Let a$a_n \in D \cap B(x, \frac{1}{2k})$. We claim that $x \in B(a_n, \frac{1}{2k}) \subset B(x, \frac{1}{k}) \subset U$. If $y \in B(a_n, \frac{1}{2k})$, then observe that

$$d(y, x) \leq d(y, a_n) + d(a_n, x) < \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} < r.$$

5.20 $\mathbb{R}_\ell$, the space $\mathbb{R}$ with lower limit topology is first countable, separable but not second countable.

This is Item 10e and may be deleted.

5.21 Let $f, g : X \to Y$ be continuous and Y be Hausdorff. Then the set $A := \{x \in X : f(x) = g(x)\}$ is closed in X.

We show that $B := X \setminus A$ is open. Let $b \in B$. Then $f(b) \neq g(b)$ and hence there exist open sets $V_1 \ni f(b)$ and $V_2 \ni g(b)$ with $V_1 \cap V_2 = \emptyset$. By continuity of f and g at b, there exist open sets $U_1 \ni b$ and $U_2 \ni b$ such that $f(U_1) \subset V_1$ and $g(U_2) \subset V_2$. Then $U_b := U_1 \cap U_2$ is an open set containing b and we have for $x \in U_b$, $f(x) \in V_1$ and $g(x) \in V_2$. Hence $f(x) \neq g(x)$ for $x \in U_b$. That is, $U_b \subset B$. Hence $B = \cup_{b \in B} U_b$ is open.

5.22 Let the hypothesis be as in the last item. Assume that D is dense in X and that $f(x) = g(x)$ for all $x \in D$. Then $f(x) = g(x)$ for all $x \in X$.

Let the notation be as in the last item. Then A is a closed set containing D. Since D is dense $\overline{D} = X \subset A$. That is, $A = X$.

An alternative proof is by contradiction. Let A be the set on which f and g agree. Then A is dense in X. Let $a \in X$ be such that $f(a) \neq g(a)$. Since Y is Hausdorff, there exist open sets $V_1 \ni f(a)$ and $V_2 \ni g(a)$ with $V_1 \cap V_2 = \emptyset$. Let $U_1 := f^{-1}(V_1)$ and $U_2 := g^{-1}(V_2)$. Then U_i are open (why?) with a as a common element. Since $U := U_1 \cap U_2 \ni a$ is a nonempty open set, and A is dense in X, there exists $b \in U \cap A$. Since $b \in A$, $f(b) = g(b)$. Since $b \in U = U_1 \cap U_2$, we have $f(b) \in V_1$ and $g(b) \in V_2$. It follows that $f(b) = g(b) \in V_1 \cap V_2 = \emptyset$, a contradiction.

6 Homeomorphisms

6.1 Let X, Y be sets. Suppose $f \colon X \to Y$ is a bijection. Assume further that one of the sets has an extra mathematical structure such as a group, vector space, metric or a topology. Then we can transfer the structure to the other set using the bijection. We look at some specific instances.

(a) Let X be a group. Then we define $y_1 \cdot y_2$ to be $f(x_1 \cdot x_2)$ where $f(x_i) = y_i$, $i = 1, 2$. It turns out that Y is group and that $f \colon X \to Y$, by virtue of the very definition of group law on Y, is a group homomorphism (and hence an isomorphism.)

(b) Let Y be a metric space. Then we set $d(x_1, x_2) := d(f(x_1), f(x_2))$. Then the metric space (X, d) is isometric to (Y, d).

(c) Let X be a topological space. Let $\mathcal{T}_X$ be the topology on X. We then define a topology $\mathcal{T}_Y$ on Y be declaring that $V \in \mathcal{T}_Y$ iff there exists $U \in \mathcal{T}_X$ such that $V = f(U)$. Then the map $f \colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a homeomorphism, a term not yet defined!

6.2 To illustrate this principle, we use the bijection $t \mapsto e^t$ from $X := \mathbb{R}$ to $Y := (0, \infty)$ to make Y into a vector space over $\mathbb{R}$. Given $y_1, y_2 \in Y$, we look at their (unique) pre-images $x_j = \log y_j$, carry out the vector addition in X, obtain $x_1 + x_2 = \log y_1 + \log y_2 = \log(y_1 y_2)$ and map it by the bijection. The result is $y_1 y_2$. Similarly, the scalar multiple of y by $\alpha \in \mathbb{R}$ is $\alpha \log y \mapsto e^{\alpha \log y} = y^\alpha$. Thus the vector addition of y_1 and y_2 is $y_1 y_2$ and the scalar multiple $\alpha \cdot y$ is y^α. The 'additive identity' is 1. Note that the map $t \mapsto e^t$ is a linear isomorphism.

6.3 A map $f \colon X \to Y$ between two topological spaces is a *homeomorphism* if (i) f is bijective, (ii) f is continuous and (iii) $f^{-1} \colon Y \to X$ is continuous.

This is the analogue of isomorphisms in Algebra. Note that there also one requires a bijective map f such that f and its inverse f^{-1} preserve the 'algebraic structures' such as group, ring, vector space structures. It turns out in the context of algebra, if f preserves the structure, then f^{-1} does automatically.

We say that two topological spaces X and Y are homeomorphic if there exists a homeomorphism $f \colon X \to Y$.

6.4 The relation of being homeomorphic is an equivalence relation among topological spaces.

6.5 Examples of homeomorphisms.

(a) Any $f \colon \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax + b$ for a nonzero $a \in \mathbb{R}$ is a homeomorphism.

(b) $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a homeomorphism.

(c) Any linear isomorphism of $\mathbb{R}^n$ is a homeomorphism.

More generally, any linear isomorphism from $(\mathbb{R}^n, \| \ \|)$ to $(\mathbb{R}^n, \| \ \|')$, where $\| \ \|$ and $\| \ \|'$ are any of the norms $\| \ \|_1$, $\| \ \|_2$ and $\| \ \|_{\max}$, is a homeomorphism.

In particular, the identity map is a homeomorphism. As a corollary, we conclude that the topologies induced by these norms are the same:

$$\mathcal{T}_{\| \ \|_1} = \mathcal{T}_{\| \ \|_2} = \mathcal{T}_{\| \ \|_{\max}}.$$

(d) Let us now look at some homeomorphisms of a normed linear space. Let $(X, \| \ \|)$ be a normed linear space. Then the maps (a) $x \mapsto \lambda x$ for $0 \neq \lambda \in \mathbb{R}$, (b) $x \mapsto x + v$, where $v \in X$ is fixed are homeomorphisms.

(e) Consider $M(n, \mathbb{R})$. Then the maps (a) $X \mapsto X^t$, (b) $X \mapsto X + A$ for fixed $A \in M(n, \mathbb{R})$ and (c) $X \mapsto AX$ for a fixed nonsingular matrix A are homeomorphisms.

(f) Any two discrete spaces are homeomorphic iff they have the same cardinality.

(g) If two metric spaces are isometric, then they are homeomorphic.

(h) In the examples of this item, the subsets are given the metric topology from the induced metric.

(i) Let X be a set with at least two elements.. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be respectively the indiscrete and discrete topology on X. Let f be the identity map on X. Then
   (i) $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ is a bijection but not continuous and hence is not a homeomorphism.
   (ii) $f : (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$ is a continuous bijection but not a homeomorphism.

(j) Keep the notation of the last example. Then any bijection of $(X, \mathcal{T}_i)$ to *itself* is a homeomorphism, where $i = 1, 2$.

(k) A few more examples.
   i. $[a, b] \simeq [0, 1]$. More generally, $[a, b] \simeq [c, d]$.
   ii. $(-1, 1) \simeq \mathbb{R}$.
   iii. $(0, 1] \simeq [1, \infty)$.
   iv. $[0, 1) \simeq (0, 1]$.
   v. Can $\mathbb{Q}$ be homeomorphic to $\mathbb{Z}$?
   vi. Is $\mathbb{N} \simeq \mathbb{Z}$?

(l) A bijective continuous map need not be a homeomorphism. Examples and a non-example:
   i. $\mathbb{R}$ with discrete topology and $\mathbb{R}$ with indiscrete topology.
   ii. $f : [0, 2\pi) \to S^1 \subset \mathbb{C}$ given by $f(t) = e^{it}$. (A more instructive exercise.)
   iii. Any bijective continuous map of a finite topological space X to itself is a homeomorphism.

(m) The spaces $(\mathbb{R}, \mathrm{VIP})$ and $(\mathbb{R}, \mathrm{Outcast})$ are not homeomorphic.
   We shall see later a lot of examples of homeomorphisms.

6.6 **Open and closed maps.** A map $f : X \to Y$ is said to be *open* if $f(U)$ is open in Y for every U open in X. A closed map is defined similarly.

6.7 A continuous map need not be an open map. Similarly, it need not be a closed map.

Example: The identity map from $(X, \mathcal{T}_2)$ to $(X, \mathcal{T}_1)$ is continuous and is neither open nor closed. Notation as in Item 5i.

A closed (respectively open) map need not be continuous.

Example: The identity map from $(X, \mathcal{T}_1)$ to $(X, \mathcal{T}_2)$ is not continuous and is an open map as well as a closed map.

6.8 A bijective continuous map is a homeomorphism iff it is an open map.

The key observation is the following. Let $f: X \to Y$ be any map. For any $B \subset Y$, we have the inverse image B under f, namely $f^{-1}(B) \subset X$. If f happens to be a bijection and if $g: Y \to X$ is its inverse then $g(B) = f^{-1}(B)$. We prove this.

Let $x \in g(B)$. Then there exists $y \in B$ such that $x = g(y)$. Now, $x \in f^{-1}(B)$ iff $f(x) \in B$. This is the case iff $f(g(y)) \in B$. Since $f \circ g$ is the identity of Y, we see that $f(x) = y \in B$. Thus $g(B) \subset f^{-1}(B)$.

The reverse inclusion is similar. Let $x \in f^{-1}(B)$. Then $y := f(x) \in B$. What is $g(y)$? We have $g(y) = g(f(x)) = x$ since $g \circ f$ is the identity on X. Hence $x = g(y) \in g(B)$.

Now let us prove the stated result. Let $f: X \to Y$ be a homeomorphism and $U \subset X$ be open. We observe that $f(U) = g^{-1}(U)$. Since g is continuous, it follows that $g^{-1}(U) = f(U)$ is open. The proof of $f(U) = g^{-1}(U)$ is similar to that of $f^{-1}(B) = g(B)$. Replace f by g and B by U in the argument and we get the result.

Let f be a continuous bijection which is also an open map. Let g be the inverse of f. To establish the continuity of g, let $U \subset X$ be open. Is $g^{-1}(U)$ open in Y? We know $g^{-1}(U) = f(U)$ by the observation made in the beginning. Since f is open, $f(U) = g^{-1}(U)$ is open.

6.9 Application: The map $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a homeomorphism.

We have already seen that any polynomial function on $\mathbb{R}$ is continuous. The function f is strictly increasing and hence is one-one. (To see that f is increasing, either apply the derivative test or observe that $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$ and the second factor on the right side is always non-negative.) It is onto due to the intermediate value theorem. Given $\alpha \in \mathbb{R}$, we are on the look-out for an $x \in \mathbb{R}$ with $x^3 = \alpha$. Let $N \in \mathbb{N}$ be such that $|x| < N$. When restricted to $[-N, N]$, the function $g(x) = x^3 - \alpha$ takes values of opposite signs.

6.10 A bijective continuous map is a homeomorphism iff it is a closed map.

This again depends on the following set theoretic observation: Let $f: X \to Y$ be *any* map. Let $B \subset Y$. Then $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$. Proof is easy. Let $x \in f^{-1}(Y \setminus B)$. That is, $f(x) \notin B$. Hence $x \notin f^{-1}(B)$. This shows that $f^{-1}(Y \setminus B) \subset X \setminus f^{-1}(B)$. To prove the reverse inclusion, let $x \in X \setminus f^{-1}(B)$. Note that this implies $f(x) \notin B$. We need to show that $x \in f^{-1}(Y \setminus B)$, that is, $f(x) \notin B$, which is true.

To prove the stated result, let $f: X \to Y$ be a homeomorphism and $K \subset X$ be closed. Note that $f(X \setminus K) = g^{-1}(X \setminus K) = Y \setminus f(K)$. Since f is a homeomorphism and $X \setminus K$ is open, we see that $f(X \setminus K) = Y \setminus f(K)$ is open. That is, $f(K)$ is closed. We leave the converse to the reader.

Application: Item 22b. We have to wait for this.

6.11 Keep the notation of Item 5i. Let X be a set with at least two elements. Can some bijection $f: (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ be a homeomorphism? How about a bijection $g: (X, \mathcal{T}_2) \to (X, \mathcal{T}_1)$?

6.12 We say that property of a topological space is a *topological property* if every space Y homeomorphic to X also has the property. Examples:

(a) The space being Hausdorff is a topological property.

(b) The space being first countable is a topological property.

(c) The space being second countable is a topological property.

(d) The space being separable is a topological property. (Item 12 is of use here.)

(e) Existence of a nonempty, proper subset which is both open and closed is a topological property.

(f) Let us say that a topological space X has BCP if every continuous real valued function is bounded. For example all closed and bounded intervals have this property. Is BCP a topological property? *Hint:* If $\varphi\colon X \to Y$ is a homeomorphism there is a "natural map" $\varphi^*\colon C(Y, \mathbb{R}) \to C(X, \mathbb{R})$ where $C(X, \mathbb{R})$ stands for the set of real valued continuous functions on X etc.

    The "adjoint" map φ^* is defined by $\varphi^*(g) := g \circ \varphi$. If φ is a bijection, then φ^* is also a bijection. If φ is a homeomorphism, then $\varphi^*(g) \in C(X, \mathbb{R})$ for any $g \in C(Y, \mathbb{R})$.

(g) Two metric spaces can be homeomorphic, but one of them could be bounded while the other is not. Hence 'being bounded' is not a topological property among metric spaces.

(h) Similarly, completeness is not a topological property among the metric spaces.

We shall see later a lot of examples of topological properties.

> The study of topology is mainly understanding topological properties and using them to assert whether given two spaces are homeomorphic or not.

7 New Topologies from the Old

7.1 We now look at some natural questions which lead us to the generation of new topologies.

7.2 Given a set X and a collection $\mathcal{S}$ of subsets of X, how to we describe the open sets in the smallest topology, say, $\mathcal{T}_{\mathcal{S}}$ that contains $\mathcal{S}$? (We assume, as this is the case that occurs in practice, that for every $x \in X$, there exists $S \in \mathcal{S}$ such that $x \in S$.) We do this in two steps.

(a) We wanted a base for some topology on X which will also contain $\mathcal{S}$. Clearly, $\mathcal{B} := \{S_1 \cap \cdots \cap S_n : S_j \in \mathcal{S}, n \in \mathbb{N}\}$ is a base for some topology and $\mathcal{S} \subset \mathcal{B}$.

(b) The topology $\mathcal{T}_{\mathcal{B}} := \{U \subset X : \forall x \in U, \exists B \in \mathcal{B}$ such that $x \in B \subset U\}$ is then the smallest topology that contains $\mathcal{S}$.

(c) Thus, we can rid of the intermediate $\mathcal{B}$ and define the topology directly in terms of $\mathcal{S}$. We say $U \in \mathcal{T}_{\mathcal{S}}$ iff for every $x \in U$, there exists $n \in \mathbb{N}$ such that we can find S_j, $1 \leq j \leq n$ with $x \in S_1 \cap \cdots \cap S_n \subset U$. One can again show directly that this is the smallest topology containing $\mathcal{S}$.

(d) $\mathcal{S}$ is called a *subbase* and $\mathcal{T}_{\mathcal{S}}$ is the topology generated by $\mathcal{S}$.

7.3 Let us look at some concrete examples:

(a) Consider $\mathcal{S} := \{(-\infty, a) : a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$. The topology generated by $\mathcal{S}$ on $\mathbb{R}$ is the usual topology.

(b) Consider $\mathbb{R}$ and $\mathcal{S} := \{\{0, x\} : x \neq 0, x \in \mathbb{R}\}$. What is the topology on $\mathbb{R}$?

(c) Let X be a set with at least 3 elements. Let $\mathcal{S}$ be the family of two-element subsets of X. The topology generated by $\mathcal{S}$ is the discrete topology.

(d) What is the topology on $\mathbb{R}^2$, if we take the subbase consisting of all straight lines in $\mathbb{R}^2$?

(e) What is the topology on $\mathbb{R}^2$, if we take the subbase consisting of all straight lines parallel to the x-axis in $\mathbb{R}^2$? Which of the following sets are open in this topology? (i) the open unit disk, $B(0,1)$, (ii) the open vertical band $\{(x,y) \in \mathbb{R}^2 : 0 < x < 1\}$, (iii) the open horizontal band $\{(x,y) \in \mathbb{R}^2 : 0 < y < 1\}$, (iv) any subset which is bounded in the Euclidean metric.

(f) What is the topology on $\mathbb{R}^2$, if we take the subbase consisting of all circles in $\mathbb{R}^2$?

(g) What is the topology on $\mathbb{R}^2$, if we take the subbase consisting of all circles, with centre at the origin, in $\mathbb{R}^2$?

(h) Consider $\mathcal{S} = \{X\}$ as a subbase on X. What topology do we get on X?

7.4 Let $f \colon X \to Y$ be *any* map between two sets. Assume that one of them is a topological space. What we wish to do is to endow the other set with an *optimal* topology in such a way that $f \colon X \to Y$ becomes a continuous map between the spaces.

(a) Let Y be a topological space. Then if we endow X with the discrete topology, then the problem is solved! But this topology has no bearing on Y and/or on the map f! So what we require is the smallest topology on X making f continuous.

(b) Let X to be a topological space. Considerations similar to the last item suggest us that we require the largest topology on Y making f continuous.

7.5 These problems arise in a very natural way.

(a) Let X be a subset a topological space Y. Then we have an *obvious* or *natural* map $i: X \to Y$, the inclusion of X in Y, that is, the restriction of the identity on Y to X.

(b) Let X be any topological space and $\sim$ an equivalence relation on X. Then as Y, we take the quotient set $X/\sim$, that is, the set of equivalence classes. Once again, we have a natural map $\pi: X \to Y$, where $\pi(x)$ is the equivalence class of x.

7.6 More general situations may also arise. Let X be a set and Y_i be topological spaces, indexed by a set I. Assume that we are given certain maps $f_i: X \to Y_i$ for each $i \in I$. We again ask for a single smallest topology on X making all the maps f_i continuous. Or the other way around, we have maps $f_j: X_j \to Y$ where X_j's topological spaces.

Typical instances of this phenomenon are:

(a) Let $\{X_j : j \in I\}$ be an indexed family of (pairwise disjoint) topological spaces. Let $X := \biguplus_{j\in I} X_j$, the disjoint union of X_j's. We have natural inclusion maps $\iota_j: X_j \to X$. We wish to endow X with the largest topology with respect to which all ι_j's are continuous.

(b) Let $\{X_i : i \in I\}$ be an indexed family of topological spaces. Let $X := \prod_{i\in I} X_i$. We have obvious maps $\pi_i(x) = x_i$, the i-th projection. We wish to equip X with the smallest topology such that each of the projections becomes continuous.

(c) Let E be a set and let $X := \mathcal{F}$ be a family of functions from E to $\mathbb{R}$. Consider the evaluation maps $\varepsilon_x(f) := f(x)$ for each $x \in E$. Thus, we have a family of maps $\varepsilon_x: X \to \mathbb{R}$ and we want the smallest topology which will make all these maps continuous.

7.7 Let us deal with various cases. Let X be a set and Y be a topological space and $f: X \to Y$ be a map. Any topology on X which makes f continuous must contain the set $\mathcal{U} := \{f^{-1}(V) : V \in \mathcal{T}_Y\}$. It turns out this collection is already a topology and hence is the smallest topology on X, as required. (We were lucky this time!)

7.8 Let us look at the concrete case in Item 5a. Then the topology on X is given by

$$\mathcal{T}_X := \{i^{-1}(V) : V \in \mathcal{T}_Y\} = \{V \cap A : V \in \mathcal{T}_Y\}.$$

The topology $\mathcal{T}_X$ is called the subspace topology on Y and any $U \in \mathcal{T}_X$ is said to be open *in* X. We say that $F \subset X$ is closed in X if its complement, $X \setminus F$, *in* X is open in X.

7.9 The following are immediate from the definition of subspace topology and

7.10 Assume that Y is ordered and is with order topology. Let $f, g: X \to Y$ be continuous. Let $h(x) := \min\{f(x), g(x)\}$. Then h is continuous. *Hint:* Item 28f.

Is $\varphi := \max\{f, g\}$ continuous?

7.11 Did you notice that all the examples above use the second version of the gluing lemma? There is a beautiful application of the 'open version' of the gluing lemma in my book "Topology of Metric Spaces". See Lemma 3.2.52 in the book.

7.12 (a) If $\mathcal{B}$ is a basis for the topology $\mathcal{T}_Y$ on Y, then $\mathcal{B}_X := \{B \cap X : B \in \mathcal{B}\}$ is a basis for the subspace topology on X. Details!

(b) If $\mathcal{B}_x$ is a local basis at $x \in Y$ for the topology $\mathcal{T}_Y$ on Y, then $\mathcal{B}_{x,X} := \{B \cap X : B \in \mathcal{B}_x\}$ is a local basis at $x \in X$ for the subspace topology on X.

7.13 Let us look at some examples to develop our intuition about subspace topology. Use the last item to identify a local basis at each point of the subset A.

(a) Consider $A = [0,1] \subset \mathbb{R}$. Then the sets $[0,1/2)$, $(1/2,1]$ and $(1/2,3/4)$ are open in in A.

(b) Consider $Y := \{(x,y) : xy = 0\} \subset \mathbb{R}^2$ be the two axes. Then the basic open sets near (0,0) are crosses (of two line segments along the x and y-axes.) At other points, just intervals around them.

(c) Let $A := \{1/n : n \in \mathbb{N}\} \cup \{0\}$. Then the basic open sets are the singletons $\{1/n\}$ for $n \in \mathbb{N}$ and $\{1/n : n \geq n_0\} \cup \{0\}$. The latter are basic opens sets near 0 in A.

(d) Let $S := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ be the unit circle in $\mathbb{R}^2$. The basic open sets in S are open arcs of the circle.

(e) Consider $A = \mathbb{Q} \subset \mathbb{R}$. Then the set $\{r \in \mathbb{Q} : -\sqrt{2} \leq r \leq \sqrt{2}\}$ is both open and closed in $\mathbb{Q}$.

(f) Let X be a metric space and $\emptyset \neq A \subset X$. Then we have two topologies on A: (i) one comes from the induced metric, call it d_A, on A and (ii) the other is the subspace topology. They are the same.

Let $\mathcal{T}_{d_A}$ denote the metric topology on A and $\mathcal{T}_A$ denote subspace topology on A. The local base at $a \in A$ for $\mathcal{T}_A$ is $\{B_{(A,d_A)}(a,r) : r > 0\}$ and the one for $\mathcal{T}_A$ is $\{B_{(X,d)}(a,r) \cap A : r > 0\}$. But, $B_{(A,d_A)}(a,r) = B_{(X,d)}(a,r) \cap A$ for each $r > 0$.

Observe the following:

$$B_{(A,d_A)}(a,r) := \{x \in A : d_A(x,a) < r\} = \{x \in A : d(a,x) < r\} = B_{(X,d)}(a,r) \cap A.$$

Hence the local bases are the same at each point $a \in A$ for both the topologies.

(g) Let $A := [0,1] \times [0,1]$. Then A has the order topology as well as the subspace topology as a subset of $\mathbb{R}^2$ with order topology. They are not the same. (Contrast this with the last item.)

Consider the set $V := \{(0,y) : 1/2 < y \leq 1\}$. Then V is is open in the subspace topology but not in the order topology on the ordered set A. Draw a picture of A and use the definitions of subspace topology and order topology. V is open in the subspace topology, since it is the intersection A with basic open set in $\mathbb{R}^2$ with an interval (in the order topology): $V = A \cap (a,b)$ where $a = (0,1/2)$ and $b = (0,2)$.

Let, if possible, A be open in the order topology on A. Then there exists an open interval (c,d) such that that point $p = (0,1) \in (c,d)$. Let $c = (x_1,y_1)$ and $d = (x_2,y_2)$. Then $x_1, x_2 \geq 0$ and $y_1, y_2 \leq 1$. Now, $(x_1,y_1) < (0,1)$ in the dictionary order. We conclude that $x_1 = 0$ and $y_1 < 1$. Similarly, $x_2 > 0$ and $y_2 \geq 0$. But an element of the form $(x_2/2, y)$ with $y \geq 0$ lies in the basic open set but not be in V.

(h) Consider $\mathbb{R}$ with VIP topology and $A = \mathbb{R}^*$. Then the subspace topology on $\mathbb{R}^*$ is the discrete topology. The subspace topology on $\mathbb{Q}$ is the VIP topology on $\mathbb{Q}$. (Do you understand this statement?)

(i) Investigate the subspace topology on $\mathbb{Q}$ considered as a subset of $\mathbb{R}$ with outcast topology.

(j) Let X be a Hausdorff space, $A \subset X$ be endowed with the subspace topology. Then A is Hausdorff.

(k) Let $[a, b] \subset \mathbb{R}_\ell$. Then $\{b\}$ is open in $[a, b]$ with the subspace topology inherited from the lower limit topology on $\mathbb{R}$.

7.14 Let A be nonempty and open in X. Then $U \subset A$ is open in A iff it is open in X.

7.15 Let $B \subset A \subset X$. Let $(X, \mathcal{T}_X)$ be a topological space. Let $\mathcal{T}_A$ denote the subspace topology on A. Let $x \in A$. Then $x \in A$ is a limit point of B **in** A iff x is a limit point of B in X.

Let x be a limit point of B in A. Let $U \in \mathcal{T}_X$ such that $x \in U$. Since $U \cap A \in \mathcal{T}_A$ is an open set containing x, $(U \cap A) \cap B \neq \emptyset$. But, $(U \cap A) \cap B = U \cap B$. Thus, x is a limit point of B in X.

Conversely, let $x \in A$ be a limit point of B in X. Let V be open in A with $x \in V$. We need to show that $V \cap B \neq \emptyset$. There exists $U \in \mathcal{T}_X$ such that $V = U \cap A$. Then $x \in U$ and since x is a limit point of B in X, we have $U \cap B \neq \emptyset$. Since $B = B \cap A$, it follows that $x \in (U \cap A) \cap B \neq \emptyset$, that is, $V \cap B \neq \emptyset$, or x is a limit point of B in A.

7.16 Let $A \subset X$. Then $F \subset A$ is closed in A iff there exists a closed set C in X such that $F = A \cap C$.

You may use the last item to prove this. Let $F = A \cap C$ where C is a closed subset of X. To show that F is closed in A, it suffices to show that if $x \in A$ is a limit point of F *in* A, then $x \in F$. By the last item, this is true. Hence F is closed in A.

To prove the converse, if F is closed in A, where do we find $C \subset X$ closed in X with $F = A \cap C$? Obvious choice is $\overline{F}$, the closure of F in X. Do we have $F = \overline{F} \cap A$? Clearly, we have $F = F \cap A \subset \overline{F} \cap A$. To prove the reverse inclusion, let $x \in \overline{F} \cap A$. Then, x is a limit point of F in A and hence by the last item, it is a limit point of F in the subspace topology. Since F is closed in the subspace topology, it contains all its limit points *in* A, in particular x.

Or, we may proceed directly as follows. (This is essentially set-theoretic exercise and so the reader should try on his own.)

Let $F = A \cap C$. We show that the complement of F in A is open. Let $U = X \setminus C$. Then U is open in X. We claim that $U \cap A = A \setminus F$. To show $(X \setminus C) \cap A \subset A \setminus F$, let $x \in (X \setminus C) \cap A$. If $x \in F$, then $x \in F = C \cap A$ and hence $x \in C$, a contradiction. For the reverse inclusion, let $x \in A \setminus F$. We need to show that $x \in X \setminus C$. If false, then $x \in C$ and hence $x \in A \cap C = F$, that is $x \in F$, a contradiction.

Assume that F is closed in A. Then $A \setminus F$ is open in A. Let U be open in X such that $A \setminus F = U \cap A$. Let $C := X \setminus U$. Then C is closed in X. We claim that $C \cap A = F$. To show that $C \cap A \subset F$, let $x \in C \cap A$. Suppose $x \notin F$, then $x \in (A \setminus F) = A \cap U$. Therefore, $x \in U$ or $U \notin C$, a contradiction. To prove the reverse inclusion, let $x \in F$. If $x \notin C$, then $x \in U$ and hence $x \in U \cap A = (A \setminus F)$, that is $x \notin F$, a contradiction.

As a specific example, the set of Item 13e is open as well as closed in $\mathbb{Q}$. (Contrast this with Item 35i.)

7.17 We shall put to use some of the concepts we learned to gain a different perspective of the limit of a sequence.

Consider the set $X := \{1/n : n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$ with the subspace topology inherited from $\mathbb{R}$. (Refer to Item 13c.) Let $f: X \to \mathbb{R}$ be a function. Note that any such f is continuous at any point of the from $1/n$. When is it continuous at 0? Let $a_n := f(1/n)$ and $a := f(0)$. We claim that f is continuous at 0 iff the sequence (a_n) converges to a.

> Let f be continuous at 0. Consider $V := (a - \varepsilon, a + \varepsilon) \ni 0$ an open set containing $a = f(0)$. Since f is continuous at 0, there exists a basic (relatively) open set $U := \{1/k : k > N\} \cup \{0\}$ such that for all $k \in U$, we have $f(k) \in (a - \varepsilon, a + \varepsilon)$. That is, for $k > N$, we have $|a_k - a| < \varepsilon$. This proves that (a_n) converges to a. Converse is along the same lines.

Consider $X = \mathbb{N} \cup \{\infty\}$, where ∞ is just a symbol representing an element not in $\mathbb{N}$. (We could have used $\star$ in place of ∞!) Let $\varphi: X \to Y := \{1/n : n \in \mathbb{N}\} \cup \{0\}$ be defined by $\varphi(n) = 1/n$ and $\varphi(\infty) = 0$. Then φ is a bijection. We endow Y with the subspace topology as a subset of $\mathbb{R}$. Using the bijection φ, we transfer the topology on Y to X. The local basic open set in Y at n are $\{n\}$, and at ∞ are $\{k : k \geq N\}$ for some $N \in \mathbb{N}$. (See Item 13c.)

Now if $f: X \to \mathbb{R}$ is a function, when is it continuous at ∞? If we restrict f to $\mathbb{N} \subset X$, we can think of f as a real sequence, say, (a_n), in $\mathbb{R}$. Do you see any relation between the continuity of f at ∞ and the convergence of (a_n)? If we replace $\mathbb{R}$ by a topological space Z, do the results (concerning the convergence of sequences in Z) continue to be true?

We shall return to this example later when we talk of one point compactifications.

7.18 Let $f: X \to Y$ be a continuous map between two topological spaces. Let $A \subset X$ be a subset endowed with the subspace topology. Then the restriction f_A of f to A gives rise to a map $f_A: A \to Y$. Is it continuous?

If $V \subset Y$ is open, then $f_A^{-1}(V) = f^{-1}(V) \cap A$ is open in A, since $f^{-1}(V)$ is open in X.

7.19 A question 'dual' to the one in the last item: Let $f: X \to Y$ be continuous. Assume that $f(X) \subset B \subset Y$. We then have an induced map $g: X \to B$ defined by $g(x) = f(x)$ for $x \in X$. Let B be given the subspace topology. Is g continuous? The answer is 'Yes.'

Let $W \subset B$ be open **in** B. Then there exists V, an open subset of Y such that $W = V \cap B$. It is easy to check that $g^{-1}(W) = f^{-1}(V)$.

> For, we have
>
> $$g^{-1}(W) := \{x \in X : g(x) \in W\} = \{x \in X : f(x) \in W\} \subset f^{-1}(V).$$
>
> On the other hand, if $x \in f^{-1}(V)$, then $f(x) \in V \cap B$, that is, $g(x) \in W$. Hence $x \in g^{-1}(W)$.

Since f is continuous, $f^{-1}(V)$ is open in X and hence $g^{-1}(W)$. (We shall return to this later when we talk of universal mapping properties. See Item 49.)

Is the converse true? If g is continuous, it f continuous? Let $V \subset Y$ be open. Since $V \cap B$ is open in B, we have $g^{-1}(V)$ is open in X. What is $g^{-1}(B)$?

$$g^{-1}(V) = \{x \in X : g(x) \in V\} = \{x \in X : f(x) \in V\} = f^{-1}(V).$$

Since g is continuous, it follows that $g^{-1}(V) = f^{-1}(V)$ is open for any open set $V \subset Y$. That is, f is continuous.

Items 18–19 are most often used when we deal with subspaces *without* explicit mention.

7.20 At this stage we are curious about the following questions.

(a) Let X and Y be topological spaces. Assume that $\{A_i \in I\}$ is a family of subsets of X such that $\cup_{i \in I} A_i = X$. Further assume that for each i, we have a continuous function $f_i \colon A \to Y$. (Here A_i's are given the subspace topology.) Can we get 'glue' them together to get a continuous function $f \colon X \to Y$ in such a way that the restriction $f|_{A_i} = f_i$ for $i \in I$?
A necessary condition is that $f_i(x) = f_j(x)$ for each $x \in A_i \cap A_j$, $i, j \in I$. This will ensure that we get a function f from the set X to Y whose restrictions to A_i are as required.

(b) Let X and Y be topological spaces. Let $f \colon X \to Y$ be a map. Assume that $\{A_i \in I\}$ is a family of subsets of X such that $\cup_{i \in I} A_i = X$ and that $f|_{A_i} \colon A_i \to Y$ is continuous. Can we conclude f is continuous?

7.21 Let us investigate the situation. Let $V \subset Y$ be open. Observe that

$$f^{-1}(V) = f^{-1}(V) \cap X = \cup_{i \in I} \left(f^{-1}(V) \cap A_i\right).$$

Each term in the union, $f^{-1}(V) \cap A_i = f_i^{-1}(V)$ is open in A_i. If we can ensure that each of these open in X then $f^{-1}(V)$ is open in X. We know a sufficient condition which will ensure this, namely, we demand each A_i is open.

Let $V \subset Y$ be closed. Since each $f^{-1}(V) \cap A_i = f_i^{-1}(V)$ is closed in A_i, to ensure $f^{-1}(V)$ is closed, we may demand that each A_i is closed. But then $f^{-1}(V)$ is a union of closed sets and it is closed if we further assume that I is finite. We have thus arrived at

7.22 Gluing Lemma:

Lemma 8. *Let X, Y be topological spaces and let $f \colon X \to Y$ be any map. Assume that $\{A_i : i \in I\}$ is a family of subsets of X whose union is X. Assume further that $f_i := f|_{A_i} \colon A_i \to Y$ is continuous for each $i \in I$. Then*
1. *f is continuous if each A_i is open.*
2. *f is continuous if each A_i is closed and I is finite.* □

7.23 Typical applications of the gluing lemma.

(a) The absolute-value function $|\ | : \mathbb{R} \to \mathbb{R}$ is continuous.

(b) Let $f, g \colon [0,1] \to X$ be continuous. Assume that $f(1) = g(0)$. Define

$$h(t) := \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(2t-1) & 1/2 \le t \le 1. \end{cases}$$

Then h is continuous.

(c) Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be defined as follows:

$$f(x) = \begin{cases} x & \text{if } \|x\| \leq 1 \\ \frac{x}{\|x\|^2} & \text{if } \|x\| > 1. \end{cases}$$

Then f is continuous.

(d) Assume that Y is ordered and is with order topology. Let $f, g\colon X \to Y$ be continuous. Let $h(x) := \min\{f(x), g(x)\}$. Then h is continuous. *Hint:* Item 28f.

Is $\varphi := \max\{f, g\}$ continuous? If you wish, you may assume $Y = \mathbb{R}$ to gain insights. Note that the application in this series is a special case of this general result.

(e) Did you notice that all the examples above use the second version of the gluing lemma? There is a beautiful application of the 'open version' of the gluing lemma in my book "Topology of Metric Spaces". See Lemma 3.2.52 in the book.

7.24 Let us consider the general case in Item 6: Let X be a set and Y_i be topological spaces, indexed by a set I. Assume that we are given certain maps $f_i\colon X \to Y_i$ for each $i \in I$. We again ask for a single smallest topology on X making all the maps f_i continuous.

We want the smallest topology $\mathcal{T}$ that contains all sets of of the form $f_i^{-1}(V_i)$ where V_i is open in X_i and $i \in I$. That is $\mathcal{T}$ is the smallest topology containing the family of sets $\mathcal{S} := \{f_i^{-1}(V_i) : V_i \in \mathcal{T}_i; i \in I\}$, where $\mathcal{T}_i$ is the topology on X_i.

There is no reason to believe that $f_i^{-1}(V_i) \cap f_j^{-1}(V_j)$ must be again of the form $f_r^{-1}(V_r)$ for some $r \in I$. Hence $\mathcal{S}$ may not be topology on X.

7.25 We now want to look at the concrete case in Item 6b. As a preliminary, we review the concept of Cartesian product.

Let $\{X_i : i \in I\}$ be an indexed family of sets. Then the Cartesian product $X := \prod_{i\in I} X_i$ is defined by

$$\prod_{i\in I} X_i := \{x\colon I \to \cup_{i\in I} X_i : x(i) \in X_i \text{ for each } i \in I\}.$$

(a) We usually write $x \in \prod_{i\in I} X_i$ as $x = (x_i)$, where $x_i := x(i)$. We shall call x_i as the i-th coordinate of x. Let $\pi_j\colon \prod_{j\in I} X_i \to X_j$ denote the map $\pi_j(x) = x(j) = x_j$. This is called the j-th projection of X onto the j-th factor X_j.

(b) As a convention, if $I = \{1, 2, \dots, n\}$, we identify X with $X_1 \times \cdots \times X_n$, that is, with the set of "ordered n-tuples" $(x_1, \dots, x_n)$. Similarly, if $I = \mathbb{N}$, we identify X with $X_1 \times X_2 \times \cdots \times X_n \times \cdots$, that is the set of ordered infinite tuples $x \mapsto (x_1, x_2, \dots, x_n, \dots)$. Details!

(c) If $V_j \subset X_j$, then $\pi_j^{-1}(V_j) = \prod_{i\in I} U_i$ where $U_i = X_i$ for $i \neq j$ and $U_j = V_j$. In particular, $\pi_1^{-1}(V_1) = V_1 \times X_2$ where $X = X_1 \times X_2$ etc.

7.26 We now give a few examples to instill some confidence to work with the concept of Cartesian products.

(a) Let $I = \mathbb{R}$ and $X_t := \mathbb{R}$ for $t \in I$. We claim that $X := \prod_{t\in I} X_t$ is the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$.. Details!

(b) Let $I = \mathbb{N}$ and $X_n := \mathbb{R}$. How do we visualize $\prod_{n\in I} X_n$? It is the set of vertical lines passing through the points $(n, 0)$ in the x-axis of $\mathbb{R}^2$.

Note that the product set is the set of all real sequences. This can be seen as in the previous example.

(c) Let $I := [0, \infty)$ and $X_t := [0, t]$ for $t \in \mathbb{R}$. How do we visualize the product? It is the region $\{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x\}$.

Does the element $(t^2)_{t\in I}$ lie in the product? Note that if $t = 2$, then the $x(t) = 4 \notin [0, t] \equiv [0, 2]$.

7.27 What we requite on $X := \prod_{i\in I} X_i$ to make the projections π_i $(i \in I)$ continuous is the smallest topology that contains

$$\mathcal{S} := \{\pi_i^{-1}(V_i) : V_i \in \mathcal{T}_i, i \in I\}.$$

This is the question we have already answered in Item 2. What is $\pi_i^{-1}(V_i)$? An element $x = (x_i) \in \prod_{i\in I} X_i$ lies in $\pi_i^{-1}(V_i)$ iff $\pi_i(x) = x_i \in V_i$. What about x_j's for $j \neq i$? No condition sis imposed on these components, which means they could be any element of X_j for $j \neq i$. Thus, $\pi_i^{-1}(V_i) = \prod_{j\in I} V_j$ where $V_j = V_i$ if $j = i$ and $V_j = X_j$ for $j \neq i$.

Can you visualize $\pi_1^{-1}((2,3))$ where $\pi_1 \colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is the projection onto the first factor?

Arguing along similar lines, we find that

$$\begin{aligned}\pi_i^{-1}(V_i) \cap \pi_j^{-1}(V_j) &= \{x = (x_r) \in \prod_{r\in I} X_r : x_i \in V_i \text{ and } x_j \in V_j\} \\ &= \prod_{r\in I} U_r \text{ where } U_r = V_r \text{ if } r \in \{i,j\} \text{ and } U_r = X_r \text{ otherwise.}\end{aligned}$$

7.28 We apply the process of Item 2.to the problem posed in Item 27. Thus we arrive at the definition of *product topology* on $\prod_{i\in I} X_i$ as follows.

As a subbase for a topology on X, we take the set

$$\mathcal{S} := \left\{ \prod_{i\in I} U_i : \text{ where } U_i = X_i \text{ for all but finitely many } i \text{ and } U_i \text{ is open in } X_i \right\}.$$

The basis for the product topology on X is finite intersections of elements from $\mathcal{S}$.

In particular, $G \subset X$ is open iff for every $x \in G$, there exists $S_1, \ldots, S_n \in \mathcal{S}$ such that $x \in S_1 \cap \cdots \cap S_n \subset G$.

> Thus, $G \subset X$ is open in the product topology iff for a given $x \in G$, there exists a finite subset $F \subset I$ and open subsets $U_j \subset X_j$ for $j \in F$ such that $x \in \prod_i V_i$ where $V_i = X_i$ for $i \notin F$, $V_j = U_j$ for $j \in F$ and $x \in \prod_{i\in I} V_i \subset G$.

7.29 Let $\emptyset \neq U_i \subsetneq X_i$, be open in X_i for $i \in I$. Then $U = \prod_{i\in I} U_i$ could never be open in X unless I is finite.

Assume I is infinite. Let $x = (x_i) \in U$. If U were open, then there exists a finite set $F \subset I$, open sets V_j for $j \in F$ such that $x \in W = \prod_i W_i \subset U$ where $W_i = X_i$ for $i \notin F$ and $W_j = V_j$ for $j \in F$. Choose an $r \in I \setminus F$. Since $U_r \neq X_r$, there exists $y_r \in X_r \setminus U_r$. Consider $z = (z_i)$ where $z_i = x_i$ for $i \neq r$ and $z_r = y_r$. Then $z \in W$ but $z \notin U$.

7.30 If I is finite, say, $I = \{1, 2, \ldots, n\}$, then the basic open sets are of the form $U_1 \times \cdots \times U_n$ where U_i is an arbitrary open set in X_i for each $1 \leq i \leq n$.

7.31 **Warning:** If, at first, we defined finite products of topological spaces with basis as in the last item, we would be tempted to use the following collection as a basis for a topology on the product $\prod_{i\in I} X_i$:

$$\mathcal{B} := \left\{ \prod_{i\in I} U_i : \text{ where } U_i \text{ is an arbitrary open set in } X_i \right\}.$$

The topology given rise to by this basis is called the box topology. Evidently, this is finer than the product topology.

> The product topology on X is the smallest topology which makes all the projection maps π_i continuous. We shall always use this topology on the product sets.

7.32 We shall see how to visualize the subbasic and basic open sets of the product topology. This will allow us to gain some geometric intuition. Pictures!

(a) Consider $X \times Y$. We visualize this the first quadrant in $\mathbb{R}^2$ where X and Y are represented by $[0, \infty)$. Then any subbasic open set if of the form $U \times Y$ or $X \times V$ where $U \subset X$ and $V \subset Y$ are open. We visualize this by a vertical strip of the from $(a, b) \times [0, \infty)$ or as a horizontal strip of the from $[0, \infty) \times (c, d)$. Hence any basic open set is represented by a rectangle of the from $(a, b) \times (c, d)$.

This can be extended to a finite product.

(b) We now consider a countable product, say $X = \prod_{n \in \mathbb{N}} X_n$. We visualize X as vertical half-lines erected at $(n, 0) \in \mathbb{R}^2$: A basic open set is therefore of the form half-lines at all points except at finitely many $n_1, \dots n_k$ and at n_j an interval of the form (a_j, b_j).

(c) Consider $X := \prod_{t \in \mathbb{R}} \mathbb{R}$. The product set can be identified with the set of functions $f : \mathbb{R} \to \mathbb{R}$. Each function can be represented by its graph in $\mathbb{R} \times \mathbb{R}$. Fix a finite set of points $\{t_1, \dots, t_n\}$ and a finite set of intervals (a_j, b_j), $1 \le j \le n$. Then the basic open set corresponding to this data is $\mathbb{R}$ at all $t \notin \{t_k : 1 \le k \le n\}$ and (a_k, b_k) if $t = t_k$, $1 \le k \le n$. Thus the elements in this basic set are functions such that $f(t_k) \in (a_k, b_k)$. We can visualize this by means of their graphs.

7.33 To have a feeling for the product topology, we look at the following results/questions:

(a) The product of Hausdorff spaces is Hausdorff.

Easy. if $x = (x_i)$ and $y = (y_i)$ are in $X = \prod_i X_i$ are distinct elements, then there exists $j \in I$ such that $x_j \ne y_j$. Since X_j is Hausdorff, there exist U_j and V_j open in X_j with $x_j \in U_j$, $y_j \in V_j$ and $U_j \cap V_j = \emptyset$. Consider the open set $U = \prod U_i$ and $V = \prod V_i$ where $U_i = X_i = V_i$ for $i \ne j$ and at j the disjoint open sets U_j and V_j. Then U and V separate x and y.

(b) A sequence (x_k) in the product space is convergent to an element x iff it converges coordinate-wise, that is, iff $\pi_i(x_k) \to \pi_i(x)$ for each $i \in I$.

To avoid confusion with indices, we use the Greek alphabet to denote elements of the index set I.

Let $x_k \in X = \prod_{\alpha \in I} X_\alpha$ converge to x. Fix $\beta \in I$. Let $x_{k\beta} := \pi_\beta(x_k)$ and $x_\beta := \pi_\beta(x)$. Let $U_\beta \ni x_\beta$ be open. Consider the subbasic open set $U = \prod U_\alpha$ where $U_\alpha = X_\alpha$ for $\alpha \ne \beta$ and $U_\alpha = U_\beta$ for $\alpha = \beta$. Then $x \in U$ and since $x_k \to x$, there exists $N \in \mathbb{N}$ such that $x_k \in U$ for $k \ge N$. It follows that $x_{k\beta} \in U_\beta$ for $k \ge N$. Hence $x_{k\beta} \to x_\beta$.

Note that this can also be proved using Item 29. Apply it to the sequence (x_n) and to the function $f = \pi_\alpha$.

To prove the converse, let $U \ni x$ be a basic open set, say, of the form $U = \prod U_\alpha$ where $U_\alpha = X_\alpha$ for $\alpha \notin F \subset I$, a finite subset and U_β are open subsets in X_β, $\beta \in F$. Since $x_{k\alpha} \to x_\alpha$, it follows that there exists N_β such that for $k \ge N_\beta$, we have $x_{k\beta} \in U_\beta$ for $\beta \in F$. Let $N = \max\{N_\beta : \beta \in F\}$. We claim that $x_k \in U$ for $k \ge N$. For, any $\alpha \notin F$, $\pi_\alpha(x_k) = x_{k\alpha} \in U_\alpha = X_\alpha$ and for $\beta \in F$, since $k \ge N$, $\pi_\beta(x_k) = x_{k\beta} \in U_\beta$.

Thus, the convergence in $\prod_{i \in I} X_i$ is "coordinate-wise convergence."

(c) Let D_i be dense in X_i for each i. Then $D := \prod_{i\in I} D_i$ is dense in X. Details!

Using our standard notation, it suffices to prove that $D \cap \pi_F^{-1}(V_F)$ is nonempty for a any finite subset of I and V_i open sets in X_i for $i \in F$. Since V_i is nonempty open set in X_i and D_i is dense in X_i, there exists $z_i \in D_i \cap V_i$ for $i \in F$. For $j \in I \setminus F$, we select an element $y_j \in D_j$. Then the element $(x_r) \in X$, where $x_r = y_r$ for $r \notin F$ and $y_r = z_r$ for $r \in F$ lies in $D \cap \pi_F^{-1}(V_F)$.

(d) Let $A_i \subset X_i$ and $A := \prod_{i\in I} A_i$. Then $\overline{A} = \prod_{i\in I} \overline{A}_i$. In particular, if each A_i is closed, then the product $A := \prod_{i\in I} A_i$ is closed in the product space X. Contrast this with Item 29.

The proofs of this runs almost like the earlier three items. We need to show that $x = (x_i)$ is a limit point of A iff each x_i is a limit point of A_i. Let x be a limit points of A. Let $U_i \ni x_i$. We need to show that $U_i \cap A_i \neq \emptyset$. Let $U := \pi_i^{-1}(U_i)$. Then $U \ni x$ and is an open set. Hence there exists $z \in U \cap A$. Clearly, $\pi(z) \in U_i \cap \pi_i(A) = U_i \cap A_i$.

Conversely, let x_i be a limit point of A_i for $i \in I$. Let $x = (x_i)$ and $U \ni x$ be an open set. We need to show that $U \cap A \neq \emptyset$. Since $x \in U$ and U is open in the product topology, there exists a finite set $F \subset I$ and a finite collection V_i of open sets in X_i for $i \in F$ such that $x \in \pi_F^{-1}(V_F) \subset U$. Note that $x_i \in V_i$ for $i \in F$. Since x_i is a limit point of A_i in X_i, there exists $z_i \in V_i \cap A_i$ for $i \in F$. Let $y = (y_j)$ be defined as $y_j \in A_j$ is arbitrary and $y_j = z_j$ if $j \in F$. Then $y \in \pi_F^{-1}(V_F) \cap A$.

(e) Let X_i be a discrete space for each i. When is $\prod_{i\in I} X_i$ is discrete?

(f) Let X, Y be metric spaces. We have a product metric on the product $X \times Y$ given by $\delta((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), d(y_1, y_2)\}$. Thus we have two topologies on $X \times Y$, namely, the topology induced by the metric δ and the product topology (got out of the metric topologies on X and Y). We claim that these two topologies are the same. Details!

It suffices to show that the identity map of $X \times Y$ is a homoeomophism of these two spaces. Let $I\colon (X \times Y, \mathcal{T}_\delta) \to X \times Y, \mathcal{T}_X \times \mathcal{T}_Y)$ be the identiy map.

Later, we shall see an easy proof. Reference?

Investigate whether the converses (wherever applicable) are true.

Optional: Investigate how many of them are true if we equip X with the box topology. Note that if D is dense in $(X, \mathcal{T}_2)$ and if $\mathcal{T}_1$ is another topology on X with $\mathcal{T}_1$ weaker than $\mathcal{T}_2$, then D is dense in $(X, \mathcal{T}_1)$.

7.34 This is a continuation the theme of Item 33.

(a) Let $f\colon Y \to \prod_{i\in I} X_i$ be a map from a topological space Y to the topological space $\prod_{i\in I} X_i$ with product topology. Then f is continuous iff each f_i, is continuous, where $f_i = \pi_i \circ f$ for $i \in I$,

If f is continuous, then f_i is the composition of two continuous functions and hence is continuous.

Assume that f_i are continuous. To prove that f is continuous, let U be an open subset of X. Since U is a union of basic open sets, it is enough to show that $f^{-1}(B)$ is open for any basic open set. But B is of the form $\cap_{j\in F} \pi_j^{-1}(U_j)$ where $F \subset I$ is finite, and U_j is open subset of X_j, $j \in F$. Since taking inverse images behaves well with set-theoretic operations, it suffices to show that $f^{-1}(\pi_j^{-1}(U_j))$ is open in Y for any $j \in I$ and any V_j open in X_j. But

$$f^{-1}(\pi_j^{-1}(U_j)) = (\pi_j \circ f)^{-1}(u_j) = f_j^{-1}(U_j), \tag{1}$$

which is open in Y by the continuity of f_j. (We used Item 2c in (1).)

An application: Let $X = \mathbb{R}^2$. Let $\mathcal{T}_P$ be the product topology on $\mathbb{R} \times \mathbb{R}$ and $\mathcal{T}$ be the standard topology. Then the identity map is a homeomorphism of $(\mathbb{R} \times \mathbb{R}, \mathcal{T}_P)$ onto $(\mathbb{R}^2, \mathcal{T})$. Hence $\mathcal{T}_P = \mathcal{T}$. Details!

(b) Let $f \colon X \to Y$ be continuous. Let $\text{Graph}(f) := \{(x, f(x)) : x \in X\}$ be the graph of f. Let $\text{Graph}(f) \subset X \times Y$ be endowed with the subspace topology. Then X is homeomorphic to $\text{Graph}(f)$.

Consider $\varphi \colon X \to \text{Graph}(f)$ given by $\varphi(x) = (x, f(x))$. Then φ is continuous as a map from X to $X \times Y$ by the last sub-item. By Item 19, $\varphi \colon X \to \text{Graph}(f)$ is also continuous. Clearly φ is a bijection and its inverse $\varphi^{-1} \colon \text{Graph}(f) \to X$ is given by $\varphi((x, f(x)) = f(x)$. That is, φ^{-1} is the restriction of the projection of $X \times Y$ onto its first factor. By the definition of product topology, the projection map is continuous on $X \times Y$. Its restriction to $\text{Graph}(f)$ is continuous by Item 18.

(c) The map $x \mapsto (x, y_0)$ of X into $X \times Y$ is a homeomorphism of X with $X \times \{y_0\}$ with the subspace topology inherited from $X \times Y$.

Argue as in the last sub-item.

(d) Let X, Y be topological spaces. Let $A \subset X$ and $B \subset Y$. Let $\mathcal{T}_A$ denote the subspace topology on A induced from the topology on X etc. Let $\mathcal{T}_A \times \mathcal{T}_B$ (respectively, $\mathcal{T}_X \times \mathcal{T}_Y$) denote the product topology on $A \times B$, (respectively, the product topology on $X \times Y$). Let $\mathcal{T}_{A\times B}$ denote the subspace topology on $A \times B$ considered as a subset of $X \times Y$. Then $\mathcal{T}_A \times \mathcal{T}_B = \mathcal{T}_{A\times B}$.

Easy if you know how to set up a notation which keeps your head clear. Let $W \in \mathcal{T}_A \times \mathcal{T}_B$. Let $x = (a, b) \in W$. There exist $U_a \in \mathcal{T}_A$ and $V_b \in \mathcal{T}_B$ such that $x = (a, b) \in U_a \times V_b \subset W$. Since $U_a \in \mathcal{T}_A$, there is a $U \in \mathcal{T}_X$ such that $U_a = U \cap A$. Similarly, $V_b = V \cap B$. Hence

$$x = (a, b) \in (U \cap A) \times (V \cap B) = (U \times V) \cap (A \times B) \subset W.$$

Since $(U \times V) \cap (A \times B) \in \mathcal{T}_{A\times B}$, it follows that $W \in \mathcal{T}_{A\times B}$.

Reverse inclusion is proved similarly. Let $W \in \mathcal{T}_{A\times B}$. Then there exists $W' \in \mathcal{T}_X \times \mathcal{T}_Y$ such that $W = W' \cap (A \times B)$. Let $x = (a, b) \in W$ so that $x \in W'$. We can find $U' \in \mathcal{T}_X$ and $V' \in \mathcal{T}_Y$ such that $x = (a, b) \in U' \times V' \subset W'$. Hence it follows that

$$x = (a, b) \in (U' \cap A) \times (V' \cap B) = (U' \times V') \cap (A \times B) \subset W' \cap (A \times B).$$

Since $(U' \cap A) \times (V' \cap B)$ is a basic open set in $\mathcal{T}_A \times \mathcal{T}_B$, it follows that $W \in \mathcal{T}_A \times \mathcal{T}_B$.

(e) Let $\Delta(X)$ denote the diagonal $\{(x, x) : x \in X \times X\} \subset X \times X$. Then X is Hausdorff iff $\Delta(X)$ is closed in $X \times X$. (We leave this as a very easy exercise.)

7.35 A frequently used observation. The projection maps $\pi_i \colon \prod_{i\in I} X_i \to X_i$ are open.

Observe that for any set $A := \prod A_i$, with A_i's nonempty we have $\pi_i(A) = A_i$.

Let $x_j \in \pi_j(A)$. Then there exists $x \in A$ such that $\pi_j(x) = x_j$. Since $x \in \prod A_i$, we have $x_j \in A_j$. Let $a_j \in A_j$. Since A_i's are non-empty, there exist $x_i \in A_i$, $i \neq j$. Consider (z_i) where $z_i = x_i$ for $\neq j$ and $z_j = a_j$. Then $z \in A$ and we have $\pi_j(z) = a_j$.

Let $U \subset \prod X_i$ be open. Let $G_i := \pi_i(U)$ and $a_i \in G_i$. Let $a \in U$ be any point such that $\pi_i(a) = a_i$. Since $a \in U$ and U is open there exists a basic open set $\pi_F^{-1}(V_F) = \prod_i V_i$ such that $a \in \pi_F^{-1}(V_F) \subset U$. We have $V_i = \pi_i\left(\prod_i V_i\right) \subset \pi_i(U)$. Hence $a_i \in V_i \subset G_i$. Thus G_i is open .

7.36 The projection maps $\pi_i \colon \prod_{i \in I} X_i \to X_i$ are not closed. For example, the projection of a rectangular hyperbola on to the x-axis is $\mathbb{R}^*$. Let $F := \{(x, y) \in \mathbb{R} \times \mathbb{R} : xy = 1\}$. Then F is closed (why?) in $\mathbb{R}^2$ with the standard topology (which is same as the product topology, why?). The projection of F on the first component, namely, $\pi_1(F) = \mathbb{R}^*$ is open in $\mathbb{R}$ and not closed in $\mathbb{R}$.

7.37 Let X and Y be sets. The set Y^X of functions from X to Y can be considered as the product space $\prod_{x \in X} Y_x$ where $Y_x = Y$ for $x \in X$ via the map $\varphi(f) = (f(x)) \in \prod_{x \in X} Y_x$. Thus, what the last line of Item 33b says is that the if we use φ to transfer the topology to Y^X, then a sequence of functions (f_n) in Y^X converges to a function $f \in Y^X$ iff $f_n(x) \to f(x)$ for each $x \in X$, that is, convergence in Y^X is pointwise convergence. Because of this, product topology is known as topology of pointwise convergence.

7.38 Is the product of first/second countable spaces first/second countable? We show that $\prod_{t \in \mathbb{R}} \mathbb{R}$ is not first countable. Note that $\mathbb{R}$ is second countable. The key idea comes from the last item. It may be worthwhile to review Item 32c.

The product space is the set of functions from $\mathbb{R}$ to $\mathbb{R}$ and the convergence is pointwise convergence. How does local base at (the constant function) 0 look like? Fix a finite subset $F \subset \mathbb{R}$ and $k \in \mathbb{N}$. Then a typical element of the local base is of the form

$$U_{F,k} := \{f \colon \mathbb{R} \to \mathbb{R} : f(t) \in (-1/k, 1/k), t \in F\}.$$

How many such basic open sets are there? As many as in the set $\mathcal{F} \times \mathbb{N}$ where $\mathcal{F}$ is the set of finite subsets of $\mathbb{R}$, that is, the cardinality of $\mathcal{F} \times \mathbb{N}$. It is intuitively clear and we expect that this space has no countable local base at $0 \in \prod_{t \in \mathbb{R})} \mathbb{R}$.

We would like to translate these ideas into a rigorous argument. An obvious method of attack is to prove this by contradiction. Consider the set

$$E := \{f \in \mathbb{R}^{\mathbb{R}} : f(t) = 0 \text{ or } 1 \;\&\; \{t \in \mathbb{R} : f(t) = 1\} \text{ is finite .}\}$$

We claim the constant function 0 is a limit point of E. For, let $U_{F,k}$ be a basic open set containing 0. Then the function $f(t) = 1$ for $t \notin F$ and $f(t) = 0$ for $t \in F$ lies in $U_{F,k} \cap E$. If the product topology were first countable, then there exists a sequence (f_n) in E such that $f_n \to f$ in the product topology. Let $F_n := \{t : f_n(t) = 0\}$. Let $A := \cup_n F_n$. Then A is countable. Observe that for $t \notin A$, $f_n(t) = 1$ and therefore $f(t) = \lim f_n(t) = 1$ for $t \notin A$. This is a contradiction

7.39 Contrast Item 42b with the following. Let E be any set and let $B(E, \mathbb{R})$ denote the set of all bounded real valued functions on E. If we endow this vector space with the norm $\|f\|_\infty := \sup_{x \in E} |f(x)|$, then $f_n \to f$ in this normed linear space iff $f_n \to f$ uniformly on E. (This is Item 28b.)

7.40 Refer to Item 38. Each of the factors in $\mathbb{R}^{\mathbb{R}}$ is a metric space. But the topology on $\mathbb{R}^{\mathbb{R}}$ is not first countable and hence there cannot be any metric d on the product $\mathbb{R}^{\mathbb{R}}$ which will induce the product topology.

7.41 In most of the examples above, we looked at subsets of the product set X which are of the form $\prod_{i\in I} A_i$, where $A_i \subset X_i$. You should be aware that not all subsets of X are of this form. For example, $S := \{(x,y) \in \mathbb{R}\times\mathbb{R} : x^2+y^2 = 1\}$, $D := \{(x,x) \in \mathbb{R}^2 : x \in \mathbb{R}\times\mathbb{R}\}$ are not a product of subsets of $\mathbb{R}$.

For, if $D = A \times B$, then $(1,1),(2,2) \in D = A\times B$. Hence $1,2 \in A$, $1,2 \in B$ and hence $(1,2) \in A \times B = D$!

7.42 It is equally important to recognize product spaces in disguise. The following are very typical of this situation.

(a) Define a topology on the set S of all real sequences such that a sequence (x_k) in S converges to $x \in S$ iff the $x_{kn} \to x_n$ as $n \to \infty$ for all k where $x_k = (x_{k1}, x_{k2}, \ldots, x_{kn}, \ldots)$. (Convergence = Coordinate-wise convergence).

(b) Let X denote the set of all real valued functions on $\mathbb{R}$. Define a topology on X such that a sequence (f_n) of functions in X converge to a function $f \in X$ iff $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$. (Convergence = point-wise convergence of functions.)

(c) Let $I = \mathbb{N}$ and $X_i = \{0,1\}$ for $i \in \mathbb{N}$. Then the product space $X := \prod_{i\in\mathbb{N}} X_i$ "is isomorphic to" the Cantor set. We have to introduce concepts and develop some more theory to explain this satisfactorily.

7.43 A problem similar to Item 33a: Let X be any set and $\mathcal{F}$ be a collection of real valued functions on X with the property that for any pair of distinct points $x, y \in X$, there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$. Then the smallest topology on X which makes all the functions in $\mathcal{F}$ continuous is Hausdorff.

7.44 Let $(X, \mathcal{T}_X)$ be a topological space and Y, a set. Let $f\colon X \to Y$ be a map. What is the largest topology on Y which makes f continuous?

Let $\mathcal{T}$ be any topology which makes $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T})$ continuous. Then for any $V \in \mathcal{T}$, we should have $f^{-1}(V) \in \mathcal{T}_X$. This suggests that we consider the collection $\mathcal{T}_Y := \{V \subset Y : f^{-1}(V) \in \mathcal{T}_X\}$. It is easy to see that this is the largest topology on Y ensuring the continuity of f.

7.45 An easy example: With the notation above, assume that $\mathcal{T}_X$ is discrete. Then $\mathcal{T}_Y$ is the discrete topology on Y.

7.46 Let X be a topological space and $\sim$ is an equivalence relation on X. Let $Y := X/\sim$ be the quotient set, that is, the set of all equivalence classes. Let $\pi\colon X \to Y$ be the quotient map $\pi(x) := [x]$, the equivalence class of x. The largest topology on Y with respect to which π is continuous is called the quotient topology on Y. It is given by

$$\left\{V : \pi^{-1}(V) \text{ is open in } X\right\}.$$

7.47 Two of the standard ways in which an equivalence relation is prescribed on a topological space are:

(i) when a group acts on the set X underlying the topology and
(ii) when we have a map from X onto a set Y.

In the first case we say $p \sim q$ iff there exists a group element g such that $g \cdot p = q$. This is same as requiring that p and q lie in the same orbit of the group.

In the second case, we say $p \sim q$ iff $f(p) = f(q)$.

7.48 It is a good idea at this juncture to read my article "Generation Topologies — A Unified View of Subspace, Product and Quotient Topologies".

7.49 Now that we have created these new objects, how do we work with them? The answer is provided by the so-called universal mapping properties.

Theorem 9 (Universal Mapping Property)**.**
(1.) *Let $f\colon X \to Y$ be a map from a set X to a topological space Y. Let X be given the topology generated by f. Then a function $h\colon Z \to X$ is continuous iff $f \circ h\colon Z \to Y$ is continuous.*

(2.) *Let $g\colon X \to Y$ be a map from a topological space X to a set Y. Let Y be endowed with the topology generated by g. Then a map $h\colon Y \to Z$ is continuous iff the map $h \circ g\colon X \to Z$ is continuous.*

(3.) *Let $\pi_i\colon X \to X_i$ be maps from the set X to topological spaces X_i for $i \in I$. Let X be given the topology generated by π_i's. Then a map $h\colon Y \to X$ is continuous iff the maps $\pi_i \circ h\colon Z \to X_i$ are continuous.*

Proof. Let us prove (1) as a sample, as the proofs are all similar and easy. To prove the nontrivial part, let us assume that the map $f \circ h$ is continuous. Let $U \subset X$ be open. We need to show that $h^{-1}(U)$ is open in Z. By the very definition of the topology on X, there exists an open set $V \subset Y$ such that $U = f^{-1}(V)$. Now

$$h^{-1}(U) = h^{-1}(f^{-1}(V)) = (f \circ h)^{-1}(V),$$

which is open by the continuity of $f \circ h$.

To prove (2), let W be open in Z. We need to show that $h^{-1}(W)$ is open in Y. By the continuity of $h \circ g$, the set

$$(h \circ g)^{-1}(W) = g^{-1}(h^{-1}(W))$$

is open in X. By the definition of topology on Y, the subset $h^{-1}(W)$ is open.

To prove (3), we observe that it is enough to show that $h^{-1}(U)$ is open for

$$U \in \mathcal{S} := \{U : \pi_i^{-1}(V_i) \text{ for an open } V_i \subset X_i \text{ for some } i \in I\}.$$

(Prove this. Refer to Item 29.) If $U = \pi_i^{-1}(V)$, we have $h^{-1}(U) := h^{-1}\left(\pi_i^{-1}(U)\right) = (\pi_i \circ h)^{-1}(U)$ is open, by the continuity of $\pi_i \circ h$. □

7.50 The most important thing to observe in the theorem is that the problem of establishing continuity of a map either from or to a newly constructed space is reduced to showing the continuity of a 'natural composite map' between the 'known spaces'. Go back to the statement and understand this remark. Also, go through the next few item.

7.51 Let us explicate the theorem in the concrete situations.

- If $f\colon X \to Y$ is the inclusion map (that is, the restriction of the identity of Y to X) of a subset X into a topological space Y, then (1) of the theorem says that a map $h\colon Z \to X$ is continuous iff we think of h as a map form the space Z to Y (taking values only in X) is continuous.
- Let X be a topological space, $\sim$ be an equivalence relation on X and $Y = X/\sim$ be the quotient set with the quotient map $\pi\colon X \to Y$. Then a map $h\colon Y \to Z$ form the quotient space Y to a space Z is continuous iff the map $h \circ \pi\colon X \to Z$ is continuous.
- Let $X := \prod_{i \in I} X_i$ is the Cartesian product of topological spaces with the product topology. Then a map $h\colon Z \to X$ can be written as $h(z) = (h_i(z))$ where the *coordinate maps* $h_i(z) := \pi_i \circ h(z)$. Thus $h\colon Z \to X$ is continuous iff the coordinate maps h_i are continuous.

7.52 The universal mapping property is the most important to deal with the newly constructed topologies. For instance, in the case of quotient space $Y = X/\sim$, giving a map $h\colon Y \to Z$ is the same as giving a map $\tilde{h}\colon X \to Z$ which is constant on the equivalence classes. Hence the continuity of h is the same as that of $\tilde{h}$. As a concrete example, let $X = \mathbb{R}$ and $x \sim y$ iff $x - y \in \mathbb{Z}$. Let $S := \{z \in \mathbb{C} : |z| = 1\}$ be with the subspace topology. We have a natural map $\tilde{h}\colon X \to S$ given by $\tilde{h}(t) := e^{2\pi i t}$. Then $\tilde{h}$ gives rise to map $h\colon Y \to S$ which is a bijection from well-known properties of the exponential map. Also, h is continuous since $\tilde{h}$ is so. Since any continuous map from a compact space to a Hausdorff space is a closed map, h is a homeomorphism. Thus Y is the circle in $\mathbb{C}$! For more such applications, see my article on Quotient spaces.

7.53 Let $X = \mathbb{R}^2$ and we let the additive group $\mathbb{Z}$ act on X as follows: $n \cdot (x, y) = (x + n, y)$. Then $(x_1, y_1) \sim (x_2, y_2)$ iff $x_1 - x_2 \in \mathbb{Z}$. Consider the set $E : -\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, y \in \mathbb{R}\}$. Note that any $(x, y) \in \mathbb{R}^2$ is equivalent to a unique point $(a, y) \in E$ if $x \notin \mathbb{Z}$ and $(x, y) \sim (0, y) \sim (1, y)$ if $x \in \mathbb{Z}$. Thus if we identify the vertical lines $x = 0$ and $x = 1$, we may be tempted to believe that the quotient space is homeomorphic to the cylinder $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$. Can we at least get a continuous bijection of $Q := X/\sim$ onto S? Consider the map $f([x, y]) := (\cos 2\pi x, \sin 2\pi x, y)$. It is easy to check that this is well defined bijection. To check continuity, we use UMP. The natural composite map is $\mathbb{R}^2 \to S$ given by $(x, y) \mapsto (\cos 2\pi x, \sin 2\pi x, y)$, which is continuous.

7.54 Real Projective Spaces: An important class of spaces which arise as quotient spaces is the class of projective spaces.

We shall construct them in three different ways. Let $X := \mathbb{R}^{n+1} \setminus \{0\}$ be the set of nonzero vectors in $\mathbb{R}^{n+1}$ with the subspace topology. Let the group $\mathbb{R}^*$ act on it via $(t, x) \mapsto tx$. Note that any group action gives rise to an equivalence relation, which in this case translates as follows: $x \sim y$ iff there exists $t \in \mathbb{R}^*$ such that $y = tx$. The orbits of the group action are the equivalence classes. Then the n-dimensional projective space (over $\mathbb{R}$), also called n-dimensional real projective space $\mathbb{P}^n(\mathbb{R})$ is the set of orbits with the quotient topology.

There is a second description of $\mathbb{P}^n(\mathbb{R})$. Observe that any orbit of the last paragraph intersects the n-dimensional sphere $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ at exactly two points. Details!

Thus the equivalence relation on $X := \mathbb{R}^{n+1} \setminus \{0\}$ induces an equivalence relation on $Y := S^n$: Given two points $u, v \in S^n$, we say that $u \sim v$ iff $v = \pm u$. This equivalence is again induced by the $\mathbb{Z}_2$ action on S^n: $0.u = u$ and $1.u = -u$. (It may be better to think of $\mathbb{Z}_2$ as the multiplicative group of square roots of unity to 'appreciate' this action!) The quotient space of Y is also denoted by $\mathbb{P}^n(\mathbb{R})$. There is an obvious bijection between the quotient set X and that of Y which turns out to be a homeomorphism. This will be proved later.

A third description is $\mathbb{P}^n(\mathbb{R})$ is the collection of all lines passing though the origin in $\mathbb{R}^{n+1}$. This does not arise by quotient construction. (Why?) However, we have an obvious bijection of this set with the quotient set of X of the first description. A line through the origin is determined as soon as we know a nonzero vector on it and any two such different by a (necessarily nonzero) scalar multiple. Now using this bijection, we transfer the topology on the $X/\sim$ to the set of lines passing through the origin. (See Item 1.)

7.55 We have an obvious map $\varphi \colon S^n \to \mathbb{P}^n(\mathbb{R})$ defined by $x \mapsto [x]$. This map is an onto continuous map. For, it is the composite of the maps $x \mapsto x$ from $S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ followed by the quotient map $x \mapsto [x]$. Note that the equivalence relation induced by f is the one we defined on S^n. Hence this map induces a homeomorphism of the quotient of S^n relative to the equivalence onto $\mathbb{P}^n(\mathbb{R})$.

See how we established the continuity of $S^n \to \mathbb{P}^n(\mathbb{R})$. It cannot be done using UMP.

7.56 **More examples of homeomorphisms.** Recall that a map $f\colon X \to Y$ between two topological spaces is a *homeomorphism* if (i) f is bijective, (ii) f is continuous and (iii) $f^{-1}\colon Y \to X$ is continuous.

(a) Any $f\colon \mathbb{R} \to \mathbb{R}$ of the form $f(x) = ax$ for a nonzero $a \in \mathbb{R}$ is a homeomorphism.

(b) Fix $v \in \mathbb{R}^n$. Then $x \mapsto x + v$ is a homeomorphism of $\mathbb{R}^n$.

(c) Fix $a \in \mathbb{R}$ nonzero. Then the map $x \mapsto ax$ is a homeomorphism of $\mathbb{R}^n$. (This and the last item can be generalized to any normed linear space.)

(d) $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a homeomorphism.

(e) Any linear isomorphism of $\mathbb{R}^n$ is a homeomorphism. (In general this is not true for a normed linear space, as such a linear isomorphism need not even be continuous!)

(f) $[a, b] \simeq [0, 1]$. More generally, $[a, b] \simeq [c, d]$.

(g) $(-1, 1) \simeq \mathbb{R}$.

(h) $(0, 1] \simeq [1, \infty)$.

(i) $[0, 1) \simeq (0, 1]$.

(j) Any two discrete spaces are homeomorphic iff they have the same cardinality.

(k) Can $\mathbb{Q}$ be homeomorphic to $\mathbb{Z}$ with the subspace topologies (induced from $\mathbb{R}$)?

(l) Is $\mathbb{N} \simeq \mathbb{Z}$ with the subspace topologies (induced from $\mathbb{R}$)?

(m) If two metric spaces are isometric, then they are homeomorphic.

(n) $B(0, 1) \simeq \mathbb{R}^n$.

(o) $S^n \setminus \{e_{n+1}\} \simeq \mathbb{R}^n$. (Refer to Example 3.3.5 on Page 73 of my book, "Topology of Metric Spaces", 2nd edition.)

(p) $f\colon X \to Y$ continuous. Then the graph of f with the subspace topology of $X \times Y$ is homeomorphic to X. Applications:

   i. $\mathbb{R}$ is homeomorphic to the parabola $y = x^2$.

   ii. $\mathbb{R}^*$ is homeomorphic to the hyperbola $xy = 1$.

(q) The product space $[-1, 1] \times S^1$ is homeomorphic to a cylinder.

(r) The annulus $\{p \in \mathbb{R}^2 : 1 \le \|p\| \le 2\}$ is homeomorphic to the cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, 1 \le z \le 2\}$.

(s) Let $f\colon X \to Y$ be a homeomorphism and let $A \subset X$. Then f induces homeomorphism between A and $f(A)$ (and between $X \setminus A$ and $f(X \setminus A)$).

   This is a very useful fact. Typical ways of applying this are:

   i. $[0, 1)$ is not homeomorphic to $(0, 1)$.

   ii. $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$.

   Both these results need connectedness at least in disguise, but can be proved at this stage using the intermediate value theorem.

   For example, let, if possible, $\varphi\colon [0, 1) \to (0, 1)$ be a homeomorphism. Then we have a homeomorphism, again denoted by $\varphi\colon (0, 1) \to (0, 1) \setminus \{\varphi(0)\}$. Then $f\colon (0, 1) \setminus \{\varphi(0)\} \to \mathbb{R}$ defined as $f(x) = x$ is a continuous function. Let $c, d \in (0, 1)$ be such that $c < \varphi(0) < d$. Let $\varphi(a) = c$ and $\varphi(b) = d$. Then $a, b \in (0, 1)$. The function $f \circ \varphi\colon (0, 1) \to \mathbb{R}$ is continuous on the interval $(0, 1)$ such that $f \circ \varphi(a) = c$, $f \circ \varphi(b) = d$. But it misses the point $\varphi(0) \in (c, d)$. This contradicts the intermediate value theorem.

(t) Homeomorphism between conic sections:

i. A circle is homeomorphic to an ellipse.

ii. A parabola is homeomorphic to a line.

iii. A (rectangular) hyperbola is homeomorphic to $\mathbb{R}^*$.

iv. A pair of intersecting lines is not homeomorphic to any of the other conic sections. More generally, a circle, a parabola, a hyperbola and a pair of intersecting lines are mutually non-homeomorphic. (We shall see a proof of this later. Meanwhile you may try to prove along this along the lines of a proof of Item 56(s)i.)

(u) Any k-dimensional subspace W of $\mathbb{R}^n$ with subspace topology is homeomorphic to $\mathbb{R}^k$. Let $\{w_i : 1 \leq i \leq k\}$ be a basis of W. Then the map $f\colon \mathbb{R}^k \to \mathbb{R}^n$ given by $f(a_1, \ldots, a_k) := a_1 w_1 + \cdots + a_k w_k$ is continuous. For, write $w_i := w_{i1} e_1 + \cdots + w_{in} e_n$ with respect to the standard basis of $\mathbb{R}^n$. What are the components f_i, $(1 \leq i \leq n)$, of f? Why is the inverse continuous? Details!

7.57 Let X and Y be metric spaces. A map $f\colon X \to Y$ is said to be *distance preserving* if for any $x_1, x_2 \in X$, we have $d(f(x_1), f(x_2)) = d(x_1, x_2)$. Any distance preserving map f is a homeomorphism of X onto $f(X)$.

A distance preserving map need to be onto. For instance, consider $X = Y = [0, \infty)$ with the standard metric. Then $f(x) := x = 1$ maps X onto $[1, \infty_)$.

7.58 An *isometry* of between metric spaces is an *onto* distance preserving map.

Any orthogonal/unitary linear transformation of a finite dimensional (real/complex) inner product space is an isometry.

7.59 In any normed linear space , any two open balls are homeomorphic. Recall that $B(x,r) = x + rB(0,1)$ and $B(y,s) = y + sB(0,1)$.

The map $u \mapsto x + ru$ is a homeomorphism of $B(0,1)$ onto $B(x,r)$.

7.60 In any normed linear space , any open ball is homeomorphic to the entire space. Enough to show that $B(0,1)$ is homeomorphic to the normed linear space X. Any nonzero x is of the from $x = tu$, $0 \leq t < 1$. Can we map $[0,1)$ to $[0,\infty)$ homeomorphically? If yes, then the finite radial line tu, $0 \leq t < 1$ will be mapped to the line segment emanating from 0 in the direction of u.

7.61 In $\mathbb{R}^n$, we have $B_\infty[0,1] \simeq B_2[0,1]$. For a complete proof, see Example 3.3.6 on Page 73 of my book on Metric Spaces.

7.62 $\mathbb{R}^m \simeq \mathbb{R}^n$ iff $m = n$. More generally, if $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are non-empty open sets which are homeomorphic, then $m = n$. This is a highly nontrivial result, known as the invariance of domain, We shall not prove this in our course!

7.63 Another most important way of proving that a map is a homeomorphism is to use the following result which you might have seen in TYBSc.

A bijective continuous map from a compact metric space to another metric space is a closed map and hence is a homeomorphism.

We shall see a more general result later in Item 22b.

8 Compact Spaces

8.1 Let X be a topological space and $A \subset X$. We say that a collection $\{U_i : i \in I\}$ of subsets of X is an open cover of A if (i) each U_i is an open subset of X and (ii) $A \subseteq \cup_{i \in I} U_i$.

Given an open cover $\{U_i : i \in I\}$ of A, by a subcover of A, we mean a subfamily $\{U_i : i \in J\}$ for some subset $J \subset I$ such that $\{U_i : i \in J\}$ is an open cover of A.

We say that the subcover is proper if J is a proper subset of I.

We say that the given open cover admits a finite subcover, if J (in the notation above) is a finite set.

For example, $\{(a,b) : a, b \in \mathbb{R}, a < b\}$ is an open cover of $\mathbb{R}$. The collection $\{(a,b) : a, b \in \mathbb{Q}, a < b\}$ is a subcover of $\mathbb{R}$.

Let X be a space with discrete topology. The family $\{A, X \setminus A\}$ is an open cover for X where A is a nonempty proper subset of X with no proper subcover.

Let X be a space with discrete topology. Let $\emptyset \neq A \subsetneq X$. Consider $\{A\} \cup \{\{x\} : x \in X \setminus A\}$. Then this collection is an open cover of X with no proper subcover.

8.2 Let $X = (0,1) \subset \mathbb{R}$ be with the metric/subspace topology. Let $U_n := (0, \frac{n-1}{n})$. Then $\{U_n : n \in \mathbb{N}\}$ is an open cover of $(0,1)$. (Why?) Can you think of a proper subcover? Can such a subcover be finite?

8.3 Examples of open covers:

(a) "Non trivial" open covers of $\mathbb{R}$:

i. $\{(-n, n) : n \in \mathbb{N}\}$.

ii. $\{(-\infty, n) : n \in \mathbb{N}\}$.

iii. $\{(-r, 2r) : r \in \mathbb{Q}^+\}$.

Do they admit finite subcovers? proper subcover?

(b) Nontrivial open covers of $(-1,1)$.

(c) In any metric space, $\{B(x, r_x) : x \in X\}$ is an open cover where $r_x > 0$ is pre-assigned for $x \in X$. Such a cover arises "naturally" in the following way: Let $f \colon X \to \mathbb{R}$ be a continuous function. Let $\varepsilon > 0$ be given. Given $x \in X$, by the continuity of f at x, there exists $r_x > 0$ such that for all y with $d(x,y) < r_x$, we have $|f(x) - f(y)| < \varepsilon$. The collection $\{B(x, r_x) : x \in X\}$ is an open cover of X.

(d) Given a Hausdorff space with at least two elements, think of a nontrivial open cover.

(e) Can you say something specific about any open cover of $\mathbb{R}$ with outcast topology?

(f) Give an open cover of $\mathbb{R}$ with VIP topology which has no proper subcover.

(g) Give a non-trivial open cover of $\mathbb{R}$ with lower limit topology. For example, $\{[a,b) : a < b\}$. Think of a non-trivial open cover which does not admit any proper subcover.

(h) Open covers of S^n:

i. $\mathbb{R}^{n+1} \setminus \{0\}$. (This is a trivial open cover!)

ii. $U = \mathbb{R}^{n+1} \setminus \{N\}$ and $V := \mathbb{R}^{n+1} \setminus \{S\}$, where N, S are north and south poles of the sphere S^n respectively.

iii. $U_i^{\pm} := \{x \in \mathbb{R}^{n+1} : x_i \gtrless 0\}$, $1 \leq i \leq n+1$.

(i) Open cover for a discrete space X. Let $A \subset X$. Look at $\{A\} \cup \{\{x\} : x \notin A\}$. This is an open cover of X. What if $A = \emptyset$?

(j) Open cover for an uncountable space with co-countable topology. If $X = \mathbb{R}$ with co-countable topology and $U = \mathbb{R} \setminus \mathbb{N}$, I would think of defining $U_n := U \cup \{n\}$ or $V_n := U \cup \{k : 1 \leq k \leq n\}$. Then $\{U_n : n \in \mathbb{N}\}$ and $\{V_n : n \in \mathbb{N}\}$ are open covers of $\mathbb{R}$. Do they admit any proper subcovers of $\mathbb{R}$?

What will you do for an countable set X with co-countable topology?

(k) Open cover for a set with co-finite topology.

8.4 A subset A of a topological space X is said to be *compact* if given any open cover $\{V_i : i \in I\}$ of A where each V_i is open in A, we can find a finite subcover. We say that X is a compact space if X is a compact subset of X.

8.5 Given an open cover $\{U_i : i \in I\}$ of A by means of open subsets of X, then we have a "natural" open cover $\{V_i : i \in I\}$ of a subset $A \subset X$ by means of subsets of A which are open *in* A and conversely. (Note the indices. "Naturality" does not mean that given V_i's, the U_i's are unique!)

The significance of this observation is that when dealing with compactness of a subset $K \subset X$ we may either work an open cover of K by means of open subsets in X or by sets open **in** K. See Items 8, 11, 11 where this observation is exploited.

8.6 Examples of compact sets.

(a) A finite subset of any space is compact. In particular, the empty set is compact.

(b) An indiscrete space is compact.

(c) A discrete space is compact iff it is finite.

(d) $\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{Z}$ are not compact.

(e) The intervals of the form (a, b), $[a, b)$, $(a, b]$, any infinite interval are not compact.

(f) $\mathbb{R}$ with lower limit topology $\mathcal{T}_L$ is not compact. Go through the proof. Can you make a general principle of which this is a special case?

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on the same set X. Assume that $\mathcal{T}_1 \leq \mathcal{T}_2$. Then if $(X, \mathcal{T}_1)$ is not compact, then $(X, \mathcal{T}_2)$ is not compact and $(X, \mathcal{T}_2)$ is compact, so is $(X, \mathcal{T}_1)$.

This is reminiscent of the comparison test for infinite series of positive terms.

(g) Any open ball in $\mathbb{R}^n$ (or in any normed linear space) is not compact.

(h) $\mathbb{R}^n$ is not compact.

(i) Any closed and bounded interval $[a, b] \subset \mathbb{R}$ is compact.

Let $\{U_i : i \in I\}$ be a collection of open sets in $\mathbb{R}$ such that $[a, b] \subset \cup_i U_i$. Consider the set

$$E := \{x \in [a, b] : \exists \text{ a finite set } F_x \subset I \text{ such that } [a, x] \subset \cup_{j \in F_x} U_j\}.$$

If $a \in U_i$, then there exists $\varepsilon > 0$ such that $x \in (a - \varepsilon, a + \varepsilon) \subset U_i$ and also $a + \varepsilon < b$. Then $a + \varepsilon/2 \in E$. Let $c =$ l.u.b. E. (Why does it exist?) We claim $c \in [a.b]$, $c = b$ and $c = b \in E$. If $c < b$, then $c \in U_j$. By the argument above, there exists $\varepsilon > 0$ such that $E \ni c + \varepsilon/2 < b$, a contradiction. Hence $c = b$. Repeat the same argument to conclude $b \in E$.

(j) $\mathbb{R}$ with VIP topology is not compact.

(k) $\mathbb{R}$ with outcast topology is compact.

(l) Any set with co-finite topology is compact.

(m) An uncountable set with co-countable topology is not compact. See Item 3j.

(n) A finite union of compact sets is compact.

(o) The intersection of two compact sets need not be compact. See, however, Item 8.

Consider $\mathbb{Z}$ with the discrete topology. Let $\{\pm\infty\}$ be two distinct elements not in $\mathbb{Z}$. Let $X = \mathbb{Z} \cup \{\pm\infty\}$. We say a subset $U \subset X$ is open if either (i) $U \subset \mathbb{Z}$ or if either of $\pm\infty$ lies in $U \subset X$, then both the elements lie in U and $X \setminus U$ is finite. It is easy to verify that this defines a topology on X. The sets $A := \mathbb{Z} \cup \{\infty\}$ and $B := \mathbb{Z} \cup \{-\infty\}$ are compact but their intersection $\mathbb{Z}$ is not compact. Note that neither A nor B is closed.

8.7 A closed subset K of a compact space X is compact.

Let $\{U_i : i \in I\}$ be an open cover of K. To exploit the compactness of X, we need an open cover of X. Clearly, if we add the open set $X \setminus K$ to the given open cover of K, we end up with an open cover, say, $\mathcal{U}$ of X. Let $\mathcal{U}_0$ be a finite subcover of X. It is possible $\mathcal{U}_0$ contains $X \setminus K$. In any case, $\mathcal{U}_0 \setminus \{X \setminus K\}$ is a finite subcover of K.

8.8 Let $K \subset X$ be a compact subset of X. Is K closed in X?

That is, is $X \setminus K$ open? The only way of doing this it for each $x \in X$, to find an open set $U_x \ni x$ such that $U_x \cap K = \emptyset$. We also need to exploit the compactness of K. That is, we need to find an open cover of K (via open sets of X), members of which do not have x. This suggests that we may require X to be Hausdorff. (All these were arrived at by students!) We have the following result.

In a Hausdorff space a compact subset is closed and hence the intersection of compact sets is compact in a Hausdorff space.

Let X be Hausdorff and $K \subset X$ be compact. We shall show that the complement $K^c := X \setminus K$ is open. Given $p \in K^c$, we need to show that there exists an open set $U_p \ni p$ with $U_p \subset K^c$. We need to exploit Hausdorffness of X and the compactness of K. This means we need to generate an open cover of K. For any $q \in K$, we have disjoint open sets $V_q \ni q$ and $U_{pq} \ni p$. Hence $\{V_q : q \in K\}$ is an open cover of K and hence there exists a finite set $\{q_1, \ldots, q_n\}$ of K such that $K \subset \cup_{i=1}^n V_{q_i}$. Let $U_p := \cap_{i=1}^n U_{pq_i}$. Then $U_p \ni p$ is an open set and it lies in K^c. Thus, $K^c = \cup_{p \in K^c} U_p$ is open. Picture!

8.9 Note that the a proof in the last item establishes the following result.

Let X be a compact Hausdorff space. Let $K \subset X$ be closed and $x \notin K$. Then there exist disjoint open sets $U_x \ni x$ and $U_K \supset K$.

A space X is said to be *regular* if K is closed in X and $x \notin K$, there exist disjoint open sets $U_x \ni x$ and $U_K \supset K$.

Hence a compact Hausdorff space is regular.

Question: In the result about a compact Hausdorff space X, can we replace x by a closed set L disjoint from K?

8.10 Let (X, d) be a metric space. We say that $A \subset X$ is *bounded* if there exist $x_0 \in X$ and $r > 0$ such that $A \subset B(x_0, r)$. The following are easily seen results about this concept:

(a) A is bounded iff for every $x_1 \in X$, there exists $R > 0$ such that $A \subset B(x_1, R)$.
Easy. Observe

$$d(a, x_1) \leq d(a, x_0) + d(x_0, x_1) < r + d(x_0, x_1).$$

Hence let $R = r + d(x_0, x_1)$.

(b) Let $(X, \| \ \|)$ be an normed linear space . Show that $A \subset X$ is bounded iff there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in A$. Easy. $A \subset B(0, M)$ for some $M > 0$ by the last subitem.

(c) Any finite set is bounded.

(d) Any open or closed ball is bounded.

(e) A is bounded iff there exists $M > 0$ such that $d(x, y) \leq M$ for all $x, y \in A$.

(f) If $A \neq \emptyset$, we set $\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}$, which is set to ∞ if the supremum does not exist. The extended real number $\operatorname{diam}(A)$ is called the diameter of A. A set A is bounded iff either $A = \emptyset$ or $\operatorname{diam}(A) < \infty$.

(g) $\operatorname{diam}(B(x, r)) \leq 2r$ and strict inequality can occur.

(h) In an normed linear space , $\operatorname{diam}(B(x, r)) = 2r$. *Hint:* Go through Item 8.

(i) Any convergent sequence in a metric space is bounded.

(j) Boundedness is not a topological property. Already seen in Item 12g.

(k) Which vector subspaces of an normed linear space are bounded subsets?

(l) The set $O(n)$ of all orthogonal matrices (that is, the set of matrices satisfying $AA^t = I = A^tA$) is a bounded subset of $M(n, \mathbb{R})$. Here $M(n, R)$ is considered as an normed linear space as in Ex. 27. Observe that $\|A\|^2 = \sum_i \left(\sum_j |a_{ij}^2|\right) = n$.

(m) The set $SL(n, \mathbb{R})$ of all $n \times n$ real matrices with determinant 1 is not bounded in $M(n, \mathbb{R})$.

(n) The set of all nilpotent matrices in $M(n, \mathbb{R})$ is not a bounded set. *Hint:* What is he "canonical form" of a nilpotent matrix?

(o) Let G be a subgroup of the multiplicative group $\mathbb{C}^*$ of the non-zero complex numbers. Assume that as a subset of $\mathbb{C}$ it is bounded. Then $|g| = 1$ for all $g \in G$.

8.11 In a metric space any compact set is bounded in X.

Let K be a compact subset of a metric space X. Fix $a \in X$. Consider $\{B(a, n) : n \in \mathbb{N}\}$ This is an open cover of X and hence the collection $\{B(a, n) : n \in \mathbb{N}\}$ has a finite subcover of K. Since $(B(a, n)$ is increasing, there exists $N \in \mathbb{N}$ such that $K \subset B(a, N)$.

Applications:

(a) $SL(n, \mathbb{R})$ is not a compact subset of $M(n, \mathbb{R})$.

(b) The set of symmetric (respectively, the skew-symmetric) matrices is not compact in $M(n, \mathbb{R})$. So is the set of matrices with trace zero.

(c) The set of nilpotent matrices in $M(n, \mathbb{R})$ is not compact.

8.12 In any topological space, any convergent sequence along with its limit is a compact subset.

Let $x_n \to x$. Given an open cover $\{U_i : i \in I\}$ of $\{x_n : n \in \mathbb{N}\} \cup \{x\}$, let $x \in U_j$. Then all but finitely many $x_n \in U_j$.

8.13 If A is a nonempty compact subset of $\mathbb{R}$, then $\sup A$ and $\inf A$ exist and they belong to A.

Let $\beta = \sup A$. Then there exists x_n such that $\beta - \frac{1}{n} < x_n \leq \beta$. Hence $x_n \to \beta$ and hence β is a limit point of A. Heine-Borel says that A is closed.

8.14 Assume that $f\colon X \to Y$ is continuous and that X is compact. Then $f(X)$ is compact. In particular, compactness is a topological property.

8.15 The product $X \times Y$ of two spaces is compact iff X and Y are compact.

To understand the proof, draw a picture of $X \times Y$ as a closed rectangle, as explained in Item 32. If $\{U_i \times V_i : i \in I\}$ is an open cover by means of basic open sets, then we have an cover of $\{x\} \times Y$, a "vertical line". Since this is compact, we have a finite subcover which turns out to be an open cover of a "band" fattening the vertical line, that is, a (super)set of the form $U_x \times Y$, $U_x \ni x$ open. (A more challenging and instructive exercise could be to carry out this in the case of an open cover of the circle $x^2 + y^2 = 1$ by means of open disks in $\mathbb{R}^2$.) These U_x's cover X and hence they have a finite subcover. Thus we end up with a finite subcover of $X \times Y$.

Let us now work out the details. WLOG, we may assume that we are given an open cover by means of basic open sets as in the last paragraph. Since the inclusion map $x \mapsto (x, y)$ is continuous (Why? See Items 34b and 34c.), $\{x\} \times Y$ is compact by Item 14. Hence there exists a finite subcover, say, $\{U_i \times V_i : i \in F_x\}$ for a finite subset $F_x \subset I$. Then $x \in U_x = \cap_{j \in F_x} U_i$ is an open set. Thus the finite subcover $\{U_i \times V_i : i \in F_x\}$ is an open cover of $U_x \times Y$. As x varies over X, we have an open cover $\{U_x : x \in X\}$ of the compact space X. Let $A \subset X$ be finite such that $\{U_x : x \in A\}$ is an open cover of X. Then the collection $\{U_i \times V_i : i \in F_x, x \in A\}$ is a finite subcover of $X \times Y$.

(Why? Let $(x, y) \in X \times Y$. Since $\{U_a : a \in A\}$ is a finite open cover of X, there exists $a \in A$ such that $x \in U_a$. Hence $(x, y) \in U_a \times Y$. Now, $\{U_j \times V_j : j \in F_a\}$ is a finite open cover of $U_a \times Y$, there exists $j \in F_a$ such that $(x, y) \in U_j \times V_j$, $j \in F_a$, $a \in A$, as claimed.)

(How many elements are there in this finite subcover? Answer: $\sum_{x \in A} |F_x|$.)

8.16 A more general result known as Tykhonoff's theorem is true, which has very far-reaching applications in analysis.

Theorem 10 (Tykhonoff). *Let $\{X_i : i \in I\}$ be a family of compact spaces. The the product space $\prod_{i \in I} X_i$ with product topology is compact.* □

For a proof, see my article on Compact spaces.

8.17 Any cube $[-R, R]^n \subset \mathbb{R}^n$ is compact. This follows from Items 6i and 15.

8.18 Let K be closed and bounded subset of $\mathbb{R}^n$. Let $R > 0$ be such that $\|x\| \leq R$ for $x \in K$. Then $|x_i| \leq \mathbb{R}$ for $x = (x_1, \ldots, x_n) \in K$. Thus, $K \subset [-R, R]^n$. Hence by the last result, $[-R, R]^n$ is compact. By Item 7, K is compact. We have thus proved the sufficiency part of the following

Theorem 11 (Heine-Borel)**.** *A subset $K \subset \mathbb{R}^n$ is compact iff K is closed and bounded.*

The necessary part follows from Items 8 and 11.

8.19 Applications of Heine-Borel theorem.

(a) Among the non-degenerate conics in $\mathbb{R}^2$, only circles and ellipses are compact.

(b) The unit sphere $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is compact.

(c) $O(n, \mathbb{R})$, the set of orthogonal matrices is compact subset of $M(n, \mathbb{R})$.

(d) The subgroup $SL(n, \mathbb{R})$ is closed and unbounded. It is not a compact subset of $M(n, \mathbb{R})$.

(e) The set of nilpotent matrices in $M(n, \mathbb{R})$ is closed and unbounded. It is not a compact subset of $M(n, \mathbb{R})$.

(f) All norms on $\mathbb{R}^n$ are equivalent.

For this beautiful application, we refer the reader to Theorem 4.3.26 on Page 105 of my book on Metric Spaces (2nd edition).

Application: Any finite dimensional vector subspace of an normed linear space is always closed. *Hints:* If two equivalent norms $\| \ \|_1$ and $\| \ \|_2$ are given on a vector space X, then $(X, \| \ \|_1)$ is complete iff $(X, \| \ \|_2)$ is complete.

8.20 The set $[a, b] \subset \mathbb{R}_\ell$ is not compact in the lower limit topology $\mathcal{T}_L$ on $\mathbb{R}$. *Hint:* See Item 13k.

Consider the open cover $\{U_n : [a, b - 1/n) : n \in \mathbb{N}\} \cup \{[b, b+1)\}$.

8.21 In general, a closed and bounded subset of a metric space need not be compact.

Standard example: If X is an infinite set with the discrete metric, then X is bounded, closed but not compact.

For another, which will throw an illuminating insight into an example you learnt in the theory of convergence of functions, see Item 34h.)

8.22 Compact sets and maps:

(a) Assume that $f : X \to Y$ is continuous and that X is compact. Then $f(X)$ is compact. In particular, compactness is a topological property.

Let $\{V_i : i \in I\}$ be an open cover of $f(X)$ with V_i open in Y. By continuity of f, each $U_i := f^{-1}(V_i)$ is open in X. We claim that $\{U_i\}$ is an open cover of X. If $x \in X$, then there exists $i \in I$ such that $f(x) \in V_i$, that is, $x \in f^{-1}(V_i) = U_i$. Since X is compact, there exists a a finite subset $F \subset I$ such that $X = \cup_{i \in F} U_i$. We claim that $f(X) \subset \cup_{i \in F} V_i$. For, if $y = f(x) \in f(X)$, then $x \in U_j$ for some $j \in F$. Hence $f(x) \in V_j$, that is, $y \in \cup_{i \in F} V_i$.

(b) Let X be compact and Y be Hausdorff. Then any continuous bijection $f : X \to Y$ is a homeomorphism.

We claim that f is a closed map. Let $C \subset X$ be a closed set. Then C is compact by Item 7. Hence $f(C)$ is a compact subset of Y by sub-item (a). Since Y is Hausdorff space, and the compact set $f(C)$ is closed in Y by Item 8.

This is a very useful result. Some applications are given below.

i. Typical applications arise in the theory of quotient spaces: The quotient space $[0, 2\pi]/\sim$ is homeomorphic to S^1.

ii. Let f be any map (not assumed to be continuous) from a compact Hausdorff space X to a compact space Y. Assume that the graph of f is closed as a subset of the product space $X \times Y$. Then f is continuous.

We have a bijection $\varphi\colon X \to \text{Graph}(f)$ given by $\varphi(x) = (x, f(x))$. If we show that φ is continuous, then as a component of φ, the function f must be continuous. To use Item 22b, the requirements that the domain and co-domain are compact are to be met and we need a continuous bijection.

Since $\text{Graph}(f)$ is closed subset of the compact space $X \times Y$, we see that $\text{Graph}(f)$ is compact. If we let $\psi := \varphi^{-1}$, then $\psi(x, f(x)) = x$ is a continuous bijection from the compact space $\text{Graph}(f)$ to the compact Hausdorff X. Hence it is a homeomorphism. We conclude its inverse φ is also continuous.

This may be called a Closed Graph Theorem, in analogy with a result bearing the same name in functional analysis: Let X and Y be complete normed linear spaces. Let $T\colon X \to Y$ be a linear map whose graph is closed in $X \times Y$. Then T is continuous.

iii. Let X be a set with two distinct topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. Assume that $\mathcal{T}_1 \subset \mathcal{T}_2$ and further that $(X, \mathcal{T}_2)$ is compact Hausdorff. Then $(X, \mathcal{T}_1)$ is compact but not Hausdorff.

is

(c) Let Y be compact. Then the projection map $\pi := \pi_X \colon X \times Y \to X$ is a closed map.

Let $K \subset X \times Y$ be closed. Let $C := \pi(K)$. We claim $X \setminus C$ is open. Let $a \in X \setminus C$. Note that for any $y \in Y$, we have $(a, y) \notin K$. Since K is closed in $X \times Y$, there exists a basic open set of the form $U_y \times V_y$ such that $(a, y) \in U_y \times V_y$ and $K \cap (U_y \times V_y) = \emptyset$. We thus end up with an open cover $\{V_y : y \in Y\}$ of the compact space Y. Hence there exists a finite subset $F_a \subset Y$ such that $\{V_y : y \in F_a\}$ is a finite subcover. Let $U := \cap_{y \in F_a} U_y$. Then $a \in U$ and U is an open set. We claim that $U \cap C = \emptyset$. If not there exists $x \in U \cap C$ and hence there exists $z \in Y$ such that $(x, z) \in K$. Now $z \in V_y$ for some $y \in F_a$. Since $U \subset U_y$, it follows that $(x, z) \in U_y \times V_y$ so that $(U_y \times V_y) \cap K \neq \emptyset$, a contradiction. Hence $a \in U \subset X \setminus C$. This proves C is closed.

(d) Let X be compact and Y be a metric space. Then any continuous map $f\colon X \to Y$ is bounded.

Let $f\colon X \to Y$ be a continuous function from a compact space X to a metric space Y. Fix $q \in Y$. Consider $V_n := B(q, n)$. Then $U_n := f^{-1}(V_n)$ is open. The sequence (U_n) is increasing and $\cup_n U_n = X$. Hence $X = U_N$ for some N, that is, $f(X) \subset B(q, N)$.

Note that this also follows form Items 14 and 11.

The converse is not true, in general. See Items 8 and 6m. For metric spaces, the converse is true. For a proof, see my article on Compact Spaces. Details!

(e) Let X be compact. Then any continuous function $f\colon X \to \mathbb{R}$ attains its bounds.

Let X be a compact space and $f\colon X \to \mathbb{R}$ be continuous. By the last sub-item $f(X)$ is a bounded subset of $\mathbb{R}$. Let $M = \sup f(X)$ and $m = \inf f(X)$. If there does not exists any $a \in X$ such that $f(a) = M$, then $U_n := \{x \in X : f(x) < M - \frac{1}{n}\}$ is open, $U_n \subset U_{n+1}$ and $\cup_n U_n = X$. (Why?) By compactness, there exists N such that $f(X) = U_N$. But then $\sup f(X) \leq M - \frac{1}{N}$, a contradiction. Similar proof establishes the existence of $b \in X$ such that $f(b) = m$.

This can also be proved using Item 14, Heine-Borel theorem and Item 13.

Applications:

i. Let X be compact and $f\colon X \to \mathbb{R}$ be continuous. Assume that $f(x) > 0$ for all $x \in X$. Then there is a $\delta > 0$ such that $f(x) \geq \delta$ for all $x \in X$.

ii. Let K be a compact and C a closed subsets of a metric space X such that $K \cap C = \emptyset$. Then $d(K, C) > 0$.

iii. Let K be a nonempty compact subset of a normed linear space X. Then there exists $x \in K$ such that $\|y\| \leq \|x\|$ for all $y \in K$.

(f) Let X and Y be metric spaces. Assume that X is compact. Then any continuous map $f\colon X \to Y$ is uniformly continuous.

Fix $\varepsilon > 0$. For each $x \in X$, let δ_x correspond to $\varepsilon/2$ and the continuity of f at x. Then $\{B(x, \delta_x/2) : x \in X\}$ is an open cover of X. Let $\{B(x_k, \delta_k/2) : 1 \leq k \leq n\}$ be a finite subcover where $\delta_k = \delta_{x_k}$. Let $\delta := \min\{\delta_k/2 : 1 \leq k \leq n\}$.

Let $s, t \in X$ be such that $d(s, t) < \delta$. If $s \in B(x_k, \delta_k/2)$, then $d(t, x_k) \leq d(t, s) + d(s, x_k) < \delta_k$. Hence that

$$d(f(s), f(t)) \leq d(f(s), f(x_k)) + d(f(x_k), t) < \varepsilon.$$

8.23 Given an open cover $\{U_i : i \in I\}$ of a metric space (X, d), we say that a positive number δ is a *Lebesgue number* of the cover, if for any subset $A \subset X$ whose diameter is less than δ, there exists $i \in I$ such that $A \subset U_i$.

If δ is a Lebesgue number of the cover and $0 < \delta' \leq \delta$, then δ' is also a Lebesgue number of the given open cover.

8.24 In general, an open cover may not have a Lebesgue number. Let $X = (0, 1)$ with the usual metric. Let $U_n := (1/n, 1)$. Then $\{U_n : n \in \mathbb{N}\}$ is an open cover of X. Does there exist a Lebesgue number for this cover?

Theorem 12 (Lebesgue Covering Lemma)**.** *Let (X, d) be a compact metric space. Let $\{U_i : i \in I\}$ be an open cover of X. Then a Lebesgue number exists for this cover.* □

We mimic the argument of Item 22f. For each $x \in X$, if $x \in U_i$, then there exists δ_x such that $B(x, \delta_x) \subset U_i$. Consider the open cover $\{B(x, \delta_x/2) : x \in X\}$ like earlier and arrive at δ, which does the job.

8.25 Use the last theorem to prove Item 22f. Note that the proofs of Item 22f and Lebesgue covering lemma are also similar.

8.26 Let X be compact and $E \subset X$ be infinite. Then E has a cluster point in X. (This is known as Bolzano-Weierstrass property.)

We shall prove this by contradiction. Given $x \in X$, x is not a cluster point of E. Hence there exists $U_x \ni x$ an open set such that $U_x \cap E$ is either empty or $\{x\}$. Now $\{U_x\}_{x \in X}$ is an open cover of X and hence there exists a finite set $A \subset X$ such that $\{U_x : x \in A\}$ is a finite subcover. Since

$$E = E \cap X = E \cap (\cup_{x \in A} U_x) = \cup_{x \in A} (E \cap U_x),$$

we see that E contains at most $|A|$ elements, a contradiction as A is infinite.

8.27 Definition of FIP: A family of subsets $\{F_i : i \in I\}$ of a set X is said to have the *finite intersection property*, (FIP, in short), if every finite collection of members of the family has a nonempty intersection. Examples:

(a) Let X be any set and (F_n) be a decreasing sequence of nonempty subsets of X. Then $\{F_n : n \in \mathbb{N}\}$ enjoys FIP.

(b) Let X be noncompact. Then there exists an open cover $\{U_i : i \in I\}$ of X which does not admit a subcover. Consider the family of closed sets $\{F_i : i \in I\}$ where $F_i := X \setminus U_i$. This family of closed sets has F.I.P.

8.28 A topological space is compact iff every family of closed sets with FIP has a nonempty intersection.

Let X be compact. Let $\{A_i : i \in I\}$ be a family of closed sets with FIP. We are required to show that $\cap_i A_i \neq \emptyset$. Assume on the contrary that $\cap_i A_i = \emptyset$. Let $U_i := X \setminus A_i$. Then $\{U_i : i \in I\}$ is an open cover of X. Since X is compact, there exists a finite set $F \subset I$ such that $\cup_{j\in F} U_j = X$. By taking complements of this equation, we obtain $\cap_{j\in F} A_j = \emptyset$. This contradicts our hypothesis that $\{A_i : i \in I\}$ enjoys FIP.

Converse is exactly along the same lines. If X has the said property, we need to show that X is compact. Let $\{U_i : i \in I\}$ be an open cover of X. Assume that it does not admit a finite subcover. Let $A_i := X \setminus U_i$. Then $\{A_i : i \in I\}$ is a family of closed set with FIP. Hence $\cap_i A_i \neq \emptyset$ which entails $\cup_i U_i \neq X$! □

This characterization is used in the proof of Tykhonoff's theorem.

Question: Can we use this to prove Tykhonoff's theorem for the product two compact spaces?

Let X and Y be compact. Let $\{K_i : i \in I\}$ be a family of closed sets with FIP. Let $A_i : \pi_X(K)i)$ and $B_i := \pi_Y(K_i)$. Since π_X and π_Y are closed maps (by Item 22c), each of A_i and B_i is closed. We claim $\{A_i : i \in I\}$ has FIP. For if there exists a finite subset $F \subset I$ such that $\cap_{i\in F} A_i = \emptyset$, then $\cap_{i\in F} K_i = \emptyset$. (For, otherwise, if $(x, y) \in \cap_{i\in F}$, then $x = \pi_X(x, y) \in \pi(K_i$ for each $i \in F$. In other words, $x \in \cap_{i\in F} A_i$.) This contradiction establishes that $\{A_i : i \in I\}$ has FIP. Since X is compact, there exists $x \in \cap_{i\in I} A_i$. By a similar argument, we conclude the existence of some $y \in \cap_{i\in I} B_i$. We may wish to claim that $(x, y) \in K_i$ for each $i \in I$. **This cannot be proved, as it stands.**

8.29 Cantor intersection theorem. This is an analogue of the nested interval theorem of real analysis.

Theorem 13. *Let X be any Hausdorff topological space. Let (K_n) be a decreasing sequence of nonempty compact subsets of X. Then $\cap_n K_n \neq \emptyset$.*

Assume the contrary. Let $U_n := X \setminus K_n$. Then each U_n is open. (Why?) (U_n) is an increasing sequence of open sets whose union is X. Hence $\{U_n : n \in \mathbb{N}\}$ is an open cover for K_1 and hence there exists N such that $K_1 \subset U_N$. That is, $K_1 \subset X \setminus K_N$. Since $K_N \neq \emptyset$, if we select $p \in K_N \subset K_1$, we arrive at a contradiction $p \in K_1 \subset K_N^c$.

8.30 A subset A of a metric space (X, d) is said to be *totally bounded* if for any given $\varepsilon > 0$, there exist a finite number of points $x_1, \ldots, x_n \in X$ such that $A \subset \cup_{k=1}^n B(x_k, \varepsilon)$.

The finite set $\{x_k : 1 \leq k \leq n\}$ is usually referred to as an ε-net for A.

8.31 Examples, non-examples and properties of totally bounded sets.

(a) Any compact subset of a metric space is totally bounded.

(b) If B is totally bounded and $A \subset B$, then A is totally bounded.

(c) If A is totally bounded, so is its closure $\overline{A}$.
If $\{x_k : 1 \le k \le n\}$ is an ε-net for A, then it is 2ε-net for $\overline{A}$.

(d) Any totally bounded subset is bounded. The converse is not true.
Standard example: an infinite set with discrete metric.
A slightly more demanding example: In ℓ^2, the orthonormal set $\{e_n : n \in \mathbb{N}\}$.
An interesting example: $f_n(x) = x^n$, $n \in \mathbb{N}$, in $(C[0,1], \| \ \|_\infty)$.

(e) Any bounded subset of $\mathbb{R}$ is totally bounded. (This is essentially Archimedean property.) In fact, any bounded subset of $\mathbb{R}^n$ is totally bounded.
One can prove this directly. Or, if $A \subset \mathbb{R}^n$ is bounded, so is $K := \overline{A}$. Hence K is closed and bounded. By Heine-Borel, K is compact and hence totally bounded. A being a subset of K is therefore totally bounded by Item 31b.

8.32 Characterization of compact metric spaces.

Theorem 14. *Let X be a metric space. Then the following are equivalent.*
1. X is compact.
2. X is complete and totally bounded.
3. (Bolzano-Weierstrass property.) *Every infinite subset of X has a cluster point in X.*
4. (Sequential compactness.) *Every sequence in X has a convergent subsequence.* □

For a proof, we refer the reader to Theorem 4.3.15 on Page 101 of my book on Metric Spaces (2nd edition).

8.33 Applications of 2nd characterization:

(a) Arzela-Ascoli theorem as a characterization of compact subsets of $(C(X), \| \ \|_\infty)$, where X is a compact metric space. (Perhaps statement only.)

(b) A subset $A \subset \ell_1$ is compact iff A is closed, bounded and is such that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{n \ge N} |x_n| < \varepsilon$ for **all** $x \in A$.

8.34 Applications of (perhaps the most useful) 4th characterization.

(a) Any continuous map from a compact space to a metric space is bounded.

(b) Any continuous real valued function on a compact space attains its bounds.

(c) Let K be a nonempty compact subset of $\mathbb{R}$. Show that $\sup K, \inf K \in K$. Deduce the last item from this.
Let $\alpha := \inf K$. Then there exists $x \in K$ such that $\alpha \le x_n < \alpha + \frac{1}{n}$. Hence $x_n \to \alpha$. Since K is closed, we obtain $\alpha \in K$.
To deduce the last result, take $K = f(X)$.

(d) Let A, B be disjoint compact subsets of a metric space. Then there exist $a \in A, b \in B$ such that $d(A, B) = d(a, b)$, and hence $d(A, B) > 0$.
This result need not be true if the sets are assumed to be closed. Consider $A := \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ and $B := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0 \text{ and } xy = 1\}$.

(e) Let K be a compact subset and C a closed set in $\mathbb{R}^n$. If $K \cap C = \emptyset$, then there exist $x \in K$ and $y \in C$ such that $d(x, y) = d(K, C)$.

It is easy to see that there exists an $x \in K$ such that $d(K, C) = d(x, C)$. To get y, observe that there exists a sequence (y_n) in C such that $d(x, y_n) \to d(x, C)$. You need to apply Bolzano-Weierstrass theorem to the sequence (y_n).

(f) Let K, C be as in the last item. Then $K + C$ is closed in $\mathbb{R}^n$. Details!

This result need not be true if the sets are assumed to be closed. Consider $A := \{(x, 0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$ and $B := \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0 \text{ and } xy = 1\}$. Then $A + B = \{(x, y) \in \mathbb{R}^2 : y > 0\}$.

This can be seen geometrically. Can also be proved rigorously! Details!

(g) Let X, Y be compact metric spaces. Then $X \times Y$ is compact.

An obvious line of attack needs a careful argument. Observe that 4th characterization applied to the sequences (x_n) and (y_n) may produce subsequences of the form (x_{2n}) and (y_{2n-1}) converging to x and y respectively. This will not help us to produce a convergent subsequence of $(x_n.y_n)$!

If $((x_n, y_n))$ is a sequence in $X \times Y$, by compactness of X, there exists a subsequence (x_{n_k}) which converges to some $x \in X$. Now consider the sequence (y_{n_k}) in the compact metric space Y. Assume a subsequence $(y_{n_{k_r}})$ converges to $y \in Y$. Then the subsequence $(x_{n_{k_r}}, y_{n_{k_r}})$ converges to (x, y) by Item 33b.

(h) Let X denote the normed linear space of all bounded real valued functions on $[0, 1]$ under the sup norm $\| \ \|_\infty$. Then the closed unit ball in X is closed and bounded but not compact.

Recall Item 28b and consider the sequence (x^n) in $C[0, 1]$.

If this has a convergent subsequence, say (x^{n_k}), then the convergence is uniform convergence (by Item 28b) and its limit is continuous. But the pointwise limit of the original sequence is the discontinuous function: $f(x) = 0$ for $0 \le x < 1$ and $f(1) = 1$. Details!

9 Connected and Path-connected Spaces

9.1 **Connected Spaces.** Look at

(a) $\mathbb{R}$, an interval,

(b) a circle, a parabola, an ellipse, two intersecting lines, a disk, a circle, a parabola or an ellipse along with a tangent line at one of its points in $\mathbb{R}^2$,

(c) a plane, a sphere, a ball in $\mathbb{R}^3$.

All of them seem to be in a "single piece." Consider now

(a) $\{-1, 1\}$, $\mathbb{Z}$, $(-1, 0) \cup (0, 1)$ in $\mathbb{R}$,

(b) two (distinct) parallel lines, a hyperbola, two disjoint open disks in $\mathbb{R}^2$,

(c) two distinct parallel planes, the set consisting of the unit ball $B(0, 1)$ along with the plane $x = 2$.

All of these seem to have more than one piece.

9.2 A topological space X is said to be *connected* if the only subsets of X which are both open and closed are $\emptyset$ and X. If there exists a subset $\emptyset \neq A \neq X$ which is both open and closed, then the space is said to be *disconnected* or not connected.

Clearly, connnectedness is a topological property.

We say that a subset A of a topological space X is connected (or a connected subset of X), if A is a connected space with the subspace topology.

9.3 If X is not connected, say $\emptyset \neq A \neq X$ is both open and closed, then $B := X \setminus A$ is such that $\emptyset \neq B \neq X$ and it is both open and closed. Hence, X is disconnected iff there exist .... (Complete this sentence.) Thus X has two "pieces" A and B!

One usually calls A or the pair (A, B) as a disconnection of X.

9.4 A topological space X is connected iff it has the following property: If U and V are nonempty open sets such that $X = U \cup V$, then $U \cap V \neq \emptyset$.

9.5 A subset A is connected iff the following condition is satisfied: If U and V are open subsets of X such that $U \cap A$ and $V \cap A$ are nonempty and $A \subset U \cup V$, then $U \cap V \cap A \neq \emptyset$.

9.6 We now give some examples. (More examples will follow once we prove a powerful characterization of connected spaces. See Items 7–8.)

(a) $\mathbb{R}$ is connected. See Item 35i. Similar proof shows that any interval is connected.

(b) $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are not connected. See Item 13e.

(c) Any discrete space with more than one element is disconnected.

(d) Any indiscrete space is connected.

(e) Is the empty set connected?

9.7 The following theorem is a powerful characterization of connected spaces. The theorem remain true if we take Z to be any discrete space with at least two elements, for instance, $\mathbb{Z} \subset \mathbb{R}$ with the subspace topology.

Theorem 15. *Consider $Z := \{\pm 1\} \subset \mathbb{R}$ with subspace topology. A topological space is connected iff any continuous map $f: X \to Z$ is a constant.* □

Let X be connected. Let $f: X \to Z$ be continuous. If $f(a) = 1$ and $f(b) = -1$, let $A := f^{-1}(\{1\})$ and $B := f^{-1}(\{-1\})$. Then A and B are non-empty, open, disjoint, with $X = A \cup B$. Hence X is not connected.

Conversely, if X is not connected, let (A, B) be a disconnection of X. Define $f = 1$ on A and $f = -1$ on B. If V is an open set in $\{\pm 1\}$, then $V = \emptyset$, $V = \{\pm 1\}$, $V = \{1\}$ or $V = \{-1\}$. Their inverse images are $\emptyset$, X, A and B respectively. Hence f is a continuous function from X onto $\{\pm 1\}$.

When we use this result to deal with connectedness of subsets in a topological space, we shall make use of Items 18–19.

9.8 Applications of the last theorem.

(a) Any interval is connected. Use intermediate value theorem.

(b) A subset of $\mathbb{R}$ is connected iff it is an interval. As one can give a direct proof of this, we have the intermediate value theorem as a corollary.

(c) Let $M(n, \mathbb{R})$ denote the set of all $n \times n$ matrices of real numbers. Then $GL(n, \mathbb{R}) := \{A \in M(n, \mathbb{R}) : \det(A) \neq 0\}$ is not connected.

(d) $O(n, \mathbb{R}) := \{A \in GL(n, \mathbb{R}) : AA^t = I\}$ is not connected.

(e) Let X be a topological space. Let A and B be two connected subsets of X such that $A \cap B \neq \emptyset$. Then $A \cup B$ is connected. Generalize this.

(f) Let X be a connected topological space and $g: X \to Y$ be a continuous map. Then $g(X)$ is connected. Applications:

   i. Any line segment in an normed linear space is connected.

   ii. The circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected. Similarly, the ellipse and parabola are connected.

   iii. $SO(2, \mathbb{R}) := \{A \in O(2, \mathbb{R}) : \det A = 1\}$ is connected.

   iv. $GL(n, \mathbb{R})$ is not connected.

   v. $O(n, \mathbb{R})$ is not connected.

(g) Let X be such that every pair of points of X lies in a connected subset. Then X is connected.

   In fact, we can weaken the hypothesis. Let $p \in X$ be fixed. Assume that for any $x \in X$, there exists a connected subset $A_x \subset X$ such that $p, x \in A_x$. Then X is connected.

   Applications:

   i. Any star-shaped subset of a normed linear space is connected. A subset E of a normed linear space X is said to be star-shaped at $p \in E$ if for any $q \in E$, the line segment $[p, q] := \{(1 - t)p + tq : 0 \leq t \leq 1\} \subset E$.

   ii. A subset C of a normed linear space is said to be convex if it is star-shaped at each of its points. Hence a convex set in a normed linear space is connected.

iii. It is easy to see that a ball $B(a, r)$ in a normed linear space is convex. If $x, y \in B(a, r)$, we have, for $t \in [0, 1]$,

$$\begin{aligned} d((1-t)x + ty, a) &= \|(1-t)(x-a) + t(y-a)\| \\ &\leq (1-t)\|x-a\| + t\|y-a\| \\ &< (1-t)r + tr = r. \end{aligned}$$

Consequently, any ball (open or closed) in a normed linear space is connected.

iv. $\mathbb{R}^2 \setminus \{0\}$ is connected.

v. $\mathbb{R}^2 \setminus \{(n, 0) : n \in \mathbb{Z}\}$ is connected.

vi. $\mathbb{R}$ with the lower limit topology $\mathcal{T}_L$ is not connected.
For, the basic open set $[a, b)$ is both open and closed. Its complement is $(-\infty, a) \cup [b, \infty)$. The set $(-\infty, a)$ is open in the standard topology of $\mathbb{R}$ and since $\mathcal{T}_L$ is finer than the standard topology, it is open in the lower limit topology. Alternatively, if $x \in (-\infty, a)$, then $x < a$ and hence x belongs to the basic open set $[x, a) \subset (-\infty, a)$.
Given any $x \in [b, \infty)$, the basic open set $[x, x+1) \subset [b, \infty)$. Hence the set $[b, \infty)$ is open in $\mathcal{T}_L$. Thus the complement of $[a, b)$ is the union of two $\mathcal{T}_L$ open sets and hence is open in $\mathcal{T}_L$.

(h) Let A be a connected subset of a space X. Let $A \subset B \subset \overline{A}$. Then B is connected.
Let $f : B \to \{\pm 1\}$ be continuous. Restricted to A, f is a constant, say 1. Let $x \in B$. We show that $f(x) = 1$. Let, if possible, $f(x) = -1$. Then there exists a set $U_x \ni x$, open in B, such that $f(U_x) \subset (-3/2, -1/2)$. Since x is a limit point of A, we can find $a \in U_x \cap A$. But then $f(a) = 1 \notin (-3/2, -1/2)$.
Application:
   - Consider the set $L := \{(t, 0) : t \in [0, 1]\}$, $A_n := \{(1/n, y) : y \in [0, 1]\}$ for $n \in \mathbb{N}$ and $A_0 := \{(0, y) : y \in [0, 1]\}$. Then $E := L \cup (\cup_n A_n)$ is connected and so its closure, $E \cup A_0$ is connected. Hence the set $E \cup \{(0, 1)\}$ is connected. ($X := E \cup A_0$ is known as the comb space.)

(i) Let X be the union of open disk in $\mathbb{R}^2$ along with the tangent line $x = 1$. It is connected.

(j) The open unit disk in $\mathbb{R}^2$ along with any subset of its boundary is connected. (This is geometrically 'obvious'.)

(k) Let $\{A_i : i \in I\}$ be a collection of connected subsets of a space X with the property that for all $i, j \in I$ we have $A_i \cap A_j \neq \emptyset$. Then $A := \cup_i A_i$ is connected.
Applications:
Any star-shaped subset of a normed linear space is connected. In particular, we have
   - Any convex subset of a normed linear space is connected.
   - Any open/closed ball in any normed linear space is connected.
   - Any vector subspace in a normed linear space (in particular $\mathbb{R}^n$) is connected.
   - Any coset of a vector subspace in a normed linear space (or $\mathbb{R}^n$) is connected.

(l) Let X be a topological space. Assume that $\{A_i : i \in I\}$ is a family of connected subsets of X. Let L be another connected subset such that $L \cap A_i \neq \emptyset$ for all $i \in I$. Show that $L \cup (\cup_{i \in I} A_i)$ is a connected subset of X.

(m) Let X and Y be topological spaces. Then the product space $X \times Y$ is connected iff both X and Y are connected.

Let $f\colon X \times Y \to \{\pm 1\}$ be a continuous function. Fix $(a,b) \in X \times Y$. We show that for any $(x,y) \in X \times Y$, we have $f(x,y) = f(a,b)$. Since $\{a\} \times Y$ is connected (why?), the restriction of f to this set is a constant. In particular, $f(a,y) = f(a,b)$. Now the subset $X \times \{y\}$ is connected and hence the restriction of f to it is a constant. In particular, $f(a,y) = f(x,y)$. Hence $f(x,y) = f(a,b)$. Picture!

Applications:

i. $\mathbb{R}^2 \setminus \{(0,0)\}$ is connected as it is the product of $(0,\infty) \times [0, 2\pi)$.

ii. A cylinder $\{(x,y,z) : x^2 + y^2 = 1\}$ is the product of circle and $\mathbb{R}$ and hence is connected.

(n) The sphere $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is connected.

The case $n = 1$ is already seen. Assume $n > 1$.

Note that $S = S_+ \cup S_-$ where $S_\pm := \{x \in \mathbb{R}^{n+1} : \pm x_{n+1} > 0\}$, union of two closed hemi-spheres. The map $\varphi\colon S_- \to B[0,1]$ given by $\varphi(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$ is a bijective continuous map whose inverse is $\psi\colon B[0,1] \subset \mathbb{R}^n \to S_-$ given by $\psi(u) = \left(u, -\sqrt{1 - \sum u_j^2}\right)$. Hence it is homeomorphism and S_- is connected. Similarly, S_+ is connected. Now the intersection of these two hemi sphere is the equator $\{x_{n+1} = 0\}$. By Item 8e, the sphere is connected.

Alternatively, connectedness of S^n can also seen as follows. Since S^n is the image the polar coordinate map, S^n is connected. For instance, $\varphi\colon [-\pi/2, \pi/2] \times [0, 2\pi] \to S^2$ is given by $\varphi(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$.

A third way of seeing this is to observe that any point x other than the north pole e_{n+1} lies on a unique great circle and appeal to Item 8g.

Applications:

i. $\mathbb{R}^n \setminus \{0\}$ is connected, $n \geq 2$.

ii. A cylinder $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ is connected.

iii. An annular region $\{x \in \mathbb{R}^n : r < \|x\| < R\}$ is connected.

9.9 Union, intersection of two connected sets need not be connected. Consider $(-1,0) \cup (0,1)$, intersection of the closed upper semi-circle with the closed lower semi-circle.

Complement of a connected set need not be connected. Look at complement of a bounded interval in $\mathbb{R}$.

9.10 We can give a direct proof of Item 8(n)i. Draw pictures in $\mathbb{R}^2$ to understand the proof below. Let $x \in \mathbb{R}^n$ be nonzero. Let $P = \{x \in \mathbb{R}^n : x_n = 1\}$. Then P is homeomorphic to $\mathbb{R}^{n-1}$ and hence is connected (as $n > 1$). We show that any non-zero x lies on a line segment which meets P. Hence by Item 8l it will follow that the set of nonzero vectors in $\mathbb{R}^n$ $(n > 1)$ is connected.

Let $x \in \mathbb{R}^n$ be nonzero. If $x_j \neq 0$ for some $j < n$, then the line $(x_1, \dots, x_j, \dots, t)$ passes through x, does not contain 0 and it meets P.

If x_n is the only nonzero coordinate, the line joining x with $(1, 0, \dots, 0, 1)$ is given by $(1-t)(0, \dots, 0, x_n) + t(1, 0, \dots, 0, 1)$. It contains the given point, does not pass through origin and it meets P.

We now prove that the sphere is connected. The continuous map $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n$ given by $x \mapsto x/\|x\|$ has the sphere as its image.

9.11 A finite metric space is connected iff is a singleton.

9.12 Let X be connected and $f\colon X \to \mathbb{R}$ be a continuous non-constant function. Show that $f(X)$ is uncountable.

Since f is non-constant, there exist points $p, q \in X$ such that $f(p) = a < b = f(q)$. Since $f(X)$ is a connected subset of $\mathbb{R}$, it is an interval.

9.13 Let X be a connected metric space with at least two elements. There X "has at least as many elements as $\mathbb{R}$." In particular, X is uncountable.

9.14 What are all the continuous functions from $f\colon \mathbb{R} \to \mathbb{R}$ that take only rational values?

9.15 Are there continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ that take irrational values at rational numbers and rational values at irrational numbers?

9.16 Let $f\colon [a,b] \to \mathbb{R}$ be continuous. "Identify" the image $f([a,b])$.

9.17 Let f be a one-one continuous function on an interval. Then f is monotone.

9.18 What are all the continuous functions from a connected space to (i) a discrete space, (ii) a finite Hausdorff space?

9.19 Let $f\colon X \to Y$ be a continuous map from a connected space X **onto** a finite Hausdorff space. What can you conclude about Y?

9.20 Let X and Y be topological spaces and $f\colon X \to Y$ be a map. We say that f is *locally constant* if for each $x \in X$, there exists an open set U_x containing x with the property that f is a constant on U_x.

Show that any locally constant function is continuous.

9.21 Let $U \subset \mathbb{R}^n$ be a nonempty open set. Let $f\colon U \to \mathbb{R}$ be a differentiable function with derivative 0. Then f is locally constant. (It need NOT be a constant function!)

9.22 Let X be connected and Y be any space. Then any locally constant function $f\colon X \to Y$ is a constant function on X.

Fix $p \in X$. We show that $f(x) = f(p)$ for $x \in X$. Define a subset $A := \{x \in X : f(x) = f(p)\}$. Then $p \in A$. Hence p is non-empty. We shall show that A is both open and closed. Since X is connected, it will follow that $A = X$.

Let $x \in A$. Since f is locally constant, there exists an open set $U \ni x$ on which f is a constant. Hence for any $z \in U$, $f(z) = f(x) = f(p)$. That is, $U \subset A$. Hence A is open.

Let q be a limit point of A. Let $U_q \ni q$ be an open set on which f is a constant. Since q is a limit point of A, there exists $a \in A \cap U_q$. Hence $f(z) = f(a)$ for all $z \in U_q$, in particular, $f(q) = f(a) = f(p)$. Hence $q \in A$, that is, A is closed. Hence $A = X$.

This is a typical way in which connectedness hypothesis is used. If a result has connectedness as hypothesis, define a set which reflects what we want to prove and show that the set so-defined is non-empty, open and closed. Learn this proof well. For another example, refer to Item 39.

9.23 The converse of the last item is true. So we have the following characterization of connected spaces.

A topological space X is connected iff any locally constant function from X to any space Y is a constant.

9.24 In Item 21, if we further assume that U is connected, then f is a constant.

This follows from Items 21–22.

9.25 **Connected Components.** In a topological space X, the relation $x \sim y$ if there exists a connected set A with $x, y \in A$ is an equivalence relation.

Reflexivity: $x \sim x$ as $x \in \{x\}$.
Symmetry: If $x, y \in C$, a connected set, then $y, x \in C$.
Transitivity: Let $x, y \in C$ and $y, z \in D$ where C and D are connected. Since $y \in C \cap D$, $C \cup D$ is connected and $x, z \in C \cup D$. That is, $x \sim z$.

The equivalence classes are called the *connected components* or components of X. The following are immediate:

(a) If C is a component, then C is a connected set.
Let $C = [x]$, the equivalence class of x. We claim that any two points $y, z \in C$ lie in a connected set. It follows from Item ????that C is connected. Since $y, z \in C$, we have $x, y \in A$ and $y, z \in B$ with A and B connected. As earlier, $A \cup B$ is a connected set with $y, z \in A \cup B$. Give Ref!

(b) Any component C is a maximal connected set in the sense that if A is connected and $C \subset A$, then $C = A$.
Let $a \in A$ and let $C = [x]$. Since $a, x \in A$, a connected set, it follows that $a \sim x$ and hence $a \in [x] = C$. Thus $A \subset C$.

(c) Any connected component is closed.
If C is a connected component, then $\overline{C}$ is also connected set with $C \subset \overline{C}$. Hence $C = \overline{C}$ by the 'maximality.'

(d) If C is a component, $x \in C$ and if A is a connected set with $x \in A$, then $A \subset C$.
Easy. $x \in C \cap A$ and hence $C \subset C \cup A$, a connected set. By maximality, we have $C \cup A = C$, that is, $a \subset C$.

9.26 Examples of components:

(a) The only component of a connected space X is X.
(b) The components of a discrete space are the singleton sets.
(c) The components of $\mathbb{Q}$ are the singleton sets. (Note that the topology on $\mathbb{Q}$ is not discrete topology. We gave two proofs of this. One is direct use of subspace topology and another used existence of non trivial convergent sequences.) Details!
(d) What are the components of $\mathbb{R}$ with VIP topology? with outcast topology?

9.27 If $f \colon X \to Y$ is a homeomorphism, then f induces a natural bijective correspondence between the components of X and those of Y: If C is a component of X, then $f(C)$ is a component of Y. Application: The pair of intersecting lines is not homeomorphic to $\mathbb{R}$. (If they are, remove the point of intersection from the pair of lines and its image from $\mathbb{R}$. Count the components.)

9.28 Connectedness can be used to settle questions on homeomorphisms:

(a) The set of irrational numbers in $\mathbb{R}$ with subspace topology is not homeomorphic to $\mathbb{R}$.

(b) A hyperbola cannot be homeomorphic to $\mathbb{R}$.

(c) $\mathbb{R}$ cannot be homeomorphic to $\mathbb{R}^2$.

(d) A pair of intersecting lines cannot be homomorphic to a parabola.

(e) The set A of two distinct parallel lines in $\mathbb{R}^2$ is not connected. Hence a pair of intersecting lines cannot be homomorphic to A.

9.29 **Path-connected Spaces.** A continuous map $\alpha\colon [a, b] \to X$ to a topological space X is called a *path*. Since any two intervals are homeomorphic, it is a standard practice to assume that $a = 0$ and $b = 1$. The point $p := \alpha(0)$ is called the initial point and $q := \alpha(1)$ is called the terminal point of the path α. We also say that p is path connected to q by the path α.

Note that if p is connected to q by a path to $\alpha\colon [0, 1] \to X$ with $\alpha(0) = p$ and $\alpha(1) = q$, then the *reverse* path $\tilde{\alpha}\colon [0, 1] \to X$ defined by $\tilde{\alpha}(t) := \alpha(1 - t)$ is a path from q to p.

Thus if p is connected to q by a path iff q is connected to p by a path. Because of this we simply say that p and q are path-connected, without specifying which is the initial point etc.

9.30 It is important not to identify the path α with its image $\alpha([0, 1])$ in X. (It is called the trace of α. Mnemonic: the trains could be different but the tracks may be the same.) The paths $\alpha, \beta\colon [0, 1] \to \mathbb{R}^2$ given by $\alpha(t) = (t, 0)$ and $\beta(t) = (t^3, 0)$ have the same trace.

9.31 Two point p and q may be connected by more than one path. Think of at least 3 different paths connecting $(-1, 0)$ to $(0, 1)$ in $\mathbb{R}^2$.

9.32 If x and y are path-connected and y and z are path-connected in a space, then x and z are path connected.

This is an application of gluing lemma. Assume that $\alpha\colon [0, 1] \to X$ connects x to y and $\beta\colon [0, 1] \to X$ connects y to z. Then the map $\gamma\colon [0, 1] \to X$ defined as

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{if } 0 \le t \le 1/2 \\ \beta(2t - 1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

Since $\alpha(1) = y = \beta(0)$, we can apply gluing lemma to conclude that γ is path connecting x to z.

9.33 We say that a topological space X is *path-connected* if any two points of X are connected by a path.

9.34 X is path connected iff there exists $p \in X$ such that any point $x \in X$ is path connected to p.

9.35 Any path connected space is connected.

Let X be path connected. Let $f: X \to \{\pm 1\}$ be continuous. Let $p, q \in X$ and $\alpha: [0,1] \to X$ be a path joining p to q. Now $f \circ \alpha: [0,1] \to \{\pm 1\}$ is continuous and hence is a constant. In particular, $f(p) = f(\alpha(0)) = f(\alpha(1)) = f(q)$, Hence f is a constant.

Or, observe that any two points lie on the trace of a path, which is connected. Hence, by Item 8g, X is connected.

9.36 The converse is not true. Two examples:

(a) Comb space: Let $L := \{(x,0) : 0 \leq x \leq 1\}$ and $A_n := \{(1/n, y) : 0 \leq y \leq 1\}$, for $n \in \mathbb{N}$. Let $P = \{(0,1)\}$. Then $L \cup (\cup_{n\in\mathbb{N}} A_n)$ is connected and its closure contains $X := L \cup (\cup_{n\in\mathbb{N}} A_n) \cup \{P\}$. Hence X is connected. It is not path connected. If possible, let γ be path joining P to $Q = (1,0) \in X$. Choose an open disk $B(P,r)$ which does not meet the x-axis. Let $[0,\delta)$ be such that $\gamma(t) \in B(P,r)$ for $t \in [0,\delta)$. Let $\gamma = (\gamma_1, \gamma_2)$. Then $\gamma_1([0,\delta))$ is a connected subset of $B(P,r) \cap X$. It follows that $\gamma_1(t) = 0$ for $t \in [0,\delta)$. Hence $\gamma_1(\delta) = 0$. We Let $t_0 := \sup\{t \in [0,1] : \gamma_1(t) = 0\}$. Then $\gamma_1(t_0) = 0$. We claim that $t_0 = 1$. If not, repeat argument using the continuity of γ at t_0. We then get there exists $s > t_0$ such that $\gamma_1(s) = 0$. Thus we conclude that $\gamma(t) = P$ for $t \in [0,1]$.

9.37 The continuous image of a path connected space is path connected.

Let $f: X \to Y$ be continuous with X path connected. Let $Y = f(X)$. Given $y_j = f(x_j) \in Y$, $j = 1, 2$. Let γ be a path connecting x_1 to x_2. Then $f \circ \gamma$ is a path connecting y_1 to y_2.

An application. The proof in Item 10 showed that $\mathbb{R}^n \setminus \{0\}$, $n \geq 2$, is path connected. Hence S^n, being a continuous image of $\mathbb{R}^n \setminus \{0\}$ is also path connected.

9.38 The product space of path connected spaces is path connected.

Let $x = (x_i), y = (y_i) \in \prod_i X_i$. Let γ_i be a path connecting x_i to y_i in the space X_i. Define $\gamma(t) := (\gamma_i(t))$. Then γ is a path connecting x to y .

An application. The third proof of the connectedness of S^n in Item 8n established its path connectedness. Hence $\mathbb{R}^n \setminus \{0\} = (0,\infty) \times S^{n-1}$ is path connected.

9.39 Any open subset of a normed linear space is connected iff it is path connected.

Let A be a connected open subset of a normed linear space X. Fix $p \in A$. It suffices to show that there is a path connecting p to any $q \in A$. (See Item 34.) Let

$$E := \{x \in A : x \text{ is path-connected to } p\}.$$

From here onwards, the proof is exactly similar to the one in Item 22.

Clearly, $p \in E$ and hence $E \neq \emptyset$. Let $x \in E$. Then there exists an open ball $B(x,r) \subset A$, since A is open. Now any $z \in B(x,r)$ is connected to x via the line segment $t \mapsto (1-t)z + tx$. Since $x \in E$, x is path connected to p. Hence by Item 32, z is path-connected to p and hence $B(x,r) \subset E$. We conclude that E is open.

Let $q \in A$ be a limit point of E **in** A. As earlier, there exists $B(q,r) \subset A$. Since q is a limit point of E, there exists $z \in B(q,r) \cap E$. Now, q is path connected by the line segment $(1-t)q + tz$ to z which in turn is path connected to p, as $z \in E$. Hence q is path connected to p, or $q \in E$. Hence E is closed.

9.40 Path components are defined in an obvious way. If C_x (resp. P_x) is the component (resp. path-component) containing $x \in X$, then $P_x \subseteq C_x$.

9.41 Going through the proof in Item 39, we are led to the concept of locally path connected spaces. First of all a definition.

10 Locally P-Spaces

10.1 Let X be a topological space and $x \in X$. A subset U is called a *neighbourhood* of x in X if there exists an open set G such that $x \in G \subset U$. Example: $[0, 1)$ is a neighbourhood of any $x \in (0, 1)$ but not of $x = 0$.

10.2 A set in a topological space is open iff it is a neighbourhood of each of its points.

10.3

Locally P spaces

General Philosophy: Let P be a topological property. We say that a space X is *locally* P (or enjoys P locally) if for each $x \in X$ and an open set $U \ni x$, there exists a neighbourhood N of x where N has the property P and $N \subset U$.

10.4 Let X be a topological space. Then X is said to be *locally path connected* if for each $x \in X$ and an open set $U \ni x$, there exists a path connected neighbourhood N of x such that $N \subset U$.

Now you can similarly define *locally connected* and *locally compact* spaces.

Do you see the need for introducing the notion of neighbourhoods? If we replace a neighbourhood by an open set in the locally P spaces, what will happen if we wanted a Hausdorff space to be locally compact?

10.5 The proof of Item 9.39 yields the following result: An open set in a locally path connected space is connected iff it is path-connected.

10.6 An important remark: In general X may have property P but it may not be locally P. For instance, the complete comb space is connected but not locally connected. (Look for a connected neighbourhood of the point $(0, 1)$.) Similarly, there exists a compact space (Item 10.17c) which is not locally compact. (Do NOT get confused with the 'bad' definition of Munkres and hence his "note" that any compact space is locally compact!)

Similarly, the space X may be locally P, but X may not enjoy P. For instance, consider $\mathbb{R}$ with discrete topology. Then it is locally connected, locally path-connected and locally compact. But it is not connected, not path connected and not compact.

10.7 A space X is locally connected iff the components of any open subset (with subspace topology) are open in X. In particular, the components of X are open. Details!

10.8 The components in a locally path connected space are open.

10.9 Let U be an open subset of a locally path connected space. Then U is connected iff it is path-connected.

10.10 In a locally path connected space, the components and path components are the same.

10.11 Can we define locally $\mathbb{R}^n$ or locally Euclidean spaces?

We say that a space X is *locally Euclidean* or locally $\mathbb{R}^n$ if for each $x \in X$, there exists a neighbourhood $U_x \ni x$ which is homeomorphic to a neighbourhood in $\mathbb{R}^n$. (Note that n is fixed.)

10.12 Can we define locally Hausdorff spaces? Is it necessarily Hausdorff?

Consider $X = \mathbb{R}^* \cup \{\theta_1, \theta_2\}$ where θ_j are two elements not in $\mathbb{R}^*$. (We shall think of them as "two zeros" or "the zero with split personality!") As a local basis for $x \in \mathbb{R}^*$, we take $\{(x - 1/k, x + 1/k) : k \in \mathbb{N}\}$. At θ_j, we take $\{(-1/k, 0) \cup \{\theta_j\} \cup (0, 1/k) : k \in \mathbb{N}\}$. Then we get a topological space which is locally Euclidean and hence it is locally Hausdorff. However, it is not Hausdorff.

10.13 Locally Compact Spaces. We say that X is locally compact if for each $x \in X$ and an open set $U \ni x$, there exists a compact neighbourhood $K \ni x$ such that $x \in K \subset U$.

Note that this is seemingly stronger definition than the one found in Munkres but is equivalent to it. See Theorem 16.

10.14 The following are descendants of Item 8.8.

(a) Let K be a compact subset of a Hausdorff space X and $x \notin K$. Then there exist disjoint open sets U and V such that $x \in U$ and $K \subset V$. (This is Item 9.)

(b) Let A and B be disjoint compact subsets of a Hausdorff space. Then there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

(c) Let X be a compact Hausdorff space. Let A and B be disjoint closed subsets of X. Then there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

10.15 A space X is said to be *normal* if given two disjoint closed sets A and B, there exist disjoint open sets $U \supset A$ and $V \supset B$.

Last item shows that a compact Hausdorff space is normal.

10.16 Another example of a normal space is any metric space.

To appreciate this, look at $A = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$, the set of axes which are asymptotes of the rectangular hyperbola $B := \{(x, y) \in \mathbb{R}^2 : xy = 1\}$.

We now prove the result. If $a \in A$, then A is a not a limit point of B. Hence $d(a, B) > 0$, by Item 3.7. There exists $r_a > 0$ such that $B(a, r_a) \cap B = \emptyset$. $U := \cup_{a \in A} B(a, r_a)$. We can do similarly for B to get a V. Do they intersect? If $z \in U \cap V$, then $z \in B(a, r_a) \cap B(b, r_b)$ for some $a \in A$ and $b \in B$. It follows that $d(a, b) \leq r_a + r_b \leq 2 \max\{r_a, r_b\}$, no contradiction! If we replaced r_x by $r_x/2$ in forming U and V, we get a contradiction! Go ahead and work out the details. (Note that we have employed a similar trick in Theorem 12 and Item 8.22f.)

10.17 Examples of locally compact spaces:

(a) $\mathbb{R}$, $\mathbb{R}^n$ are locally compact.

(b) $\mathbb{Q}$ is not locally compact.

(c) A compact space need not be locally compact. Example: Consider $\mathbb{Q}$ with the usual topology, adjoin an extra element, say ∞. The neighbourhoods of $x \in \mathbb{Q}$ are either the neighbourhoods of x in $\mathbb{Q}$ or ∞ added to the standard neighbourhoods. The neighbourhoods of ∞ are complements in $\mathbb{Q}$ of a finite subset of F along with ∞.

(d) An normed linear space is locally compact iff it is finite dimensional. (One way is easy; the proof of the other is omitted.)

(e) A locally compact metric space need not be complete. A trivial example is $(0,1)$!

Theorem 16. . *The following are equivalent for a Hausdorff space:*

1. *X is locally compact.*
2. *For every $x \in X$ and a neighbourhood U of x, there exists an open set V such that $x \in V$, $\overline{V}$ is compact and $\overline{V} \subset U$.*
3. *Each $x \in X$ has a compact neighbourhood.*

Proof. (1) $\implies$ (2): Let $x \in X$ and K be a compact neighbourhood of x. Then there exists an open set $V \subset K$ with $x \in V$. Since X is Hausdorff, K is closed. Hence $\overline{V} \subset K$. Hence $\overline{V}$ being a closed subset of a compact set K, is compact.

(2) $\implies$ (3): Take $\overline{V}$ of (2).

(3) $\implies$ (1): □ Details!

Since locally compact spaces such as $\mathbb{R}^n$ arise quite often, whenever we say X is locally compact, we shall assume that X is Hausdorff also.

10.18 Local compactness is a topological property. In fact, more is true: Let $f\colon X \to Y$ be a continuous open map of a locally compact space X onto Y. Then Y is locally compact.

Let $y \in Y$ and $V \ni y$ be open. Let $x \in X$ be such that $f(x) = y$. Then $U := f^{-1}(V)$ is an open set with $x \in U$. Since X is locally compact, there exists an open set $W \ni x$ with $\overline{W}$ compact and $W \subset U$. Since f is open $f(W) \ni y$ is open, since $\overline{W}$ is compact, $f(\overline{W})$ is compact. Thus, $f(\overline{W})$ is a compact neighbourhood of y.

10.19 A closed (respectively open) subspace of a locally compact space is locally compact.

10.20 A Hausdorff topological space X is called an *n-dimensional topological manifold* if for each $p \in X$, we can find an open set $U_p \ni p$ such that U_p is homeomorphic to an open subset of $\mathbb{R}^n$ for n fixed. Thus, a manifold is a Hausdorff space which is locally Euclidean.

Typical examples are (i) open subset of $\mathbb{R}^n$ and (ii) $S^n \subset \mathbb{R}^{n+1}$. A non-example is a pair of intersecting lines in $\mathbb{R}^2$. Modern topology deals mostly with manifolds.

10.21 An interesting example of a topological manifold is $\mathbb{P}^n(\mathbb{R})$. Give Ref!

To see this, let $U_i := \{[(x_1, x_2, , \dots, x_{n+1})] : x_i \neq 0\}$, $1 \leq i \leq n+1$. Note that U_i is "well-defined". It is open in $\mathbb{P}^n(\mathbb{R})$. Also, $\cup_i U_i = \mathbb{P}^n(\mathbb{R})$. The maps $\varphi_i\colon U_i \to \mathbb{R}^n$ given by $[(x_1, \dots, x_n)] \mapsto (x_1/x_i, \dots x_{i-1}/x_i, x_{i+1}/x_i, \dots, x_{n+1}/x_i)$ are well-defined and homeomorphisms.

10.22 The following class of examples arise in Differential Geometry and Differential topology. Let $f\colon \mathbb{R}^{n+k} \to \mathbb{R}^k$ be a continuously differentiable map. Assume that $M := f^{-1}(0) \neq$. We also assume that for each $p \in M$, the derivative $Df(p)\colon \mathbb{R}^{n+k} \to \mathbb{R}^k$ is of rank k. Let us write $p = (a,b) \in \mathbb{R}^n \times \mathbb{R}^k$. Then the implicit function theorem says that for each $p \in M$, there exist open sets $W \ni p$ and $U \ni a$ and a continuously differentiable function $h\colon U \to \mathbb{R}^k$ such that

$$U \cap M = \text{Graph}(h) := \{(x, h(x)) : x \in U\}.$$

It follows from Item 7.34b on Page 51 that M is an n-dimensional manifold.

10.23 Any topological manifold is locally path-connected and locally compact.

11 One Point Compactification

11.1 Given $X = (0,1] \subset \mathbb{R}$, by adding just the point 0, we can make it to be compact. Note that $(0,1]$ is dense in $[0,1]$.

Similarly, the subspace topology on the set $\{1/n : n \in \mathbb{N}\} \subset \mathbb{R}$ is discrete. If we add the point 0 to it, then the resulting space is compact in which the original set is dense.

Can we so something similar to any locally compact, non-compact Hausdorff space X?

That is, can we add a new point, which is denoted by ∞ to X and obtain a compact Hausdorff space? Let us work backwards. Assume $X_\infty := X \cup \{\infty\}$ is compact Hausdorff. We would like to retain open subset of X in tact. So we need to provide a local base at ∞. If $U \ni \infty$ is an open set , then $X_\infty \setminus U$ is a closed subset of the compact space X_∞ and hence is compact. But, it is in fact a subset of X. This suggests a way of defining a local base at ∞, namely, a subset $U \ni \infty$ is open if $X_\infty \setminus U$ is a compact subset of X.

11.2 **One point compactification.** Given a locally compact noncompact Hausdorff space X, let $X_\infty := X \cup \{\infty\}$ where $\infty \notin X$. Let $\mathcal{T}$ denote the topology on X. Consider

$$\mathcal{T}_\infty := \mathcal{T} \cup \{V \subset X_\infty : X_\infty \setminus V \text{ is a compact subset of } X.\}.$$

Then Details!

(i) $\mathcal{T}_\infty$ is a Hausdorff topology on X_∞.
(ii) The subspace topology on X is $\mathcal{T}$.
(iii) $(X_\infty, \mathcal{T}_\infty)$ is compact.
(iv) X is dense in X_∞.

11.3 Let X be noncompact, locally compact Hausdorff space. Let Y be a compact Hausdorff space. Assume that there exists $q \in Y$ and a homeomorphism $f : X \to Y \setminus \{q\}$. Then the one point compactification X_∞ of X is homeomorphic to Y. Details!

Extend f to $g : X_\infty \to Y$ by setting $g(\infty) = q$. Since g is a bijection of the compact space onto the Hausdorff space, it suffices to show that g is continuous at ∞. Let $V \ni q = f(\infty)$ be open. Then $L := Y \setminus V$ is closed in Y and hence compact subset of Y. Since f is a homeomorphism, $K := f^{-1}(L)$ is a compact subset of X. If we let $U := \{\infty\} \cup X \setminus K$, then $g(U) = V$. This establishes the continuity of g at ∞.

11.4 Examples:

(a) Consider $f : (0, 2\pi) \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$ be defined by $f(t) = e^{it}$. Then f is a homeomorphism of $(0, 2\pi)$ onto $S^1 \setminus \{1\}$. Hence the one point compactification of $(0, 2\pi)$ is S^1. Since homeomorphic (non-compact, locally compact Hausdorff) spaces have homeomorphic one-point compactifications, it follows that the one-point compactifications of (a, b) and $\mathbb{R}$ are S^1. More generally, we have the next item.

(b) We claim that the one point compactification of $\mathbb{R}^n$ is (homeomorphic to) S^n. Details!
Recall that we have shown (?!) that $S^n \setminus \{e_{n+1}\}$ is homeomorphic to $\mathbb{R}^n$ in Item 7.56o

(c) Let $x : \mathbb{N} \to X$ be a sequence in X. Then $x_n \to x_\infty$ iff the function $x : \mathbb{N}_\infty \to X$ defined by $x(n) = x_n$ and $x(\infty) = x_\infty$ is continuous at ∞.
Application: Use this to give another solution of Item 33b.

(d) Let X be an infinite discrete space. What is its one point compactification? Details!

11.5 Functions vanishing at infinity: Let X be a locally compact Hausdorff space. A continuous function $f\colon X \to \mathbb{R}$ is said to *vanish at infinity* if for any given $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ for $x \notin K$. (We can also define continuous function vanishing at ∞ for functions taking values in a normed linear space in an obvious way.)

A continuous function $f\colon X \to \mathbb{R}$ vanishes at infinity iff it extends to a continuous function $f_\infty\colon X_\infty \to \mathbb{R}$ with $f_\infty(\infty) = 0$.

(a) Let $f\colon X \to \mathbb{R}$ be given. Its *support* is by definition the **closure** of the set $\{x \in X : f(x) \neq 0\}$, that is,

$$\operatorname{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}.$$

We say that f has compact support if the support of f is compact. Evidently, any continuous function with compact support vanishes at infinity.

(b) What are the entire functions $f\colon \mathbb{C} \to \mathbb{C}$ which vanish at infinity?

11.6 A closely related concept is proper maps between (locally compact Hausdorff) spaces. See Appendix J.

This concept is so important that **any** proof of Fundamental theorem of algebra has to either directly or indirectly use the fact that the any non-constant polynomial with complex coefficients when considered as a map from $\mathbb{C}$ to $\mathbb{C}$ is proper.

12 Baire Category Theorems

12.1 A subset $A \subset X$ of a topological space is said to be *nowhere dense* in X, if given any nonempty open set U, we can find a nonempty open subset $V \subset U$ such that $A \cap V = \emptyset$.

This definition is equivalent to the standard one found in all text-books: A is nowhere dense in X iff the interior of the closure of A in X is empty: $\text{Int}\,(\overline{A}) = \emptyset$.

Let A be nowhere dense. We need to show that $\text{Int}\,(\overline{A}) = \emptyset$. Assume the contrary. Let $x \in \text{Int}\,(\overline{A}$. Then there exists an open set U such that $x \in U \subset \overline{A}$. Let V be an open set with $x \in V$. Let W be any open subset such that $x \in W \subset V$. Since $x \in \overline{A}$, x is a limit point of A and hence $W \cap A \neq \emptyset$. Thus A is not nowhere dense.

To prove the converse, let us assume that $\text{Int}\,(\overline{A}) =$. If A is not nowhere dense, then there exists a nonempty open set U such that for any nonempty open subset $V \subset U$, we have $V \cap A \neq \emptyset$. In particular, if $x \in U$, we claim that $x \in \text{Int}\,(\overline{A})$ and hence $U \subset \overline{A}$. If $V \ni x$ is an open set, then $x \in V \cap U$ is a nonempty open subset U and hence $(V \cap U) \cap A \neq \emptyset$. Thus $x \in \overline{A}$.

12.2 Prototype examples of nowhere dense sets:

(a) Let V be any proper vector subspace of $\mathbb{R}^n$. More generally, any proper vector subspace of a normed linear space .

(b) The set of zeros of any polynomial map $\mathbb{R}^n \to \mathbb{R}$.

12.3 **Baire Category theorem.** We shall give the formulation of Baire category theorem in a form which will be more useful than the one which uses the notion of category.

Theorem 17. *Let (X, d) be a complete metric space.*
(1) *Let U_n be open dense subsets of X, for $n \in \mathbb{N}$. Then $\cap_n U_n$ is dense. (In particular, $\cap_n U_n$ is non-empty.)*
(2) *X cannot be a countable union of nowhere dense closed subsets F_n.*

We first observe that both the statements are equivalent. For, G is open and dense iff its complement $F := X \setminus G$ is closed and nowhere dense. Hence any one of them follows from the other by taking complements. So, we confine ourselves to proving the first. In fact, we shall show that $\cap_n U_n$ is dense in X.

The basic idea is to get into a situation like nested interval theorem. Since we need to exploit completeness, we need to produce a Cauchy sequence whose limit is likely to be in the intersection of U_n's. If we have a nested sequence of open balls, say, $(B(x_n, r_n))$ such that $B(x_n, r_n) \subset B(x_{n-1}, r_{n-1})$, we get a sequence (x_n). If we wish to show that it is Cauchy, the only obvious estimate available (for $n > m$) is

$$d(x_n, x_m) \leq d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \leq \sum_{k=m}^{n} r_k.$$

Thus we are lead to make the sequence (r_n) of radii as the terms of a convergent series. The standard way of doing this is to demand $0 < r_n < 2^{-n}$.

We can also replace (U_n) by a nested sequence (V_n) of open dense sets. Define $V_1 := U_1$. Having defined V_n, define $V_{n+1} = U_{n+1} \cap V_n$. Clearly, V_1 is open dense. Assume that we have shown V_n is open dense. Let U be any nonempty open set. We need to show that $U \cap V_{n+1} \neq \emptyset$. Observe that

$$U \cap V_{n+1} = (U \cap V_n) \cap U_{n+1}.$$

Since by induction $U \cap V_n$ is nonempty open, it must have nonempty intersection with the dense set U_{n+1}. Thus we have produce a nested sequence (V_n) of open dense sets with $\cap_n V_n = \cap_n U_n$.

We now show that $\cap_n V_n$ is dense. Let $B(p,r)$ be an open ball. Since V_1 is dense, there exists $x_1 \in V_1 \cap B(p,r)$. Since the intersection is open, there exists a positive $r_1 < 1/2$ such that $B[x_1, r_1] \subset V_1 \cap B(p,r)$. Repeating the same argument with $B(x_1, r_1) \cap V_2$, we find x_2 and $0 < r_2 < 2^{-2}$ such that

$$B[x_2, r_2] \subset B(x_1, r_1) \cap V_2 \subset V_2 \cap V_1 \cap B(p,r) = V_2 \cap B(p,r).$$

By induction (What is the induction hypothesis?) we get sequences (x_n) and (r_n) such that Details!

$$B[x_n, r_n] \subset B(x_{n-1}, r_{n-1}) \cap V_n \subset V_n \cap B(p,r).$$

Clearly (x_n) is Cauchy. Since X is complete, (x_n) converges, say, to $x \in X$. Observe that $\{x_k : k \geq n\} \subset B[x_n, r_n]$. Hence x is a limit point of the closed ball $B[x_n, r_n]$ so that $x \in B[x_n, r_n]$. Since this is true for all n, we obtain $x \in B(p,r) \cap (\cap_n V_n)$. The theorem is proved. □

12.4 A most often used corollary of Baire's theorem is the following: If a complete metric space X can be written as a countable union of closed sets F_n, then at least one F_n will have a nonempty interior.

12.5 Applications:

(a) $\mathbb{R}^n$ cannot written as the union of a countable family of its proper vector subspaces. In particular, $\mathbb{R}^2$ is not the union of a countable family of lines through the origin.

(b) No infinite dimensional complete normed linear space can be countable dimensional. (Algebraic sense!)

(c) There can exist no metric d on $\mathbb{Q}$ such that d induces the usual topology on $\mathbb{Q}$ and $(\mathbb{Q}, d)$ is complete.

(d) Let (X,d) be complete and $f_n \colon X \to \mathbb{R}$ be a sequence of continuous functions. Assume that $f_n \to f$ pointwise on X. Then the set $A := \{x \in X : f \text{ is continuous at } x\}$ is dense in X.

Proof. Our proof is a beautiful application of both versions of Baire's theorem. Fix $\varepsilon > 0$. Define, for each $k \in \mathbb{N}$,

$$E_k(\varepsilon) := \{x \in X : |f_n(x) - f_m(x)| \leq \varepsilon, \text{ for all } m, n \geq k\}.$$

Then we claim that $E_k(\varepsilon)$ is closed for each k.

Reason: Fix $m, n \geq k$. Since $|f_n - f_m|$ is continuous, the set

$$E_k^{m,n}(\varepsilon) := \{x \in X : |f_n(x) - f_m(x)| \leq \varepsilon\}$$

is a closed subset of X. Now, since $E_k(\varepsilon) = \cap_{m,n \geq k} E_k^{m,n}(\varepsilon)$, the claim follows.

It is easy to show that $X = \cup_k E_k(\varepsilon)$.

Reason: Let $x_0 \in X$. Since $f_n(x_0) \to f(x_0)$, the sequence $(f_n(x_0))$ is Cauchy. Hence for the given $\varepsilon > 0$, there exists k_0 such that for $m, n \geq k_0$, we have $|f_m(x_0) - f_n(x_0)| \leq \varepsilon$. Hence we conclude that $x_0 \in E_{k_0}(\varepsilon)$.

Since $X$is a complete metric space, at least one of $E_k(\varepsilon)$ should have nonempty interior. Let $U_\varepsilon := \cup_k \text{Int}\,(E_k(\varepsilon))$. Then U_ε is a nonempty open subset of X.

Let $U_n := U_{1/n}$. We claim that each U_n is dense in X.

Reason: It is enough if we show that every closed ball $B := B[x, r]$ meets U_n non-trivially. (Why?)

Reason: To show a set A is dense in a metric space, it suffices to show that $A \cap B(x, r) \neq \emptyset$ for any $x \in X$and $r > 0$. Assume that $A \cap B[z, \rho] \neq \emptyset$ for any $z \in X$ and $\rho > 0$. Then given any $B(x, r)$, we may take $z = x$ and $\rho = r/2$. Then $\emptyset \neq A \cap B[x, \rho] \subset A \cap B(x, r)$.

Observe that the closed set (and hence a complete metric space) B is the union of a countable family of closed sets: $B = \cup_n (B \cap E_k(1/n))$. By Baire, at least one of them has nonempty interior, say, $\text{Int}\,(B \cap E_k(1/n)) \neq \emptyset$. Since $\text{Int}\,(B \cap E_k(1/n)) \subset B \cap \text{Int}\, E_k(1/n)$, it follows that $B[x, r] \cap U_n \neq \emptyset$ and hence the claim is proved.

Let $D := \cap_n U_n$. By Baire, D is dense in X. We claim that every $x \in D$ is a point of continuity of f.

Reason: Fix $p \in D$. Let $\varepsilon > 0$ be given. Choose $N \gg 0$ such that $1/N < \varepsilon$. Since $p \in D$, $p \in U_N$ and hence there exists $k \in \mathbb{N}$ such that $p \in \text{Int}\,(E_k(1/N)$. By continuity of f_k at p, there exists an open neighbourhood V of p contained in $\text{Int}\, E_k(1/N)$ such that

$$|f_k(x) - f_k(p)| < \varepsilon, \text{ for all } x \in V. \tag{2}$$

For $x \in V$, since $V \subset E_k(1/N)$, by the definition of $E_k(\varepsilon)$'s, we have

$$|f_m(x) - f_k(x)| \leq 1/N, \text{ for all } m \geq k. \tag{3}$$

Letting $m \to \infty$ in the above equation, we obtain

$$|f(x) - f_k(x)| \leq 1/N, \text{ for all } x \in V. \tag{4}$$

We are now ready for the kill. We claim that $|f(x) - f(p)| < 3\varepsilon$ for $x \in V$.

$$\begin{aligned} |f(x) - f(p)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(p)| + |f_k(p) - f(p)| \\ &\leq 1/N + \varepsilon + 1/N \\ &< 3\varepsilon. \end{aligned}$$

This shows that f is continuous at every point of D.

□

12.6 An amusing exercise: Let (x_n) be any sequence of real numbers. Show that the set $\{x \in \mathbb{R} : x \neq x_n, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$. Hence conclude that $\mathbb{R}$ is uncountable. Details!

12.7 **Baire category theorem for locally compact spaces.** Let X be a locally compact Hausdorff space. Let (U_n) be a sequence of open dense sets in X. Then $\cap_n U_n$ is dense in X.

Let G be a nonempty open set in X. We need to prove that there exists $x \in G$ such that $x \in U_n$ for all n. The strategy is to mimic the proof in the case of metric spaces replacing open balls by the existence of open sets V such that $\overline{V}$ is compact and $x \in V \subset \overline{V} \subset U$ for any given open set U and $x \in U$ and then invoking Cantor intersection theorem for a decreasing sequence of compact sets.

Since G is a nonempty open set and U_1 is dense, there exists $x_1 \in G \cap U_1$. Since $G \cap U_1$ is open, $x \in G \cap U_1$ and X is locally compact hausdorff space, there exists an open set V_1 such that $x \in V_1$, $\overline{V}_1$ is compact and $\overline{V}_1 \subset G \cap U_1$. Assume, by way of induction, that we have chosen x_i, $V_i \ni x_i$, $\overline{V}_i$ is compact and that $x_i \in V_i \subset \overline{V}_i \subset V_{i-1} \cap U_i$, for $1 \leq i \leq n$.

Now given a nonempty open set V_n, since $V_n \cap U_{n+1}$ is nonempty, there exists $x_{n+1} \in V_n \cap U_{n+1}$. Since X is locally compact and hausdorff, there exists an open set $V_{n+1} \ni x_{n+1}$ such that $\overline{V}_{n+1}$ is compact and $x_{n+1} \in V_{n+1} \subset \overline{V}_{n+1} \subset V_n \cap U_{n+1}$. Let $K_n := \overline{V}_n$. Thus we have a decreasing sequence (K_n) of nonempty compact subsets. Hence by Cantor intersection theorem, there exists $x \in \cap_n K_n$. Since $x \in K_n = \overline{V}_n \subset U_n$, it follows that $x \in \cap U_n$. Also, $x \in K_1 \subset U$. □

12.8 Locally closed sets: A subset A of a topological space is *locally closed* if for every $a \in A$, there exists an open set U_a in X such that $a \in U_a$ and $U_a \cap A$ is closed **in** U_a.

(a) A characterization of locally closed sets: $A \subset X$ is locally closed iff there exist an open set U and a closed set C such that $A = U \cap C$.

(b) The characterizations gives us easy examples of locally closed sets: $[0, 1)$ is neither closed nor open in $\mathbb{R}$ but is locally closed in $\mathbb{R}$.

13 Completely Regular and Normal Spaces

13.1 Separation axioms. They deal with separating various kinds of disjoint objects by means of disjoint open sets that contain the given objects. The prominent ones are given below.

(a) Hausdorff spaces: Given two distinct points $x \neq y$, if we can find open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

(b) Regular spaces: Given a point x and a closed set F with $x \notin F$, there exist open sets U and V such that $x \in U$ and $V \subset V$ with $U \cap V = \emptyset$.

(c) Normal spaces: Given two disjoint closed sets A, B, there exist open sets U, V such that $A \subset U$ and $B \subset V$ with $U \cap V = \emptyset$.

13.2 There are counterparts of these separation axioms in which we require that the objects be 'separated by continuous functions'.

(a) A space X is said to be *completely Hausdorff* if given two distinct points $x \neq y$ in X there exists a continuous function $f : X \to \mathbb{R}$ such that $f(x) \neq f(y)$.

(b) Completely regular spaces: Given two disjoint (nonempty) closed sets, we can find a continuous function $f : X \to \mathbb{R}$ such that $f = 0$ on A and $f = 1$ on B.

(c) Completely normal spaces: A space X is said to be *completely normal* if given two nonempty disjoint closed subsets A, B of X, there exists a continuous function $f : X \to \mathbb{R}$ such that $f = a$ on A and $f = b$ on B with $a \neq b$.

(d) Clearly, a completely Hausdorff (respectivley, completely regular, complletely normal) space is Hausdorff, (respectively regular, normal).

These 'complete' spaces will be useful for analysts since they assure that there is an 'abundant' supply of real valued continuous functions on the given space!

13.3 Some standard examples and facts concerning the above concepts:

(a) Examples of regular spaces.

i. Any metric space is regular.
Let A be a closed subset of a metric space X and $x \notin A$. Let $U := X \setminus A$. Then U is open and $x \in U$. hence there exists $r > 0$ such that $B(x, 3r) \subset U$. Then the open sets $B(x, r)$ and $X \setminus B[x, 2r]$ are open set which separate x and A.

ii. Any locally compact Hausdorff space is regular. Let A be a closed subset of a locally compact space X and $x \notin A$. Then $x \in X \setminus A$ and hence there exists an open set U such that $\overline{U}$ is compact and $x \in U \subset \overline{U} \subset X \setminus A$. The open sets U and $X \setminus \overline{A}$ separate x and A.

(b) Examples of normal spaces.

i. Any metric space is normal.
We give two proofs of this.
Let A and B disjoint closed subsets of a metric space X. For each $a \in A$, since $a \notin B$, a is not a limit point of B. hence there exists $r_a > 0$ such that $B(a, 2r_a) \cap B = \emptyset$. Similar analysis holds for each $b \in B$. Now consider $U := \cup_{a \in A} B(a, r_a)$ and $V := \cup_{b \in B} B(b, r_b)$. Then U and V are open sets

containing A and B respectively. If $x \in U \cap V$. then $x \in B(a, r_a) \cap B(b, r_b)$ for some $a \in A$ and $b \in B$. We observe

$$d(a,b) \leq d(a,x) + d(x,b) < r_a + r_b \leq 2\max\{r_a, r_b\}.$$

Thus, $a \in B(b, 2r_b)$ if $r_b \geq r_a$ or $b \in B(a, 2r_a)$ if $r_a \geq r_b$. This contradicts our choice of r_a etc. Hence $U \cap V = \emptyset$.

The second is based on Urysohn's lemma for metric spaces. See Item 13.5.

ii. Any compact Hausdorff space is normal.

We adapt the argument which showed that in a Hausdorff space, compact sets are closed. Let A and B be disjoint closed subsets of a compact Hausdorff space X. Fix $x \in A$. For each $b \in B$, there exist open sets $U_b \ni a$ and $V_b \ni b$ such that $U_b \cap V_b = \emptyset$. Since B is compact, the open cover $\{V_b : b \in B\}$ admits a finite subcover, say, $B \subset V := \cup_{b \in F} V_b$ for a finite subset $F \subset B$. Consider $U_a := \cap_{b \in F} U_b$. Then U_a, being a finite intersection of open sets, is open and $a \in U_a$. Clearly, $U_a \cap V = \emptyset$. Note that this argument shows that a compact Hausdorff space is regular.

Given $a \in A$, by the last paragraph, there exist open sets $U_a \ni a$ and $V_a \supset B$ such that $U_a \cap V_a = \emptyset$. Now the open cover $\{U_a : a \in A\}$ of A admits a finite subcover, say, $\{U_a : a \in G\}$ for a finite subset $G \subset A$. Let $U := \cup_{a \in G} U_a$ and $V := \cap_{a \in G} V_a$. It is easy to see that U and V separate A and B.

(c) A normal space in which all singleton sets are closed is regular.

13.4 The most important result about normal spaces is the Urysohn's lemma in Item 13.9. It says that a space is normal iff it is completely normal.

13.5 We prove Urysohn's lemma in the case of a metric space. Look at

$$f(x) := \frac{d(x,A)}{d(x,A) + d(x,B)}.$$

Note that f makes sense, as the denominator is nonzero. For, $d(x,A) + d(x,B) = 0$ implies that each of the non-negative terms is zero. That is, $d(x,A) = 0$ and $d(x,B) = 0$. Hence x is a limit point of the closed sets A and B (Item ???) and hence $x \in A$ and $x \in B$, a contradiction. By Item ??, f is continuous. Clearly, $f(x) = 0$ iff $x \in A$ and $f(x) = 1$ iff $x \in B$. (This is stronger than what is required!)

13.6 A key fact needed for Urysohn's lemma for normal spaces is the following observation.

Lemma 18. *A space X is a normal space iff for each closed set F and an open set V containing A there exists an open set U such that $F \subset U \subset \overline{U} \subset V$.*

Let X be normal and F, V as above. Then F and $X \setminus V$ are disjoint closed sets. By normality of X there exist open sets U and W such that $F \subset U$ and $X \setminus V \subset W$ and $U \cap W = \emptyset$. Since $U \subset X \setminus W$ and $X \setminus W$ is closed, we see that $\overline{U} \subset X \setminus W \subset V$. Thus U is as required.

To see the converse, let A and B disjoint closed subsets of X. Let $V_1 := X \setminus B$. Then $V_1 \supset A$ is an open subset. Hence by hypothesis, there exists U_1 such that $A \subset U_1 \subset \overline{U_1} \subset V_1$. Let $V_2 = X \setminus \overline{U_1}$. Then $V_2 \supset B$ is an open set. Let U_2 be an open set such that $B \subset U_2 \subset \overline{U}_2 \subset X \setminus U_1$. We claim that $U_1 \cap U_2 = \emptyset$. For if $x \in U_1 \cap U_2$, then $x \in U_1$ and $x \in U_2 \subset X \setminus \overline{U_1}$ and hence $x \notin \overline{U_1}$. Since $U_1 \subset \overline{U_1}$, this is a contradiction. $\square$

13.7 A key step in the proof of Urysohn's lemma is the construction of a sequence (U_n) of open sets indexed by dyadic rationals in $(0, 1)$. We bisect $(0, 1)$ successively. At n-th stage we have 2^n subintervals. To proceed to the $n + 1$-th stage, we insert an open set (provided by the lemma) indexed by the midpoints of the subintervals

Lemma 19. *Let X be a normal space. If A and B are closed subsets of X, for each dyadic rational $r = k2^{-n} \in (0, 1)$, there is an open set U_r with the following properties: (i) $A \subset U_r \subset X \setminus B$, (ii) $\overline{U}_r \subset U_s$ for $r < s$.*

Let $U := X \setminus B$. Since X is normal, there exist disjoint open sets V and W such that $A \subset V$ and $B \subset W$. Let $U_{1/2} = V$. Then, since $X \setminus W$ is closed, we have

$$A \subset U_{1/2} \subset \overline{U}_{1/2} \subset X \setminus W \subset X \setminus B = U.$$

Applying the same lemma once again to the open set $U_{1/2}$ containing A, that is to the pair $(A, U_{1/2})$ and to the pair $(\overline{U}_{1/2}, U)$, we get open sets $U_{1/4}$ and $U_{3/4}$ such that

$$A \subset U_{1/4} \subset \overline{U}_{1/4} \subset U_{1/2} \subset \overline{U}_{1/2} \subset U_{3/4} \subset \overline{U}_{3/4} \subset U.$$

Continuing this manner, we construct, for each dyadic rational $r \in (0, 1)$, an open set U_r with the following properties:

(i) $\overline{U}_r \subset U_s$, $0 < r < s \leq 1$.
(ii) $A \subset U_r$, $0 < r < 1$.
(iii) $U_r \subset U$, $0 < r < 1$.

More formally, we proceed as follows. We select U_r for $r = k2^{-n}$ by induction on n. That is, at n-th stage we have

$$A \subset U_{\frac{1}{2^n}} \subset \cdots \subset U_{\frac{k-1}{2^n}} \subset \overline{U}_{\frac{k-1}{2^n}} \subset U_{\frac{k}{2^n}} \subset \cdots \subset U_{\frac{2^n-1}{2^n}} \subset \overline{U}_{\frac{2^n-1}{2^n}} \subset U. \tag{5}$$

Assume that we have chosen U_r for $r = k2^{-n}$, $0 < k < 2^n$, $1 \leq n \leq N$. To find U_r for $r = \frac{2j+1}{2^{N+1}} = \frac{1}{2}\left(\frac{j}{2^N} + \frac{j+1}{2^N}\right)$,the midpoint, $0 \leq j < 2^N$, observe that $\overline{U}_{\frac{j}{2^N}} \subset U_{\frac{j+1}{2^N}}$. So once again applying the last lemma to the pair $(\overline{U}_{\frac{j}{2^N}}, U_{\frac{j+1}{2^N}})$, we obtain an open set U_r such that

$$\overline{U}_{\frac{j}{2^N}} \subset U_r \subset \overline{U}_r \subset U_{\frac{j+1}{2^N}}.$$

We claim that U_r's are as desired. We need only show that $r < s$ implies $\overline{U}_r \subset U_s$, the rest being obvious. Let $r = k/2^m$ and $s = l/2^n$. Cast both with the same denominator. If $m \leq n$, then $r = \frac{2^{n-m}k}{2^n}$ and $s = l/2^n$. Since $r < s$ iff $2^{n-m}k < l$, the property (i) follows from (5). If $m > n$, similar argument establishes the result. □

13.8 We consider the U_r's as the level sets of a function to "recover" the function. Let us explain this with an example.

Let $f \colon X \to \mathbb{R}$ be a function on a set X. Fix $c \in \mathbb{R}$. The set $X_c := \{x \in X : f(x) < c\}$ is called the level set of f at the level c.

Let $f \colon X := \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 + y^2$. Then $X_c \neq \emptyset$ iff $c > 0$. If $c > 0$, then the level set $X_c = B(0, \sqrt{c})$. Thus given the family $(X_{r^2}) := (B(x, r))_{r \geq 0}$ and given $(a, b) \in \mathbb{R}^2$, can we define a function $f \colon X \to \mathbb{R}$ such that $f(a, b) = a^2 + b^2$? A moment's thought tells us to define $f(a, b) := \inf\{c : (a, b) \in X_c\}$.

This example helps us understand the construction in Urysohn's lemma.

13.9 We are now ready to prove

Theorem 20. Urysohn's Lemma. *A space X is a normal space iff the following is true: For any two disjoint closed subsets A and B of X there exists a continuous function $f\colon X \to [0,1]$ such that $f = 0$ on A and $f = 1$ on B.*

Let U_r's be as in the lemma of the last item. The function f is defined as follows:

$$f(x) := \begin{cases} 1, & \text{if } x \text{ lies in no } U_r \\ \inf\{r : x \in U_r\}, & \text{otherwise.} \end{cases}$$

The continuity of f falls into three cases:
(i): The continuity of f at x when $f(x) = 1$. If $x \notin \overline{U}_r$, then $f(x) \geq r$.
(ii): The continuity of f at x when $f(x) = 0$. If $x \in U_r$, then $f(x) \leq r$.
(iii): The continuity at other points. If $x \in U_s \setminus \overline{U}_r$, then $r \leq f(x) \leq s$.

We shall attend to (iii). Let $0 < f(x) < 1$. Let $\varepsilon > 0$ be given. By the density of the dyadic rationals (Item 5.21), we can find two dyadic rational numbers r, s such that $f(x) - \varepsilon < r < f(x) < s < f(x) + \varepsilon$. Consider the open set $U_s \setminus \overline{U}_r$. If z lies in this set, then $f(z) \leq s$ and $f(z) \geq r$ by the very definition of f. Thus $x \in U_s \setminus \overline{U}_r$ and $f(U_s \setminus \overline{U}_r) \subset [r, s] \subset (f(x) - \varepsilon, f(x) + \varepsilon)$. Therefore f is continuous at x. The other cases are treated in a similar vein.

13.10 Note that Urysohn's lemma does **not** say that $f^{-1}(0) = A$ and $f^{-1}(1) = B$. It simply says that $A \subset f^{-1}(0)$ etc.

Contrast this with our version of Urysohn's lemma for metric spaces in Item 13.5. Our construction says that $f(x) = 0$ iff $x \in A$ etc.

13.11 Let X be a discrete space. Consider $A \subset X$ and $B := X \setminus A$. What function is constructed via Uryson?

More generally, if A and B are disjoint subsets of a discre space X, what are all the possible Urysohn's functions?

13.12 Let $h\colon [0,1] \to [a,b]$ be a homeomorphism, say, $h(t) := a + (b-a)t$. Let f be as in Urysohn's lemma. The composition $h \circ f\colon X \to [a,b]$ is such that $f = a$ on A and $f = b$ on B.

13.13 We now prove Tietze extension theorem, an important tool for analysts and topologists. *Let X be a normal space. Let $A \subset X$ be nonempty closed. Let $f\colon A \to [-1,1]$ be continuous. Then there exists an extension $F\colon X \to [-1,1]$, that is, there exists a continuous function $F\colon X \to [-1,1]$ such that $F(a) = f(a)$ for all $a \in A$.*

Proof. The subsets

$$B_1 := \{x \in A : f(x) \geq 1/3\} \text{ and } C_1 := \{x \in A : f(x) \leq -1/3\}$$

are closed subsets of A and hence of X. Hence by Urysohn's lemma, there exits a continuous function $f_1\colon X \to [-1/3, 1/3]$ such that $f = 1/3$ on B_1 and $f = -1/3$ on C_1. Clearly, $|f(x) - f_1(x)| \leq 2/3$ for $x \in A$. Hence $f - f_1$, restricted to A takes values in $[-2/3, 2/3]$.

We now repeat the proces repalcing f by $f - f_1$. We divide the interval $[-2/3, 2/3]$ into three equal parts and define

$$B_2 := \left\{x \in A : (f - f_1)(x) \geq \frac{1}{3} \times \frac{2}{3}\right\} \text{ and } C_2 := \left\{x \in A : f(x) \leq -\frac{1}{3} \times \frac{2}{3}\right\}.$$

By Urysohn, there exists a continuous function $f_2 \colon X \to [-2/9, 2/9]$ such that $f_2(x) = 2/9$ on B_2 and $f_2(x) = -2/9$ on C_2. It is clear that we have

$$|((f - f_1) - f_2)(x)| \leq \left(\frac{2}{3}\right)^2 \text{ for all } x \in A.$$

Continuing this process we have a sequence (f_n) of continuous functions on X such that (i) $f_n \colon X \to [-\frac{1}{3}(2/3)^{n-1}, \frac{1}{3}(2/3)^{n-1}]$ and (ii)

$$|(f - (f_1 + f_2 - \ldots + f_n))(x)| \leq \left(\frac{2}{3}\right)^n \text{ for all } x \in A. \qquad (6)$$

We let $F(x) := \sum_n f_n(x)$ for $x \in A$. Since $|f_n(x)| \leq M_n := \frac{1}{3}(2/3)^{n-1}$ for $x \in X$, and since $\sum_n M_n$ is convergent it follows by Weierstrass M-test that the series is uniformly convergent to a continuous function F on X. The dsiplayed inequality (6) shows that the partial sums s_n of the series (defining F) converge to f on A. Thus F is as required. □

13.14 We may replace the interval $[-1, 1]$ by any interval $[a, b]$ in the theorem. Justify this.

13.15 We may replace the interval $[-1, 1]$ by $\mathbb{R}$ in the theorem.

Let $h \colon \mathbb{R} \to (-1, 1)$ be a homeomorphism. Consider the function $g := h \circ f$. We may assume that g takes value in $[-1, 1]$. By the theorem, there exists a continuous extension, say, G taking values in $[-1, 1]$ which extends g. Let $A_1 := \{x \in X : |G(x)| = 1\}$. Since G is an extension of g, if $x \in A$, then $|G(x)| = |g(x)| < 1$ and hence $x \notin A_1$. Thus A and A_1 are disjoint closed subsets of X. Hence by Urysohn's lemma, there exists a continuous function λ such that $\lambda(x) = 1$ on A and $\lambda(x) = 0$ on A_1. The function $\lambda(x)G(x)$ takes values in $(-1, 1)$ and extends g. The function $h^{-1} \circ (\lambda G)$ is the desired extension of f.

13.16 Exercises.

(a) Let X be a normal space and F a closed subset. Assume that $f \colon F \to (-R, R)$ be a continuous function. Then f can be extended to a continuous function from X to $(-R, R)$. *Hint:* You may need Urysohn's lemma.

(b) Let X be a normal space and F a closed subset. Assume that $f \colon F \to \mathbb{R}$ be a continuous function. Then f can be extended to a continuous function from X to $\mathbb{R}$. *Hint:* $\mathbb{R}$ is homeomorphic to $(-1, 1)$.

(c) Assuming Tietze extension theorem, prove Urysohn's lemma.

Consider $f \colon A \cup B \to \mathbb{R}$ where $f = 0$ on A and 1 on B.

(d) A topological space is normal iff every continuous function from a closed subset to [0,1] extends to a continuous function from X to [0,1].

By the last item, Urysohn's lemma is valid for X. Use Item **??**.

(e) Let A be a closed subset of a normal space X. Let $f: A \to S^n$ be continuous. Show that there exists an open set $U \supset A$ (U depends on f) and an extension g of f to U.

(f) Show that with the notation of Exer. 198 that f may not extend to all of X. *Hint:* What happens (i) if $n = 0$ and X is connected or (ii) if $X := B[0,1] \subset \mathbb{R}^{n+1}$, $A := S^n$ and f is the identity?

13.17 **An example for practice.**

Consider $X = \mathbb{R}$ with the topology $\mathcal{T}$ consisting of sets of the form $U \setminus A$ where U is open in the standard topology $\mathcal{T}_d$ and $A \subset \mathbb{R}$ is any countable subset.

(a) A subset F is closed in $\mathcal{T}$ if $F = E \cup B$ where E is closed in $\mathcal{T}_d$ and B is a countable subset.

(b) If $E = U \setminus A$ is open in $\mathcal{T}$, show that the closure of E in $\mathcal{T}$ is the closure of E in $\mathcal{T}_d$.

(c) Is $\mathbb{Q}$ dense in $(X, \mathcal{T})$?

(d) What are compact subsets in $\mathcal{T}$?

(e) Show that any open cover of $(X, \mathcal{T})$ admits a countable subcover.

(f) Show that $(X, \mathcal{T})$ is not first countable.

(g) Show that any countable subset is closed in $\mathcal{T}$ and hence $(X, \mathcal{T})$ is not separable.

(h) Show that $(X, \mathcal{T})$ is connected but not path-connected.

14 Homotopy

1. Let X be a topological space. A *loop* in X is a path $\alpha\colon [0,1] \to X$ with $\alpha(0) = \alpha(1)$. We say that α s a loop based at $\alpha(0)$.

   Recall that if $\alpha, \beta\colon [0,1] \to X$ are paths such that $\alpha(1) = \beta(0)$, then their *join* $\alpha * \beta$ is defined by
$$\alpha * \beta(t) := \begin{cases} \alpha(2t) & \text{for } 0 \leq t \leq 1/2 \\ \beta(2t-1) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$
   Then, $\alpha * \beta$ is continuous (by gluing lemma) and we say that it is got by concatenation.

   Standard Notation in homotopy theory: Let $I = [0,1]$.

2. Let X, Y be topological spaces. Let $f, g\colon X \to Y$ be continuous maps. We say that they are homotopic if there exists a continuous map $F\colon X \times I \to Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. We say that $f_t(x) := F(x,t)$ for $t \in I$ and $x \in X$.

   The map F is called a homotopy from f to g and we write $f \overset{F}{\simeq} g$.

   If $f(a) = g(a)$ for all $a \in A \subset X$ and if the homotopy F is such that $F(a,t) = f(a)$ for all $t \in I$ and $a \in A$, we say that f is homotopic to to g relative to A. We denote this by $f \overset{F}{\simeq} g$ rel A.

   If α and β are paths in X with the same initial and terminal points, then saying that α is homotopic to β relative to $\{0,1\}$ is the same as saying that all the intermediate paths $\alpha_t(s) := F(s,t)$ have the same initial and terminal points, that is, they satisfy $F(0,t) = \alpha(0)$ and $F(1,t) = \alpha(1)$.

3. Examples:

   (a) Let $C \subset \mathbb{R}^n$ be convex. Let $f, g\colon X \to C$ be continuous maps. Then the map $F(x,t) := (1-t)f(x) + tg(x)$ is a homotopy from f to g. If f and g agree on a set $A \subset X$, then F is a homotopy relative to A.

   (b) Let $f, g\colon X \to S^n$ be continuous maps such that $f(x) \neq -g(x)$ for $x \in X$. Then the map
$$F(x,t) := \frac{(1-t)f(x) + tg(x)}{\|(1-t)f(x) + tg(x)\|}$$
   is a homotopy from f to g.

   (c) The map $f\colon S^1 := \{z \in \mathbb{C} : |z| = 1\} \to S^1$ defined by $f(z) = -z$ is homotopic to the identity map $g(z) = z$.

   (d) Let $f\colon X \to S^n$ be a continuous map which is not onto. Then it is null-homotopic, that is, homotopic to a constant map.

   (e) Consider $X := \{p \in \mathbb{R}^2 : 1 \leq \|p\| \leq 2\}$. Let α be 'the inner circle' and β be the ellipse lying in X and circumscribing α. Assume that they both start and end at $(0,1)$. They are homotopic in X. (Note that X is not convex.)

4. The relation of homotopy between the continuous maps from a space X to another space Y is an equivalence relation.

For, if $f \overset{F}{\simeq} g$ and $g \overset{G}{\simeq} h$, then

$$H(x,t) := \begin{cases} F(x,2t) & 0 \le t \le 1/2 \\ G(x,2t-1) & 1/2 \le t \le 1, \end{cases}$$

is a homotopy from f to h.

5. The relation of homotopy between the continuous maps from a space X to another space Y relative to a subset $A \subset X$ is an equivalence relation among maps that agree on A.

6. Homotopy behaves well with respect to composition of maps.

   (a) Let $f, g\colon X \to Y$ be homotopic relative to a set $A \subset X$ via the homotopy F. Let $h\colon Y \to Z$ be a map. Then $h \circ f \overset{h \circ F}{\simeq} h \circ g$ relative to A.

   (b) Let $f\colon X \to Y$ be a map. Assume that $g, h\colon Y \to Z$ are homotopic relative to $B \subset Y$ via a homotopy G. Then $g \circ f \overset{F}{\simeq} h \circ f$ relative to $f^{-1}(B)$, where $F(x,t) := G(f(x),t)$.

7. Fix a base point $p \in X$. Let α be a loop at p. The equivalence class $\langle \alpha \rangle$ of all loops based at p homotopic to α relative to $\{0,1\}$ is called a *homotopy class*. The collection of homotopy classes of loops at p is denoted by $\pi_1(X,p)$.

8. Construction of the fundamental group. We make $\pi_1(X,p)$ into a group as follows. For $\langle \alpha \rangle, \langle \beta \rangle \in \pi_1(X,p)$, we let $\langle \alpha \rangle * \langle \beta \rangle := \langle \alpha * \beta \rangle$.

   (a) The above multiplication is well-defined.

   For, $\alpha \overset{F}{\simeq}$ and $\beta \overset{G}{\simeq} \beta'$, then $\alpha * \beta \overset{H}{\simeq} \alpha * \beta'$ where $H(s,t) := \begin{cases} F(2s,t) & 0 \le s \le 1/2 \\ G(2s-1,t) & 1/2 \le s \le 1. \end{cases}$

   (b) The multiplication is associative.
   First of all, we compute

$$((\alpha * \beta) * \gamma)(s) \quad = \quad \begin{cases} \alpha(4s) & 0 \le s \le 1/4 \\ \beta(4s-1) & 1/4 \le s \le 1/2 \\ \gamma(2s-1) & 1/2 \le s \le 1 \end{cases}$$

$$(\alpha * (\beta * \gamma))(s) \quad = \quad \begin{cases} \alpha(2s) & 0 \le s \le 1/2 \\ \beta(4s-2) & 1/2 \le s \le 3/4 \\ \gamma(4s-3) & 3/4 \le s \le 1 \end{cases}.$$

   Define $f\colon I \to I$ by setting

$$f(s) := \begin{cases} 2s & 0 \le s \le 1/4 \\ s + \dfrac{1}{4} & 1/4 \le s \le 1/2 \\ (s+1)/2 & 1/2 \le s \le 1 \end{cases}.$$

Since $f(0) = 0$ and $f(1) = 1$, we see that $f \simeq 1_I$, that is, f is homotopic to the identity map 1_I of I relative to $\{0, 1\}$. We have

$$\begin{aligned}(\alpha * \beta) * \gamma &= (\alpha * (\beta * \gamma)) \circ f \\ &\simeq (\alpha * (\beta * \gamma)) \circ 1_I \\ &= \alpha * (\beta * \gamma).\end{aligned}$$

(c) Existence of the identity. Let $e = e_p$ denote the constant loop at p: $e(t) = p$ for $0 \leq t \leq 1$. Then $\langle e \rangle$ serves as the identity for the multiplication. Again, proceeding as earlier, we have

$$\begin{aligned}e * \alpha(s) &= \begin{cases} e(2s) & 0 \leq s \leq 1/2 \\ \alpha(2s-1) & 1/2 \leq s \leq 1 \end{cases} \\ e * \alpha &= \alpha \circ f \\ \text{where } f(s) &= \begin{cases} 0 & 0 \leq s \leq 1/2 \\ 2s-1 & 1/2 \leq s \leq 1. \end{cases}\end{aligned}$$

Thus we have

$$e * \alpha = \alpha \circ f \simeq \alpha \circ 1_I \text{ rel } I = \alpha.$$

Similarly, one shows that $\alpha * e \simeq \alpha$.

(d) Existence of inverse. The inverse of $\langle \alpha \rangle$ is $\langle \alpha^{-1} \rangle$, where α^{-1} is the reverse path defined by $\alpha^{-1}(s) := \alpha(1-s)$.

i. The inverse s well-defined. If $\alpha \overset{F}{\simeq} \beta$ relative to $\{0,1\}$, then $\alpha^{-1} \overset{G}{\simeq} \beta^{-1}$ relative to $\{0, 1\}$ where $G(s,t) := F(1-s, t)$.

ii. We show that $\alpha * \alpha^{-1} = \alpha \circ f$ where

$$f(s) = \begin{cases} 2s & 0 \leq s \leq 1/2 \\ 2-2s & 1/2 \leq s \leq 1. \end{cases}$$

Now, $f \simeq g$ relative to $\{0,1\}$ where $g(s) = 0$ for $0 \leq s \leq 1$. Hence,

$$\alpha * \alpha^{-1} = \alpha \circ f \simeq \alpha \circ g \text{ rel } \{0, 1\} = e.$$

One similarly, shows that $\alpha^{-1} \circ \alpha \simeq e$.

(e) Explicit homotopies can also be given. (Of what use?)

i. Existence of identity.

- $\alpha * e \simeq \alpha$ via

$$H(s,t) := \begin{cases} \alpha\left(\frac{2t}{s+1}\right) & s \geq 2t-1 \\ p & s \leq 2t-1. \end{cases}$$

- $e * \alpha \simeq \alpha$ via

$$H(s,t) = \begin{cases} p & s \geq 2t \\ \alpha\left(\frac{2t-s}{2-s}\right) & s \leq 2t \end{cases}$$

ii. Existence of inverse. $\alpha * \alpha^{-1} \simeq e$ via

$$H(s,t) = \begin{cases} \alpha(2t) & s \geq 2t \\ \alpha(s) & s \leq 2t \text{ and } s \leq 2-2t \\ \alpha(2-2t) & s \geq 2-2t \end{cases}$$

iii. Associativity. $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$ via

$$H(s,t) = \begin{cases} \alpha(\frac{4t}{s+1}) & 4t-1 \leq s \\ \beta(4t-s-1) & 4t-2 \leq s \leq 4t-1 \\ \gamma(\frac{4t-2s}{2-s} - 1) & s \leq 4t-2. \end{cases}$$

I have not verified these, simply copied from a book!

9. Let α, β be two paths such that $\alpha(1) = \beta(0)$. Then proceeding as in the last item, we show the following, as the same homotpies work as they take care of the end points!

   (a) If $\alpha' \simeq \alpha$ relative to $\{0,1\}$ and If $\beta' \simeq \beta$ relative to $\{0,1\}$, then $\alpha * \beta \simeq \alpha' * \beta'$ relative to $\{0,1\}$.

   (b) If α, β, γ are paths such that $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ make sense, then

   $$\alpha * (\beta * \gamma) \simeq (\alpha * \beta) * g \text{ relative to } \{0,1\}.$$

   (c) We have $\alpha \circ \alpha^{-1} \simeq e_{\alpha(0)}$ relative to $\{0,1\}$ and $\alpha^{-1} \circ \alpha \simeq e_{\alpha(1)}$ relative to $\{0,1\}$.

10. If X is path connected, then $\pi_1(X,p)$ is isomorphic to $\pi_1(X,q)$ for $p,q \in X$. This isomorphism depends on the choice of path joining p and q.

15 Covering Space

1. Let $p\colon E \to B$ be a continuous map. An open subset $U \subset B$ is said to be *evenly covered* by p if $p^{-1}(U)$ is the union $\cup_i V_i$ of disjoint open subsets V_i of E such that the restriction p_i of p to V_i is a homeomorphism of V_i onto U.

   We say that p is a *covering map* if (i) p is onto and (ii) each $b \in B$ has an open neighbourhood U_b which is evenly covered by p.

   The set $p^{-1}(b)$ is called the *fibre* over b.

   The sets V_i are called *sheets* of $p^{-1}(U)$.

   E is called the total space and B, the base of the covering map p.

2. Properties of a covering map.

   (a) Any covering map is open.

   (b) Each of the fibres $p^{-1}(b)$ is discrete.

   (c) Each $b \in B$ has an open neighbourhood U such that $p^{-1}(U)$ is homeomorphic to $p^{-1}(b) \times U$.

3. Examples. Details!

   (a) The exponential map $p\colon \mathbb{R} \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$ is a covering.

   (b) The quotient map $\pi\colon S^n \to \mathbb{P}^n(\mathbb{R})$ is a covering.

   (c) Products of covering maps is again a covering map. (precise statement?)

   (d) Consider the exponential map $\exp\colon \mathbb{C} \to \mathbb{C}^*$. The open set $U := \mathbb{C}^*$ is not evenly covered by exp.

   In fact, an open set $U \subset \mathbb{C}^*$ is evenly covered by the exponential map iff there exists a continuous logarithm L on U, that is, a continuous map $L\colon U \to \mathbb{C}$ such that $\exp(L(z)) = z$ for all $z \in U$.

   Note however that $\exp\colon \mathbb{C} \to \mathbb{C}^*$ is a covering map.

4. Let $p\colon E \to B$ a covering map. Let $f\colon X \to B$ be continuous map. Then a map $g\colon X \to E$ such that $p \circ g = f$ is called a *lift* of f. One has the following commutative diagram. (Figure?)

5. Uniqueness of lifts. Details!

   Theorem 21. *Let $p\colon E \to B$ be a covering map and X a connected space. Let $f\colon X \to B$ be a map. If $g, h\colon X \to E$ are lifts of f such that $g(x) = h(x)$ for some $x \in X$, then $g = h$.* □

6. Path lifting lemma. Details!

   Theorem 22. *Let $p\colon E \to B$ be a covering map. Let $c\colon I \to B$ be a path. Let $e_0 \in E$ be such that $p(e_0) = c(0)$. then there exists a unique path $\gamma\colon I \to E$ such that $\gamma(0) = e_0$ and $p\colon \gamma = c$.* □

7. A Version of homotopy lifting lemma: Details!

Theorem 23. *Let $p\colon E \to B$ be a covering map. Let $F\colon I \times I \to B$ be a continuous map. Let $e_0 \in p^{-1}(F(0,0))$. Then there exists a unique lift $G\colon I \times I \to E$ of F such that $G(0,0) = e_0$.* □

8. Let (E, e) and (B, b) be topological spaces with base points e and b respectively. Let $p\colon E \to B$ be a covering map. If c is a loop at b and γ is its lift through e, we cannot conclude that γ is a loop at e but $p(\gamma(1)) = b$, that is, $\gamma(1) \in p^{-1}(b)$. Example: Consider the spaces $(\mathbb{R}, 0)$ and $(S^1, 1)$. A lift of $c(t) = e^{2\pi i t}$ is $\gamma(t) = t$ in $\mathbb{R}$.

9. Let c_0 and c_1 be homotopic loops at b with F as a a homotopy. We thus get a lift $G\colon I \times I \to E$ of F such that $G(0,0) = e$ and $p(G(s,t)) = c_t(s)$, for $(s,t) \in I \times I$. Let $\gamma_t(s) := G(s,t)$. Then all these paths start at e and have the same end point $\gamma_0(1)$.

   As a corollary, if $\langle c \rangle \in \pi_1(B, b)$ and γ is a lift of c through e, then

   $$\pi_1(B, b) \to \pi^{-1}(b) \text{ defined by } \varphi\colon \langle c \rangle \mapsto \gamma(1) \tag{7}$$

   is well-defined.

10. Simply connected space. We say a path-connected topological space X is *simply connected* if $\pi_1(X, x)$ is trivial for some (and hence for any) $x \in X$. Examples:

    (a) Any convex subset of $\mathbb{R}^n$ is simply connected.

    (b) The parabola $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$ is not convex but simply connected.

    (c) We shall show below (Item 13) that S^n for $n \geq 2$ is simply connected.

11. Let $p\colon (E, e) \to (B, b)$ be a covering map. Assume that E is simply connected. Then the map defined in (7) is a bijection of $\pi_1(B, b)$ with $\pi^{-1}(b)$. Details!

    As a corollary (under the above hypothesis), for any $q \in \pi^{-1}(b)$, if we let γ_y be a path joining e to y, then given a loop c at p, we have a unique $q \in \pi^{-1}(b)$ such that c is homotopic to $p \circ \gamma_y$.

12. Applications.

    (a) Fundamental group of $\mathbb{P}^n(\mathbb{R})$ $(n \geq 2)$. For $n \geq 2$, $\pi_1(\mathbb{P}^n(\mathbb{R}), [e_1])$ is isomorphic to $\mathbb{Z}_2$.

    (b) Fundamental group of S^1 is isomorphic to $\mathbb{Z}$. The following are the main steps.

       i. Given $\langle c \rangle \in \pi_1(S^1, 1)$, $\varphi(\langle c \rangle) \in \mathbb{Z}$. The integer $\langle c \rangle$ is called the index of c.

       ii. The map $\langle c \rangle \mapsto \varphi(\langle c \rangle)$ is a group homomorphism of $\pi_1(S^1, 1)$ to $\mathbb{Z}$.

13. Let X be a space, U, V be simply connected open subsets of X such that (i) $X = U \cup V$ and (ii) $U \cap V$ is path connected. Then X is simply connected.

    Application. S^n is simply connected for $n \geq 2$.

14. Applications of the index of loops in S^1. Details!

    (a) No retraction theorem. There is no continuous map $f\colon B^2 \to S^1$ such that $f(z) = z$ for $z \in S^1$. □

    (b) Brouwer fixed point theorem. Any continuous map of B^2 to itself has a fixed point. □

(c) Borsuk-Ulam theorem. Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map. Then there exist antipodal points $\pm v \in S^2$ such that $f(v) = f(-v)$. □

This has a physical interpretation.

(d) Ham-Sandwich theorem. Let A, B, C be bounded connected open subsets of $\mathbb{R}^3$. Then there exists a plane in $\mathbb{R}^3$ that divides each of the sets into two subsets of equal volume. □

Proof of this relied on some intuitively obvious facts on volumes.

(e) Fundamental theorem of algebra. □

For proofs, you may refer to my relevant articles in *Expository Articles*.
Give exact references in the Appendix. Give Ref!

To add as appendices:

1. Finite sets
2. Cardinality
3. Subspace Topology
4. Quotient Topology
5. Generating Topologies
6. Tykonoff's theorem
7. Compact Spaces
8. Connected Spaces
9. Existence of Continuous Functions
10. Proper maps
11. Covering spaces
12. Topological groups
13. Discrete Subgroups of $\mathbb{R}^n$.
14. Characterization of Compact Metric Spaces
15. Vector fields on spheres
16. $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ for $n \neq m$ via dimension theory.
17. Maps into punctured plane
18. Winding Numbers
19. No Retraction of $\mathbb{R}^2$ onto S^1.

A Concepts Summary

- About sets: Open, closed, dense, nowhere dense subsets. Bounded and totally bounded subsets in metric space.
- Examples: Metric spaces, $\mathbb{R}$ and $\mathbb{R}^n$ with the Euclidean topologies, discrete and indiscrete spaces, $\mathbb{R}$ with the lower limit topology, $\mathbb{R}^2$ with order topology, co-finite topology, co-countable topology, VIP topology, outcast topology.
- About Points: interior point, boundary point, limit point, cluster point and isolated point
- About maps: continuity, homeomorphism, open maps, closed maps and proper maps. Universal mapping properties of the constructions.
- About Spaces: compact, connected, path-connected spaces; their local versions: locally compact, locally connected and locally path-connected spaces.
- Bases: base for the topology and base for a topology; countability axioms and separability.
- Separation Axioms: Hausdorff spaces, regular spaces and normal spaces; their counterparts where separation is by means of continuous functions.
- Constructions: Subspace, product and quotient spaces
- Some of the major/most useful results:
  1. Product of compact/connected spaces, Heine-Bore theorem
  2. Urysohn's lemma
  3. A bijective continuous map of a compact space onto a Hausdorff space is a homeomorphism.
  4. Baire Category theorem
  5. A closed subset of a compact space is compact and a a compact subset of a Hausdorff space is closed.
  6. Characterization of compact metric spaces
  7. Any continuous map from a compact space to a metric space is bounded, such a map to $\mathbb{R}$ has a maximum and minimum
  8. A map from Y to the product space is continuous iff each of its component maps is continuous.
  9. Cantor intersection theorem.

B Finite Sets

For $n \in \mathbb{N}$, let $I_n := \{1, 2, \ldots, n\}$ be the initial segment.

Definition 24. A set A is said to be finite if either $A = \emptyset$ or there exists a bijection $f\colon A \to \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.

Lemma 25. *If $m < n$, there is no one-to-one map of I_n into I_m.*

Proof. Let $m = 1$ and $n > 1$. No map $f\colon I_n \to I_1 = \{1\}$ can be 1-1. For, $f(1) = f(n) = 1$ and $n \neq 1$. Thus the result is true for $m = 1$.

Let $P(m)$ be the statements: *Given $n > m$, no map $f\colon I_n \to I_m$ can be 1-1.*

Thus we have seen P(1) is true. Assume the result $P(m)$. Consider $m+1$. Let $n > m+1$. Let $f\colon I_n \to I_{m+1}$ be 1-1. There are two possibilities for $f(n)$.

Case 1: Let $f(n) = m + 1$. Then consider the map $g\colon I_{n-1} \to I_m$ given by $g(j) = f(j)$. Then g is 1-1 and hence I_m is not true.

Case 2: Let $f(n) = r < m+1$. Then there is at most one $1 \leq k < n$ such that $f(k) = m+1$. We define $g\colon I_{n-1} \to I_m$ by setting $g(j) = f(j)$ for $j \neq k$ and $g(k) = r = f(n)$. Then g is 1-1 and hence $P(m)$ is not true.

Thus we conclude that such an f cannot exist. In other words, $P(m + 1)$ is also true. By the principle of induction, we conclude that $P(m)$ is true for all m and hence the lemma is proved. □

Lemma 26. *If $m < n$, then there is no onto map $f\colon I_m \to I_n$.*

Proof. Let $f\colon I_m \to I_n$ be onto where $m < n$. We define $g\colon I_n \to I_m$ as follows: Let $r \in I_n$. Let $i := \min f^{-1}(r)$. We set $g(r) = i$. Then g is 1-1: If $g(r) = g(s)$, then there exists $k \in I_m$ such that $f(k) = r, s$, i.e., f is not a function! □

Corollary 27. *If $f\colon I_m \to I_n$ is a bijection, then $m = n$.*

Definition 28. A finite set A is said to have n elements, if there is a bijection $f\colon A \to I_n$. Note that in view of the corollary this is well-defined. For any finite set A, we let $|A|$ denote the number of elements in A.

Lemma 29. *Let $f\colon A \to I_n$ be 1-1. Then A is finite, and we have $|A| \leq n$.*

Proof. Let $r_1 = \min\{f(a) : a \in A\} \subset \mathbb{N}$, $r_2 = \min\{f(A) \setminus \{r_1\}\}$. Note that $r_1 \geq 1$ and $r_2 > r_1$ so that $r_2 \geq 2$. We proceed by induction to construct $r_1 < r_2 < \cdots < r_k$ where $r_k \geq k$. This process will stop at some stage in the sense that $f(A) \setminus \{r_j : 1 \leq j \leq k\} = \emptyset$ for some $k \leq n$. For, otherwise, if $k > n$, then $r_k \geq k > n$. This contradicts the fact that $r_k \in I_n$. We now construct a bijection $g\colon I_k \to A$ as follows: $g(i) = a$ where $f(a) = r_i$. One easily checks that g is a bijection. □

Corollary 30. *If A is finite and $B \subset A$, then B is finite and $|B| \leq |A|$.*

Proof. Let $f\colon A \to I_n$ be a bijection. Then the composition of the inclusion $B \hookrightarrow A$ followed by f is a 1-1 map of B into I_n. By Lemma 29, the result follows. □

Lemma 31. *Let $f\colon I_n \to A$ be onto. Then A is finite and $|A| \le n$.*

Proof. Define $g\colon A \to I_n$ by setting $g(a) = \min f^{-1}(a)$. Then g is 1-1 and the result follows from the last corollary. □

Proposition 32 (Pigeonhole Principle)**.** *Let $m, n \in \mathbb{N}$ be such that $m < n$. If $f\colon I_n \to I_m$ is a map, then there exists $i, j \in I_n$ such that $i \ne j$ and $f(i) = f(j)$.* □

Ex. 33. Let A and B be finite sets with $A \cap B = \emptyset$. Show that $A \cup B$ is finite. What is the number of elements in $A \cup B$?

Ex. 34. Let X be a finite set. Let $f\colon X \to X$ be map. Show that the following are equivalent:
(a) f is a bijection.
(b) f is one-one.
(c) f is onto.

Ex. 35. Let A and B be finite sets and $f\colon A \to B$ be a map. Prove the following:
(a) If f is one-one, than $|A| \le |B|$.
(b) If f is onto, than $|A| \ge |B|$.
(c) If $f\colon A \to B$ and $g\colon B \to A$ are one-one, then $|A| = |B|$ and f, and g are bijections.

C Cardinality and Countability

The primitive idea of "counting" a set is to set up a bijection with a known set. The words 'calculus' and 'calculation' have their origin with such a correspondence with a pile of stones!

Definition 36. We say that two sets A and B have the *same cardinality* if there is a bijection from one onto the other. (Intuitively, this means that A and B "have the same number of elements." Because of this we may even say that A and B are *equinumerous.*) Note that "having the same cardinality" is "an equivalence relation."

Example 37. (i) $\mathbb{N}$ and $2\mathbb{N}$, the set of even positive integers have the same cardinality.
(ii) Any two closed intervals $[a, b]$ and $[c, d]$ have the same cardinality.
(iii) Any two open intervals (a, b) and (c, d) have the same cardinality.
(iv) $(-1, 1)$ and $\mathbb{R}$ have the same cardinality. *Hint:* Consider the map $x \mapsto \dfrac{x}{1-x^2}$. Or, observe that $\tan\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a bijection.
(v) $\mathbb{Z}$ and $\mathbb{N}$ have the same cardinality.
(vi) $\mathbb{R}$ and $(0, \infty)$ have the same cardinality.
(vii) (0,1) and $(1, \infty)$ have the same cardinality.

Lemma 38 (Knaster)**.** *Let $F\colon P(X) \to P(X)$ be a map. Assume that it is increasing in the sense that if $A \subseteq B$, then $F(A) \subseteq F(B)$. Then F has a fixed point, that is, there exists $S \subset X$ such that $F(S) = S$.*

Hint: Consider the set $\mathcal{C} := \{C \subseteq X : C \subseteq F(C)\}$. Let S be the union of all members of $\mathcal{C}$. Then $F(S) = S$. □

The next theorem is very useful. See Example 40.

Theorem 39 (Schroeder-Bernstein)**.** *Let A and B be sets. Assume that $f\colon A \to B$ and $g\colon B \to A$ be one-one. Then there exists a bijection $h\colon A \to B$.*

Hint: Consider $F\colon P(A) \to P(A)$ given by $F(C) := A \setminus g(B \setminus f(C))$. Apply the last lemma. □

Example 40. (i) $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ have the same cardinality. *Hint:* The map $f\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $f(m, n) := 2^m 3^n$ is one-one. For an explicit bijection, see Example 45.
(ii) There exists a bijection between the intervals $[a, b]$ and (c, d). *Hint:* The interval $[a, b]$ and a closed subinterval of (c, d) have the same cardinality by Example 37.
(iii) The sets $A := [0, 1)$ and $B := A \times A$ have the same cardinality. *Hint:* Use non-recurring decimal expansion to get a one-map of B into A. For example, consider $g(0.x_1x_2\ldots, 0.y_1y_2\ldots) := 0.x_1y_1x_2y_2\ldots$.

For any $n \in \mathbb{N}$, let I_n denote the subset $\{k : 1 \leq k \leq n\}$ of $\mathbb{N}$.

Definition 41. A set A is said to be *finite* if either $A = \emptyset$ or there is a bijection $f\colon A \to I_n$ for some $n \in \mathbb{N}$.

A set which is not finite is said to be *infinite.*

Theorem 42. *Let A be a finite set. Let $f\colon A \to I_m$ and $g\colon A \to I_n$ be bijections. Then $m = n$.* □

Definition 43. If A is finite with $f\colon A \to I_n$ is a bijection, then n is unique by the last theorem. (Note that f need not be unique.) We say that A has n elements. If A is empty, we say that A has zero elements.

Definition 44. A set A is said to be countable if either A is finite or if there exists a bijection $f\colon A \to N$. A set of the latter type is said to be *countably infinite*.

Example 45. (i) $\mathbb{Z}_+$, $\mathbb{Z}$ are countably infinite.
(ii) Any infinite subset of $\mathbb{N}$ is countably infinite.
(iii) $\mathbb{N}\times\mathbb{N}$ is countably infinite. *Hint:* Consider the map $f(m,n) := \dfrac{(m+n-1)(m+n-2)}{2} + n$. How did one arrive at this map? What is the inverse of this map? The inverse is given by $m \mapsto (\frac{n(n-1)}{2} - m + 1, m - \frac{(n-1)(n-2)}{2})$ where $\frac{(n-2)(n-1)}{2} < m \leq \frac{n(n-1)}{2}$.

Proposition 46. *Let A be a set. The following are equivalent.*
(i) *A is countable.*
(ii) *There is a one-one map of A into $\mathbb{N}$.*
(iii) *There is an onto map from $\mathbb{N}$ onto A.* □

Corollary 47. (i) *A subset of a countable set is countable.*
(ii) *Let I be a countable set and let A_i be countable for each $i \in I$. Then $A := \cup_{i\in I} A_i$ is countable, that is, a countable union of countable set s is countable.*
(iii) *A finite product of countable sets is countable.* □

Example 48. The set of positive rational numbers is countable.

Example 49. A complex number is said to be an *algebraic number* if it is a root of a polynomial with integer coefficients. The set of algebraic numbers is countable. *Hint:* Show that the set of polynomials with integer coefficient is countable.

Lemma 50. *The set of functions from $\mathbb{N}$ to $\{0,1\}$ is not countable.* □

Theorem 51 (Cantor)**.** *Let X be any set and $P(X)$, the power set of X. There is no onto function from $P(X)$ onto X.* □

Example 52. $\mathbb{R}$ is uncountable. *Hint:* Enough to show that $[0,1]$ is uncountable. Use Nested interval theorem. □

A complex number is *transcendental* if it is not algebraic.

Corollary 53. *The set of transcendental numbers is uncountable.* □

Remark 54. Why is the last result historically important?

Theorem 55. *The following are equivalent for a set X.*
(i) *The set X is infinite.*
(ii) *There exists an infinitely countable subset S of X.*
(iii) *There exists a proper subset Y of X such that X and Y have the same cardinality.* □

Reference:

J.R. Munkres, *Topology*, especially Sections 1.6, 1.7 and 1.9

D Subspace Topology

Let $Y \subset X$ of a topological space $(X, \mathcal{T})$. We say that a set $V \subset Y$ is open **in** Y if there exists an open set U in X such that $V = U \cap Y$. Let $\mathcal{T}_Y$ denote the set of all subsets $V \subset Y$ which are open in Y. Then $\mathcal{T}_Y$ is a topology on Y. It is called the subspace topology. Given below are some examples-cum-exercises which will help you master this concept. We concentrate on "basic" open sets in Y, that is, those sets whose arbitrary unions will produce all elements of $\mathcal{T}_Y$. In the following any $\mathbb{R}^n$ is endowed with the standard topology coming from the Euclidean metric $(x, y) := \sqrt{\sum_{i=1}^n (x_i^2 - y_i^2)}$.

Ex. 56. Let (X, d) be a metric space. If we restrict d to $Y \times Y$ we get a metric on Y. Observe that $B_Y(y, r) := B(y, r) \cap Y$, where $B_Y(y, r) := \{z \in Y : d(z, y) < r\}$. The collection $\{B_Y(y, r) : y \in Y, r > 0\}$ is a family of basic open sets for $\mathcal{T}_Y$.

Ex. 57. Let $Y := [0, \infty) \subset \mathbb{R}$. Then the sets of the form $[0, x)$ with $x > 0$ are open in Y. In fact, the basic open sets are $[0, r)$, $r > 0$ and sets of the form (a, b), $0 < a < b$.

Ex. 58. Let $Y := \{(x, 0) : x \in \mathbb{R}\} \subset \mathbb{R}^2$. Then the sets of the form $(a, b) \times \{0\}$ are basic open sets.

Ex. 59. Let $S := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ be the unit circle in $\mathbb{R}^2$. The basic open sets in S are open arcs of the circle.

Ex. 60. Consider two circles in $\mathbb{R}^2$ which 'touch' (or, which are tangential) at the origin. Then the basic open sets around the origin are two arcs (through the origin) of the two circles.

Ex. 61. Consider $Y := \{(x, y) : xy = 0\} \subset \mathbb{R}^2$ be the two axes. Then the basic open sets near (0,0) are crosses (of two line segments along the x and y-axes.) At other points, just intervals around them.

Ex. 62. This is a generalization of Ex. 58. It requires the knowledge of product topology.

Let X and Y be topological spaces. We consider the product topology on $X \times Y$. Fix $y_0 \in Y$. Let $S := X \times \{y_0\}$. Then the basic open sets of S are of the form $U \times \{y_0\}$ where U is an arbitrary open set in X.

Ex. 63. Let $Y \subset X$ be open in X. Then $Z \subset Y$ is open in Y iff Z is open in X.

The result is not true if Y is not open in X.

Ex. 64. Let $Y \subset X$ be closed in X. Then $Z \subset Y$ is closed in Y iff Z is closed in X.

The result is not true if Y is not closed in X

Ex. 65. Let $A := \{1/n : n \in \mathbb{N}\} \cup \{0\}$. Then the basic open sets are the singletons $\{1/n\}$ for $n \in \mathbb{N}$ and $\{1/n : n \geq n_0\} \cup \{0\}$. The latter are basic opens sets near 0 in A.

Ex. 66. $\mathbb{Z} \subset \mathbb{R}$ has discrete topology as the subspace topology.

Ex. 67. The basic open sets in $\mathbb{Q}$ with the subspace topology from $\mathbb{R}$ are of the form $(a, b)_{\mathbb{Q}} := \{x \in \mathbb{Q} : a < x < b\}$ for $a, b \in \mathbb{R}$.

Is the collection $\{(a, b)_{\mathbb{Q}} : a, b \in \mathbb{Q}\}$ a family of basic open sets in $\mathbb{Q}$?

Ex. 68. Let $A \subset X$. If the subspace topology on A is the discrete topology on A, then every $a \in A$ is an *isolated* point in X, that is, there exists an open set $U_a \ni a$ in X such that $U_a \cap A = \{a\}$.

Ex. 69. Let $Y := \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ be the first quadrant in $\mathbb{R}^2$. Let $A := \{(x, y) \in Y : 0 \leq x < 1, 0 \leq y < 1\}$. Is A open in Y?

Ex. 70. Let $Y \subset X$. Let f be the restriction of the identity of X to Y. Then $f\colon (Y, \mathcal{T}_Y) \to (X, \mathcal{T})$ is continuous.

We can say more. If $\mathcal{T}'$ is a topology on Y such that $f\colon (Y, \mathcal{T}') \to (X, \mathcal{T})$ is continuous, then $\mathcal{T}_Y \subset \mathcal{T}'$. Thus, the subspace topology is the smallest topology w.r.t. which the natural inclusion map f is continuous.

Ex. 71. Let X and Y be topological spaces. Let $f\colon X \to Y$ be a (not necessarily continuous) map. Let $G(f) := \{(x, f(x)) : x \in X\} \subset X \times Y$ be the graph of f. Let $X \times Y$ be equipped with the product topology. Let $G(f) \subset X \times Y$ be given with the subspace topology. What are the basic open sets of $G(f)$?

E Generating Topologies — A Unified View of Subspace, Product and Quotient Topologies

Many constructions of new topologies in point set topology arise out of the following type of questions.

(i) Let X be a topological space and Y a nonempty set. Assume that a map $f\colon X \to Y$ is given. Can one find a topology $\mathcal{T}_Y$ such that the function $f\colon X \to (Y, \mathcal{T}_Y)$ is continuous.

(ii) The situation is reversed now. Assume that X is a nonempty set and Y a topological space. We are given a map $g\colon X \to Y$. We wish to endow X with a topology $\mathcal{T}_X$ so that $g\colon (X, \mathcal{T}_X) \to Y$ becomes continuous.

Both the questions have trivial answers. In case (i), we may take the topology on Y to be the indiscrete topology $\mathcal{T}_Y := \{\emptyset, Y\}$. In case (ii), we can endow X with the discrete topology. If we look at the questions a little more closely, it transpires that we need to ask for 'the largest topology' on Y in Case (i) while in Case (ii), we should go for 'the smallest topology' on X so that the respective maps (f or g) become continuous.

These questions and their generalizations occur very naturally in various contexts in topology. In abstract setup, we have 'very natural' maps from a set into another set and one of them is topological space. We are looking for an 'optimal' topology on the set with no topology. Let us look at two such natural instances.

Example 72. Let X be a topological space. Let $\sim$ be an equivalence relation on X. Let $Y := X/\sim$ be the set of equivalence classes of $\sim$ in X. We then have a natural map, namely, the quotient map $\pi : x \mapsto [x]$, that is, each $x \in X$ is mapped to its equivalence class.

Example 73. If X is a subset of a topological space Y, we have the natural map of inclusion of X into Y, namely, g is the restriction of the identity map Y to X. Here we are looking for the 'smallest topology' on X so that g is continuous.

We now return to answer the general questions posed above.

Case (i). If $\mathcal{T}_Y$ is a topology on Y so that f is continuous, then for any $V \in \mathcal{T}_Y$, the set $f^{-1}(V)$ must be open in X. This suggests us the following definition. We say a subset $V \subset Y$ is open if and only if $f^{-1}(V)$ is open in X. If we set

$$\mathcal{T}_Y := \{V \subset Y : f^{-1}(V) \text{ is open in } X\},$$

then it is easily verified that $\mathcal{T}_Y$ is a topology on Y such that $f\colon X \to (Y, \mathcal{T}_Y)$ is continuous. It is also the largest topology with this property. For, if $\mathcal{T}'_Y$ is another such topology and if $W \in \mathcal{T}'_Y$, then $f^{-1}(W)$ must be open in X. Hence, $W \in \mathcal{T}_Y$.

Case (ii). Reasoning as in Case (i), we arrive at the following.

$$\mathcal{T}_X := \{U \subset X : \text{ there exists an open set } V \subset Y \text{ such that } U = f^{-1}(V)\}.$$

It is easy to see that $\mathcal{T}_X$ is a topology on X such that g becomes continuous. It is the smallest topology with this property. For, if $\mathcal{T}'_X$ is another topology with this property, then for any open set $V \subset Y$, the set $f^{-1}(V)$ must be in $\mathcal{T}'_X$. Thus, any arbitrary subset of $\mathcal{T}_X$ belongs to $\mathcal{T}'_X$.

Let us apply these constructions to the specific examples cited above. In Example 1, according to our construction, a set $U \subset X$ is open iff there exists an open set $V \subset Y$ such that $U = g^{-1}(V)$. Since g is Id_Y restricted to X, we see that $g^{-1}(V) = V \cap X$. Thus the open sets in X are all of the open $V \cap X$, as V varies over all open sets in Y. This topology is known as the subspace topology of X!

In Example 2, if $\pi\colon X \to X/\sim$ is the quotient map, then $V \subset Y$ is open iff $\pi^{-1}(V)$ is open in X. This topology is known as the quotient topology on the quotient set $X/\sim$.

Now that we have created these new objects, how do we work with them? The answer is provided by the so-called universal mapping properties. Before stating them precisely, let us see what kind of working knowledge we need about these topologies. If X and Y are given as in Case (i), and we define the topology on Y as above, the natural questions would be: if Z is a topological space and if we are given a map h either from Z to Y or from Y to Z, how do we know the map h is continuous. The Universal mapping property gives a definitive answer to precisely one of the question by transferring the onus of proving the continuity of h to that of a 'natural composite' map. Note that if $h\colon Y \to Z$ is a map, then the composite $h \circ f\colon X \to Z$ makes sense. Universal mapping property of the topology on Y says that if $h\colon Y \to Z$ is a map, then h is continuous iff the natural composite $h \circ f\colon X \to Z$ is continuous. In Case (ii), the natural composite map is from $Z \to X$ followed by the given map g from X to Y. Universal mapping property of the topology on X says that if $h\colon Z \to X$ is a map, then h is continuous iff the natural composite $g \circ h\colon Z \to Y$ is continuous. These are easily proved and we prove these and more as Theorem after generalizing the Cases (i) and (ii). Note that we keep mum on the continuity $h\colon Z \to Y$ in Case (i) and that of $h\colon Z \to X$!

Thus we have dealt with both the Cases rather easily and satisfactorily. We can generalize these questions further and at least one of them is of immense interest.

Case I. Let $\{X_i : i \in I\}$ be a family of topological spaces and Y a nonempty set. Assume that we are given maps $f_i\colon X_i \to Y$ for each $i \in I$. We are interested in finding a topology on Y so that each of the maps f_i is continuous. As pointed out earlier, we need to look for the largest topology on Y with this property. Again arguing as in Case (i), we arrive at the following collection of sets which must be declared to be open sets in Y:

$$\mathcal{O} := \{V \subset Y : f_i^{-1}(V) \text{ is open for some } i \in I\}.$$

Unfortunately this time, this collection need not be a topology on Y! Before we see how to resolve this difficulty, let us consider

Case II. Let now X be a set and Y_i be topological spaces for $i \in I$. Let $f_i\colon X \to Y_i$ be given. We look for the smallest topology on X so that the maps f_i are continuous. Proceeding as earlier, we are led to conclude that any topology on X which makes f_i continuous must contain the collection

$$\mathcal{O} := \{U \subset X : f_i^{-1}(V_i) \text{ for some open set } V_i \subset Y_i, \text{ where } i \in I\}.$$

Once again there is no guarantee that this collection is a topology on X.

Thus, in both the cases, we are faced with the following type of problem. We have a set X and we have a collection $\mathcal{O}$ of subsets of X and we are looking for the smallest topology $\mathcal{T}$ on X which will contain $\mathcal{O}$. In theory, there is a smallest topology containing $\mathcal{O}$, namely, the intersection of all topologies which contain $\mathcal{O}$. (Note that $P(X)$, the power set of X is a topology on X which contains $\mathcal{O}$ and hence we are not taking the intersection over an empty collection!) What we would like to have is a practical way of dealing with this topology. Here is a neat description of the topology.

Definition 74. Given a collection $\mathcal{O} \subset P(X)$, we say a subset $U \subset X$ is open if given $x \in U$, there exists a finite collection $G_1, \dots, G_n$ of members of $\mathcal{O}$ such that

$$x \in G_1 \cap \cdots \cap G_n \subset U.$$

It is an easy exercise to show that the collection $\mathcal{T}$ of all open sets (according to this definition) is a topology on X which contains $\mathcal{O}$. It is also clear that it is the smallest topology that contains $\mathcal{O}$. This topology is said to be the topology generated by $\mathcal{O}$.

Remark 75. It is not hard to motivate this definition. We shall relegate the motivation to the end of this article. It is more important to see this construction of the topology in a concrete context.

Example 76. Let X_i be topological spaces, $i \in I$. Let $X := \prod_{i\in I} X_i$be the Cartesian product of the sets X_i. We then have natural maps $\pi\colon X \to X_i$, the projections on the i-th factor: $\pi_i(x) = x_i$. (Recall that

$$X := \{x\colon I \to \cup_{i\in I} X_i \text{ such that } x(i) \in X_i\}.$$

For $x \in X$, x_i stands for $x(i)$.) Then any set in the collection $\mathcal{O}$ of Case II is of the form $\pi_i^{-1}(U_i)$ where U_i is an open subset of X_i and $i \in I$. Note that $\pi^{-1}(U_i) := \prod_{j\in I} G_j$ where $G_j = X_j$ for $j \neq i$ and $G_i = U_i$. Thus a subset $U \subset X$ is open iff for each $x \in U$, we can find $i_1, \dots, i_n \in I$ and open sets $U_{i_k} \subset X_{i_k}$ for $1 \leq k \leq n$ such that

$$x \in \prod_j G_j \subset U, \text{ where } G_j = X_j \text{ for } j \neq i_k, \text{ and } G_{i_k} = U_{i_k}, 1 \leq k \leq n.$$

(Note that $\prod_j G_j = \cap_k \pi_{i_k}^{-1}(U_{i_k})$.) This topology on X is known as the product topology. To be able to work with this, we must have a universal mapping property of this topology. What is it? The only natural composite maps are of the form $Y \to X$ followed by $X \to X_i$. Thus, we may predict that $f\colon Y \to X$ is continuous iff $\pi_i \circ f\colon Y \to X_i$ are continuous for all $i \in I$. These adumbrations are formulated precisely in the next theorem and explained in detail in the couple of remarks that follow the theorem. The topologies constructed above in Cases (i)-(ii) and Case (II) are referred to as the topologies generated by the respective maps whose continuity was sought after.

Theorem 77 (Universal Mapping Property)**.**
(1.) *Let $f\colon X \to Y$ be a map from a set X to a topological space Y. Let X be given the topology generated by f. Then a function $h\colon Z \to X$ is continuous iff $f \circ h\colon Z \to Y$ is continuous.*

(2.) *Let $g\colon X \to Y$ be a map from a topological space X to a set Y. Let Y be endowed with the topology generated by g. Then a map $h\colon Y \to Z$ is continuous iff the map $h \circ g\colon X \to Z$ is continuous.*

(3.) *Let* $\pi_i \colon X \to X_i$ *be maps from the set* X *to topological spaces* X_i *for* $i \in I$. *Let* X *be given the topology generated by* π_i*'s. Then a map* $h \colon Y \to X$ *is continuous iff the maps* $\pi_i \circ h \colon Z \to X_i$ *are continuous.*

Proof. Let us prove (1) as a sample, as the proofs are all similar and easy. To prove the nontrivial part, let us assume that the map $f \circ h$ is continuous. Let $U \subset X$ be open. We need to show that $h^{-1}(U)$ is open in Z. By the very definition of the topology on X, there exists an open set $V \subset Y$ such that $U = f^{-1}(V)$. Now

$$h^{-1}(U) = h^{-1}(f^{-1}(V)) = (f \circ h)^{-1}(V),$$

which is open by the continuity of $f \circ h$.

To prove (2), let W be open in Z. We need to show that $h^{-1}(W)$ is open in Y. By the continuity of $h \circ g$, the set

$$(h \circ g)^{-1}(W) = g^{-1}(h^{-1}(W))$$

is open in X. By the definition of topology on Y, the subset $h^{-1}(W)$ is open.

To prove (3), we observe that it is enough to show that $h^{-1}(U)$ is open for

$$U \in \mathcal{O} := \{U : \pi_i^{-1}(V_i) \text{ for an open } V_i \subset X_i \text{ for some } i \in I\}.$$

(Prove this. Or, see Remark 82.) If $U = \pi_i^{-1}(V)$, we have $h^{-1}(U) := h^{-1}\left(\pi_i^{-1}(U)\right) = (\pi_i \circ h)^{-1}(U)$ is open, by the continuity of $\pi_i \circ h$. □

Remark 78. The most important thing to observe in the theorem is that the problem of establishing continuity of a map either from or to a newly constructed space is reduced to showing the continuity of a 'natural composite map' between the 'known spaces'. Go back to the statement and understand this remark. Also, go through the next remark.

Remark 79. Let us explicate the theorem in the concrete situations.

If $f \colon X \to Y$ is the inclusion map (that is, the restriction of the identity of Y to X) of a subset X into a topological space Y, then (1) of the theorem says that a map $h \colon Z \to X$ is continuous iff we think of h as a map form the space Z to Y (taking values only in X) is continuous.

Let X be a topological space, $\sim$ be an equivalence relation on X and $Y = X/\sim$ be the quotient set with the quotient map $\pi \colon X \to Y$. Then a map $h \colon Y \to Z$ form the quotient space Y to a space Z is continuous iff the map $h \circ \pi \colon X \to Z$ is continuous.

Let $X := \prod_{i \in I} X_i$ is the Cartesian product of topological spaces with the product topology as in Example 76. Then a map $h \colon Z \to X$ can be written as $h(z) = (h_i(z))$ where the *coordinate maps* $h_i(z) := \pi_i \circ h(z)$. Thus $h \colon Z \to X$ is continuous iff the coordinate maps h_i are continuous.

To complete the story we should say something about the way $\mathcal{B}$ was got out of $\mathcal{O}$. We start with the following definition.

Definition 80. Let $(X, \mathcal{T})$ be a topological space. Then a collection $\mathcal{B}$ of open sets is called a *base* for the topology if it satisfies the following two conditions:

(i) for each $U \in \mathcal{T}$ and $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$.
(ii) Given $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

A typical and motivating example: the family of open sets $\{B(x,r) : x \in X, r > 0\}$ is a base for the metric topology on X

This definition leads to the following question. Given a set X and a collection $\mathcal{B} \subset P(X)$ of subsets of X when is $\mathcal{B}$ a base for *some* topology on X? Since condition (i) of the definition is any way will be used to declare open sets, the decisive condition is (ii). We state this as a proposition.

Proposition 81. *Let X be a nonempty set and $\mathcal{B}$ be a collection of subsets of X. Then there exists a topology $\mathcal{T}$ on X for which $\mathcal{B}$ is a base if $\mathcal{B}$ satisfies the following condition:*

For all $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$.

Proof. Declare $U \subset X$ to be open if for each $x \in U$, there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. It is easy to check that the collection of open sets is a topology $\mathcal{T}$ on X and that $\mathcal{B}$ is a base for $\mathcal{T}$. □

If a family $\mathcal{O}$ of subsets of a set X is given and if we are looking for the smallest topology on X which contains $\mathcal{O}$, we need only find a family $\mathcal{B}$ which could be base for a topology on X. The family

$$\mathcal{B} := \{B_1 \cap B_2 \cap \cdots \cap B_n : B_i \in \mathcal{O}, n \in \mathbb{N}\}$$

is a base for some topology on X. Go back to Definition 74 and try to see how we side-stepped the intermediate construction of $\mathcal{B}$ and defined the topology straight away starting from $\mathcal{O}$.

Remark 82. If $h \colon Z \to X$ is any map from a topological space Z to X, to show that h is continuous, it suffices to verify that $h^{-1}(B)$ is open for $B \in \mathcal{O}$. If $B \in \mathcal{B}$ is of the form $B = B_1 \cap \cdots \cap B_n$ with $B_i \in \mathcal{O}$ then $h^{-1}(B) = \cap_i h^{-1}(B_i)$ is the intersection of a finite number of open sets and hence is open in Z. Now any open set U in X is an arbitrary union of members from $\mathcal{B}$. For, $U = \cup_{x \in U} B_x$ where for each $x \in U$, $B_x \in \mathcal{B}$ is chosen so that $x \in B_x \subset U$. Clearly, $h^{-1}(U) = \cup_x h^{-1}(B_x)$ is open in Z.

Remark 83. The universal mapping property is the most important to deal with the newly constructed topologies. For instance, in the case of quotient space $Y = X/\sim$, giving a map $h \colon Y \to Z$ is the same as giving a map $\tilde{h} \colon X \to Z$ which is constant on the equivalence classes. Hence the continuity of h is the same as that of $\tilde{h}$. As a concrete example, let $X = \mathbb{R}$ and $x \sim y$ iff $x - y \in \mathbb{Z}$. Let $S := \{z \in \mathbb{C} : |z| = 1\}$ be with the subspace topology. We have a natural map $\tilde{h} \colon X \to S$ given by $\tilde{h}(t) := e^{2\pi i t}$. Then $\tilde{h}$ gives rise to map $h \colon Y \to S$ which is a bijection from well-known properties of the exponential map. Also, h is continuous since $\tilde{h}$ is so. Since any continuous map from a compact space to a Hausdorff space is a closed map, h is a homeomorphism. Thus Y is the circle in $\mathbb{C}$! For more such applications, see my article on Quotient spaces.

F Quotient Topology

This article is devoted to the mathematical formulation of gluing geometric objects to get new geometric objects. For example, one may form a circle from a closed line segment by bending it around and gluing the ends together. Or, one can form a cylinder from a rectangle by bending the rectangle around and gluing two opposite sides together. If we further bend the cylinder around and glue the two circular rims together we get a torus or a cycle tube. In this article, we concentrate on some of the very basic results of the theory which will enable the reader to deal with quotient spaces with confidence. The theory is full of pathologies and often text-books and teachers tend to frighten the beginner with the macabre rather than emphasizing the positive aspects and initiating him into a working knowledge of quotient spaces. This article attempts to make it easy for a student to learn quotient spaces.

Let X be a set and $\sim$ be an equivalence relation on X. Let $X/\sim$ be the quotient set or the set of equivalence classes of $\sim$. Let $\pi\colon X \to X/\sim$ be the *quotient map* defined by $\pi(x) = [x]$, the equivalence class of x. If we further assume that X is a topological space, we then want to introduce a topology on the quotient set so that the quotient map π is continuous. Note that the indiscrete topology on $X/\sim$ will be one such. However we would like to have the largest possible topology on $X/\sim$ with this property. If τ is such a topology and V is open in $X/\sim$, then $\pi^{-1}(V)$ must be open in X. This suggests the following

Definition 84. With the notation as above, we define τ to be the set of $V \subset X/\sim$ such that $\pi^{-1}(V)$ is open in X. It is easy to check that τ is indeed a topology on the quotient set. The space $(X/\sim, \tau)$ is called the quotient space of X relative to the equivalence $\sim$.

We record the following fact which is an immediate consequence of the definition of the quotient topology.

Proposition 85. *Let X be a topological space and $\sim$ an equivalence relation on X. Then the quotient topology on $X/\sim$ is the largest topology for which the natural quotient map $\pi\colon X \to X/\sim$ is continuous.* □

The next theorem, though easy, is quite often used to check the continuity of maps from quotient spaces to others.

Theorem 86 (Universal Mapping Property)**.** *Let $\pi\colon X \to X/\sim$ be a quotient map. A map $f\colon X/\sim \to Y$ is continuous iff $f \circ \pi$ is continuous.*

Proof. If f is continuous, then certainly so is $f \circ \pi$. To prove the converse, let V be an open set of Y. Then $(f \circ \pi)^{-1}(V) = \pi^{-1}(f^{-1}(V))$ is an open subset of X. By the definition of quotient topology, $f^{-1}(V)$ is open set in $X/\sim$. Hence f is continuous. □

The next theorem tells us how to generate quotient spaces.

Theorem 87. *Let $f\colon X \to Y$ be continuous. Let $\sim$ be the equivalence relation on X defined by $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. Then there exists a continuous function $g\colon X/\sim \to Y$ such that $f = g \circ \pi$.*

Proof. Define $g([x]) = f(x)$ for any $x \in [x] \in X/\sim$. Then g is well-defined and $g \circ \pi = f$. Thm. 86 assures the continuity of g. □

The following result allows us to identify quotient spaces with other concrete spaces.

Theorem 88. *Let X and Y be compact. Assume further that Y is hausdorff. Let $f\colon X \to Y$ be a surjective continuous map. Define the equivalence relation $\sim$ by declaring $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. Then $X/\sim$ is homeomorphic to Y.*

Proof. $X/\sim$ is compact, being the image of X under the continuous map π. Let $g\colon X/\sim \to Y$ be the continuous function defined by $g([x]) = f(x)$. Then g is continuous, 1-1 and onto. Hence by Exer. 89 g is a homeomorphism. □

Ex. 89. Let $f\colon X \to Y$ be a 1-1 continuous map from a compact space to a Hausdorff space. Then f is a homeomorphism of X onto $f(X)$.

We give four simple applications to illustrate the power of Thm. 88.

Example 90. Consider the quotient space obtained from $[0, 1]$ got by identifying the end points 0 and 1. That is, Y is the space $X/\sim$ where $X := [0, 1]$ and the equivalence classes are $\{t\}$ for $0 < t < 1$ and $\{0, 1\}$. Consider the map $f\colon [0, 1] \to S^1$ given by $f(t) := e^{2\pi it}$. Y and S^1 are homeomorphic by Thm. 88.

Example 91. We show that the quotient space got by identifying two of the opposite sides of a rectangle is homeomorphic to a cylinder. Let $X := \{(u, v) \in \mathbb{R}^2 : -\pi \le u \le \pi, -1 \le v \le 1\}$. Let $Y := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, |z| \le 1\}$. On X we define the equivalence relation by setting $(u, v) \sim (u', v')$ iff $u = \pm\pi = u'$ and $v = v'$ if $u = \pm\pi = u'$ and otherwise $u = u'$ and $v = v'$. Consider the map $f\colon X \to Y$ given by $f(u, v) := (\cos u, \sin u, v)$. Then the level sets of f are precisely the equivalence classes. f induces a homeomorphism via Thm. 88.

Example 92. Let $X := [0, 1]\times[0, 1]$ and $f\colon X \to S^1\times S^1$ be defined by $f(s, t) := (e^{2\pi is}, e^{2\pi it})$. The level sets of f are the singletons inside the square, the pairs of opposite points on the interiors of the bounding intervals and the set of four vertices of X. Thus we see that the quotient space of a square obtained from the equivalence is a torus.

For any space X and a subset A of X, the space X/A stands for the quotient space of X with respect to the equivalence: $x_1 \sim x_2$ iff $x_1 = x_2$ or $x_1, x_2 \in A$. Thus X/A is the space obtained from X by collapsing A to a single point.

Example 93. Consider the map $f\colon B[0, 1] \subset \mathbb{R}^n \to S^n \subset \mathbb{R}^{n+1}$ defined by setting

$$f(tx) := (\cos\pi(1 - t), x_1 \sin\pi(1 - t), \dots, x_n \sin\pi(1 - t)),$$

where $x \in S^{n-1}$ and $0 \le t \le 1$. Then $f(S^{n-1}) = e_0 = (1, 0, \dots, 0) \in S^n$. f induces a homeomorphism of $B[0, 1]/S^{n-1}$ with S^n. See also Exer. 95 below.

Ex. 94. Let F be a closed subset of a compact Hausdorff space X. Prove that the quotient space obtained from X by identifying F to a single point is homeomorphic to the one-point compactification of $X \setminus F$.

Ex. 95. Let B be the closed unit ball in $\mathbb{R}^n$. Prove that the quotient space obtained from B by identifying its boundary S^{n-1} to a point is homeomorphic to the n-sphere.

Ex. 96. Let X denote the union of circles (in $\mathbb{R}^2$) centred at $(0, 1/n)$ and of radius $1/n$ with the subspace topology of $\mathbb{R}^2$. Let Y denote the quotient space $\mathbb{R}/\mathbb{Z}$ obtained from $\mathbb{R}$ by collapsing all of $\mathbb{Z}$ to a single point. Show that X and Y are not homeomorphic.

Thm. 88 is a special case of the next theorem which can be used the same way as the former was used in the examples above. Thm. 86, Thm. 88 and Thm. 97 are thus the most useful results in practice.

Theorem 97. *Let $f: X \to Y$ be an open (or closed) continuous surjective map. Then Y is homeomorphic to the quotient space of X obtained by identifying each level set of f to a point.*

Proof. Argue as in Thm. 88. □

The following exercises introduce some of the important quotient spaces. They will help the reader understand the concept of quotient spaces well.

Ex. 98 (Real Projective Spaces)**.** Let $\mathbb{P}^n(\mathbb{R})$ be the n-dimensional projective space over $\mathbb{R}$. It is the quotient space obtained from S^n with respect to the equivalence relation $x \sim y$ iff $x = \pm y$. Prove the following:
(a) $\mathbb{P}^n(\mathbb{R})$ is a compact Hausdorff space.
(b) The projection $\pi: S^n \to \mathbb{P}^n(\mathbb{R})$ is a local homeomorphism, that is, each $x \in S^n$ has an open neighbourhood which is mapped homeomorphically onto an open neighbourhood of $\pi(x)$ by π.
(c) $\mathbb{P}^1$ is homeomorphic to S^1.
(d) $\mathbb{P}^n(\mathbb{R})$ is homeomorphic to the quotient space obtained from the closed unit ball B in $\mathbb{R}^n$ by identifying the antipodal points of its boundary S^{n-1}.
(e) On $X := \mathbb{R}^{n+1} \setminus \{0\}$ introduce the equivalence relation $x \sim y$ iff there is a nonzero $\alpha \in \mathbb{R}$ such that $x = \alpha y$. Show that $X/\sim$ is homeomorphic to $\mathbb{P}^n(\mathbb{R})$. (Thm. 97 may be of use here.)

Ex. 99 (Complex Projective Spaces)**.** Think of $X := S^{2n+1}$ as a subset of $\mathbb{C}^{n+1}$:

$$S^{2n+1} = \{z \in \mathbb{C}^{n+1} : \sum_i |z_i|^2 = 1\}.$$

Define an equivalence relation on X by declaring that $z \sim w$ iff there exists a $\lambda \in S^1 \subset \mathbb{C}$ such that $z = \lambda w$. The resulting quotient space, denoted by $\mathbb{P}^n(\mathbb{C})$ is called the n-dimensional complex projective space. Prove the following:
(a) $\mathbb{P}^n(\mathbb{C})$ is a compact Hausdorff space.
(b) $\mathbb{P}^1(\mathbb{C})$ is homeomorphic to S^2.
(c) Let π denote the projection $\pi: X \to \mathbb{P}^n(\mathbb{C})$. Show that $\pi^{-1}(x)$ is homeomorphic to S^1 for all $x \in \mathbb{P}^n(\mathbb{C})$. Show that each $x \in \mathbb{P}^n(\mathbb{C})$ has a neighbourhood U such that $\pi^{-1}(U)$ is homeomorphic to $U \times S^1$.
(d) Let $Y := \mathbb{C}^{n+1} \setminus \{0\}$. Let $\sim$ be the equivalence relation defined by $x \sim y$ iff there exists a nonzero $\lambda \in \mathbb{C}$ such that $x = \lambda y$. Then the quotient space $Y/\sim$ is homeomorphic to $\mathbb{P}^n(\mathbb{C})$. Thus $\mathbb{P}^n(\mathbb{C})$ can be regarded as the set of one-dimensional subspaces of $\mathbb{C}^{n+1}$.

Quotient spaces are full of pathologies. It is necessary to recognize which of the topological properties of X are inherited by $X/\sim$ and which are not. The following exercises deal with this concern.

Ex. 100. Let X be a topological space and $X/\sim$ be its quotient with respect to an equivalence relation. Prove the following:
(a) If X is compact, so is $X/\sim$.
(b) If X is connected, so is $X/\sim$.
(c) If X is path connected, so is $X/\sim$.

Ex. 101. Pathologies
(a) Let $\sim$ be an equivalence relation on a space X. Prove that $X/\sim$ is a T_1-space iff each equivalence class is closed. Give an example of a T_1 space X and a quotient space of X which is not T_1.
(b) Define an equivalence relation on $X := [0,1] \times [0,1]$ by setting $(s,t) \sim (s',t')$ iff $t = t' > 0$. Describe the quotient space and show that it is not Hausdorff.

One of the most important ways of defining equivalence relations is by means of group actions. So it should not be a surprise that many quotient spaces arise as the quotients of group actions on spaces. Below we indicate some of these instances.

Definition 102. We say a group G acts on a space X if there is a map $\varphi\colon G \times X \to X$ with the following properties (Below we write $g \cdot x$ for $\varphi(g,x)$):
(i) $e \cdot x = x$ for the identity $e \in G$ and $x \in X$.
(ii) For $g, h \in G$ and $x \in X$, we have $(gh) \cdot x = g \cdot (h \cdot x)$.

Note that these conditions mean that for each $g \in G$ the map $\varphi_g : x \mapsto g \cdot x$ is a homeomorphism of X onto itself. Thus we have a group homomorphism of G into the group of homeomorphisms of X.

We say X is a G-space if an action of G on X is given.

On any G-space, we have a natural equivalence: $x \sim y$ iff there exists a $g \in G$ such that $y = g \cdot x$. The equivalence classes are called the *orbits* of G, for, $[x] \equiv \{g \cdot x : g \in G\}$. The corresponding quotient space $X/\sim$ is denoted by X/G.

Ex. 103. Let $X = \mathbb{R}$ and $G = \mathbb{Z}$. Let G act on X by $n \cdot x = x + n$. Then the quotient space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1.

Ex. 104. Let $X = \mathbb{R}^2$ and $G = \mathbb{Z}$. G acts on X by $n \cdot (x,y) = (x+n, y)$. Show that the resulting quotient space is the infinite cylinder $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$.

Ex. 105. Let $X = S^n$ and $G = \mathbb{Z}_2$, the multiplicative group of two elements. If -1 is the nontrivial element of G, then define $-1 \cdot x := -x$. Then X/G is $\mathbb{P}^n(\mathbb{C})$.

In a similar way, if we let $G := \mathbb{R}^*$, the multiplicative group of nonzero reals act on $X := \mathbb{R}^{n+1} \setminus \{0\}$ via scalar multiplication, then X/G is the n-dimensional real projective space.

Ex. 106. Let $X := S^{2n+1} \subset \mathbb{C}^{n+1}$. Let $S^1 \subset \mathbb{C}$ act on X by

$$e^{it} \cdot (z_1, \ldots, z_{n+1}) := (e^{it} z_1, \ldots, e^{it} z_{n+1}).$$

The quotient S^{2n+1}/S^1 is homeomorphic to $\mathbb{P}^n(\mathbb{C})$.

Let $Y := \mathbb{C}^{n+1} \setminus \{0\}$. Let $G := \mathbb{C}^*$, the multiplicative group of complex numbers act on Y via scalar multiplication. The quotient Y/G is $\mathbb{P}^n(\mathbb{C})$.

Ex. 107. Let $X = \mathbb{R}^2$ and $G = \mathbb{Z}^2$. The action is $(m,n) \cdot (x,y) = (x+m, y+n)$. The quotient $\mathbb{R}^2/\mathbb{Z}^2$ is homeomorphic to the torus in $\mathbb{R}^3$ got by revolving around the z-axis a circle of unit radius centered at $(2,0,0)$ and of radius 1 in the (x,z)-plane. *Hint*: Consider the map $(u,v) \mapsto ((2+\cos 2\pi u)\cos 2\pi v, (2+\cos 2\pi u)\sin 2\pi v, \sin 2\pi u)$.

Ex. 108 (Möbius Strip)**.** On the unit square X we define the equivalence relation as follows:

$$(x,y) \sim (x',y') \iff (x,y) = (x',y') \text{ or } \{x,x'\} = \{0,1\} \text{ and } y = 1-y'.$$

Thus two points of opposite vertical sides are identified *cross-wise*. The quotient space is known as the Möbius strip.

Let $Y := \{(x,y) \in \mathbb{R}^2 : -1/2 \leq y \leq 1/2\}$. Let $\mathbb{Z}$ act on Y by $m \cdot (x,y) := (m+x, (-1)^m y)$. Show that the quotient space $Y/\mathbb{Z}$ is homeomorphic to the Möbius strip.

Ex. 109 (Klein Bottle)**.** Let X be the unit square. Define an equivalence relation on X whose nontrivial relations are given by

$$(0,y) \sim (1,y) \quad \text{and} \quad (x,0) \sim (1-x,1).$$

The quotient space is called the *Klein's bottle*.

Let $Y = \mathbb{R}^2$. Let $\varphi, \psi \colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\varphi(x,y) := (x+1, y), \qquad \psi(x,y) := (1-x, y+1).$$

Thus φ is a translation parallel to the x-axis and ψ is a glide reflection along the line $x = 1/2$. Show that φ and ψ are homeomorphisms of $\mathbb{R}^2$, $\psi \circ \varphi = \varphi^{-1} \circ \psi$ so that $G := \{\varphi^m \psi^{2n} \psi^\epsilon : m,n \in \mathbb{Z}, \epsilon \in \{0,1\}\}$ is a group of homeomorphisms of $\mathbb{R}^2$. Show that Y/G is homeomorphic to the Klein bottle.

Ex. 110. Can you identify the quotient spaces X/G?
(a) Let $S^1 \subset \mathbb{C}$ act on S^2 via rotations about the z-axis.
(b) Let $\mathbb{Z}_n$ act on S^2 via rotations by an angle which is a multiple of $2\pi/n$.
(c) Let $O(n)$, the orthogonal group, act on $\mathbb{R}^n$ via the usual linear action.

Ex. 111 (Lens Spaces)**.** Consider $S^3 \subset \mathbb{C}^2$. Let p and q be relatively prime integers. Let a generator $g \in \mathbb{Z}_p$ act on S^3 by $g \cdot (z_1, z_2) := (e^{2\pi i/p} z_1, e^{2\pi q i/p} z_2)$. The quotient space is denoted by $L(p,q)$ and called a Lens space. Show that $L(2,1)$ is homeomorphic to $\P^3(\mathbb{R})$. If p divides $q - q'$, then $L(p,q)$ is homeomorphic to $L(p,q')$.

If $L(p,q)$ and $L(p',q')$ are homeomorphic then $p = p'$. *Hint:* What is the fundamental group of X/G if X is simply connected and G acts properly discontinuosly? (G acts *properly discontinuously* on X if for any $x \in X$ there exists a neighbourhood U of x such that $g \cdot U \cap U = \emptyset$ for every $g \neq e$ in G.)

Ex. 112. Let $X := \mathbb{R}^n \setminus \{0\}$. Fix any real number $\alpha \notin \{0, \pm 1\}$. Let $G := \mathbb{Z}$ act on X by $m \cdot x := \alpha^m x$. Show that G acts properly discontinuously. Identify the quotient X/G.

Hausdorff Quotient Spaces

Definition 113. Let $\sim$ be an open equivalence on a space X. For any set $A \subset X$, we let $[A]$ stand for the set of all elements $x \in X$ which are equivalent to some element of A. The equivalence is called *open* if $[A]$ is open whenever A is open in X.

Ex. 114. An equivalence relation $\sim$ on a topological space is open iff the quotient map $\pi \colon X \to X/{\sim}$ is open. *Hint:* Observe that $[A] = \pi^{-1}(\pi(A))$.

Proposition 115. *Let $\sim$ be an equivalence on a topological space X. Then $R := \{(x, y) : x \sim y\}$ is closed in $X \times X$ iff the quotient space $X/{\sim}$ is hausdorff.*

Proof. Assume that $X/{\sim}$ is hausdorff and that $(x, y) \notin R$. Then there exist disjoint neighbourhoods U of $\pi(x)$ and V of $\pi(y)$. We denote by $\tilde{U}$ and $\tilde{V}$ the open sets $\pi^{-1}(U)$ and $\pi^{-1}(V)$, which contain x and y respectively. If the open set $\tilde{U} \times \tilde{V}$ containing (x, y) intersects R, then it must contain a point (x', y') for which $x' \sim y'$, so that $\pi(x') = \pi(y')$, contrary to the assumption that $U \cap V = \emptyset$. This contradiction shows that $\tilde{U} \times \tilde{V}$ does not intersect R. Hence R is closed.

Conversely, suppose that R is closed. Given any distinct pair of points $\pi(x)$ and $\pi(y)$ in $X/{\sim}$, there is an open set of the form $\tilde{U} \times \tilde{V}$ containing (x, y) and having no points in R. It follows that $U := \pi(\tilde{U})$ and $V := \pi(\tilde{V})$ are disjoint. Exer. 114 and hypothesis imply that U and V are open. Thus $X/{\sim}$ is hausdorff. $\square$

Example 116. We show a typical application of the last result by proving that $\mathbb{P}^n(\mathbb{R})$ is hausdorff. Let $X = \mathbb{R}^{n+1} \setminus \{0\}$. Let the equivalence be as in (e) of Exer. 98. We first show that $\pi \colon X \to \mathbb{P}^n(\mathbb{R})$is open. If $\alpha \in \mathbb{R}$ is nonzero, the map $\varphi_\alpha \colon X \to X$ given be $\varphi_\alpha(x) = \alpha x$ is a homeomorphism. If $U \subset X$ is open, then $[U] = \cup \varphi_\alpha(U)$ where the union is taken over all nonzero reals. Since each $\varphi_\alpha(U)$ is open, their union $[U]$ is also open. By Exer. 114, π is open.

We now show that $\mathbb{P}^n(\mathbb{R})$ is hausdorff. Consider the function $f \colon X \times X \to \mathbb{R}$ is given by $f(x, y) := \sum_{i \neq j}(x_i y_j - x_j y_i)^2$. Then f is continuous and vanishes iff $y = \alpha x$ for some nonzero real α, that is, iff $x \sim y$. Thus $R = \{(x, y) : x \sim y\} = f^{-1}(0)$ is a closed subset of $X \times X$. By Prop. 115, $\mathbb{P}^n(\mathbb{R})$ is hausdorff.

Collapsing and Attaching

We now discuss two construction which are very important Algebraic Topology and which arise as quotient spaces.

Example 117. Let A be a nonempty subset of a topological space. Define $\sim$ to be the equivalence relation on X that identifies all points of A with each other:

$$x \sim y \iff (x = y) \text{ or } (x \in A \text{ and } y \in A).$$

The points of the quotient set $X/\sim$ are the singletons $\{x\}$ for $x \notin A$ and the distinguished point A. The quotient space is most often denoted by X/A. One says that it is obtained by *collapsing* A to a single point.

Let $\pi\colon X \to X/A$ be the quotient map. Then π maps the open subspace $x \setminus A$ of X onto the complement $(X \setminus A) \setminus \{A\}$ of the special point A of X/A. Thus the complement of A in X/A is open, as its inverse image under π is $X \setminus A$.

In fact, π induces a homeomorphism $X \setminus A \cong (X/A) \setminus \{A\}$. Note that the continuous map π is one-one on $X \setminus A$. If $U \subset (X \setminus A)$ is open in $X \setminus A$, then $\pi^{-1}(\pi(U)) = U$ is open in X. Therefore, $\pi(U)$ is open in X/A. We thus see that X/A contains a point whose complement is the same as $X \setminus A$ topologically.

We now consider a specific example. Let $X = B[0,1] \subset \mathbb{R}^2$, the closed unit disc in the plane and $A = S^1$, its boundary. Can you imagine what the quotient space look like? Imagine a circular piece of rubber with a drawstring along its boundary. When the string is drawn tight, a kind of spherical bag results. Therefore, we should expect $X/A \cong S^2$, the unit sphere in $\mathbb{R}^3$.

How do we prove this rigorously? Let p be the north pole of S^2. Geometrically thinking, can we find a map $f\colon X \to S^2$ that sends each point of S^1 to p and maps $X \setminus S^2$ injectively onto $S^2 \setminus \{p\}$? The induced map on X/A would be as required. Look at the closed unit disk $D := \{(x,y,z) \in \mathbb{R}^2 : x^2 + y^2 \leq \pi, z = -1\}$ in the plane tangent to S^2 at the south pole $-p$. We wrap it S^2 by wrapping each radial line segment in this disk onto a meridian of S^2. If you still remember cylindrical and spherical coordinates, what we plan to do is to send a point $P \in D$ with cylindrical coordinates $(r, \theta, -1)$ to a point on S^2 whose spherical coordinates are $(1, \pi - r, \theta)$. Thus f will be the composite of two maps: one is the homeomorphism $(x,y) \mapsto (\pi x, \pi y, -1)$ of X onto D and the second is as described earlier. Since the resulting map is from a compact space to a Hausdorff space, it is closed. It is clearly continuous.

Example 118. Let X and Y be topological spaces and $A \subset X$ be nonempty and closed. Let $f\colon A \to Y$ continuous. Imagine joining X and Y together by gluing each point $a \in A$ to $f(a) \in Y$. The resulting space should be a topological space Z which contains (homeomorphic) copies of $X \setminus A$ and Y in which each $y \in f(A)$ represents an identification of all $a \in f^{-1}(y)$ with y. Our aim is to construct such a space.

Before doing this, we need the notion of sum of two topological spaces. Consider two topological spaces X and Y. Assume that as sets they are disjoint. Consider the union $Z = X \cup Y$. We wish to endow a topology on Z in an obvious manner: call $W \subset Z$ open iff $W \cap X$ and $W \cap Y$ are open in X and Y respectively. Note that this is the same as saying any open set $W \subset Z$ is of the form $W = U \cup V$ with U and V being open in X and Y respectively. One easily shows that this defines a topology on Z in such a way that both X and Y are open subspaces in Z. The space Z is called the *topological sum* of X and Y

If X and Y are not disjoint, we resort to a standard trick. In stead of X and Y, we consider $X_1 := X \times \{1\}$ and $Y := Y \times \{2\}$. The open sets of X_1 are $U \times \{1\}$. Similarly, we define open sets in Y_1. Clearly, $X_1 \cong X$ and $Y_1 \cong Y$. Let Z be the topological sum of X_1 and Y_1. The space Z is denoted as $X_1 + Y_1$. Via the maps $x \mapsto (x,1)$ and $y \mapsto (y,2)$, we see that X and Y are open subsets of Z.

Let us now return to our original notation. Let us first assume that X and Y are disjoint. Then the space $X + Y$ contains both X and Y as open, closed subspaces. We define an equivalence relation $\sim$ on $X+Y$ as follows:

$$u \sim v \iff (u = v) \text{ or } (u \in A \ \&\ v = f(u)) \text{ or } (v \in A \ \&\ u = f(v)) \\ \text{or } (u \in A \ \&\ v \in A \ \&\ f(u) = f(v)).$$

We let $Z = (X+Y)/\sim$. The standard notation for Z is $X \cup_f Y$. We say that $X \cup_f Y$ is obtained by *attaching* X to Y by f. The map f is called the attaching map.

We now show that $X \cup_f Y$ has the desired properties. Let $\pi\colon X+Y \to X \cup_f Y$ be the quotient map. Clearly, $\pi(a) = \pi(f(a))$ for any $a \in A$. Also, we have

$$X \cup_f Y = \pi(X) \cup \pi(Y) = \pi(X \setminus A) \cup \pi(A) \cup \pi(Y),$$

as well as

$$\pi(X) \cap \pi(Y) = \pi(A) = \pi(f(A)).$$

We make the following claims: (i) $\pi(Y)$ is closed in $X \cup_f Y$ and π maps Y homeomorphically onto $\pi(Y)$, (ii) $\pi(X \setminus A)$ is open in $X \cup_f Y$ and π maps $X \setminus A$ homeomorphically onto $\pi(X \setminus A)$.

Proof of Claim (i): The set $\pi(Y)$ is closed since $\pi^{-1}(\pi(Y)) = A \cup Y$ is closed in $X+Y$. The restriction of π to Y is continuous and injective. If F is closed in Y, then $\pi(F)$ is closed in $\pi(Y)$ since $\pi^{-1}(\pi(F)) = A \cup F$ is closed in $X+Y$. This proves (i). The proof of (ii) is similar.

We now take up a specific example. Let $X = [0,1]$ and $A = \{0,1\}$. Let $Y = B[0,1]$, the closed unit disk in $\mathbb{R}^2$. Let $f\colon A \to Y$ be such that $f(0) = f(1) = 0 \in Y$. Can you imagine the space $X \cup_f Y$? It looks line a circle touching a unit disk tangentially at the origin of the disk. More precisely, we show that $X \cup_f Y$ is homeomorphic to the subspace

$$Z = \{(x,y,z) \in \mathbb{R}^3 : z = 0, x^2+y^2 \le 1\} \cup \{(x,y,z) \in \mathbb{R}^3 : x = 0, y^2 + (z-1)^2 = 1\}.$$

See Figure??? To complete the proof, consider the map $g\colon X+Y \to Z$ given by

$$\begin{aligned} g(x) &= (0, \sin 2\pi x, 1 - \cos 2\pi x), \text{ for } x \in X, \\ g(y) &= (y_1, y_2, 0), \text{ for } y \in Y. \end{aligned}$$

The general case is done considering X_1 and Y_1 as earlier and form their sum. Let $h\colon X \to X_1 + Y_1$ and $k\colon Y \to X_1 + Y_1$ be the imbeddings defined earlier. Define an equivalence relation on $X_1 + Y_1$ using these imbeddings and so on. The details are left to the reader.

G Tychonoff's Theorem

Theorem 119 (Tychonoff)**.** *The product of compact spaces is compact. That is, if $X = \prod X_\alpha$ where each X_α is compact, X is compact.*

Proof. Let $\mathcal{F}_0$ be a family of closed sets in X with the finite intersection property (f.i.p). We shall show that there is a point common to all the sets $F \in \mathcal{F}_0$.

We apply Zorn's lemma to get a family $\mathcal{F} \subseteq \mathcal{F}_0$ of (not necessarily closed) sets in X with finite intersection property: Two families $\mathcal{F}$ and $\mathcal{G}$ are related iff $\mathcal{F} \subseteq \mathcal{G}$. Now let $\mathcal{C}$ be any totally ordered chain of families with finite intersection property. That is, if there exists $\mathcal{F}, \mathcal{G} \in \mathcal{C}$, then either $\mathcal{F} \subseteq \mathcal{G}$ or $\mathcal{G} \subseteq \mathcal{F}$. This chain has an upper bound, viz., $\mathcal{H} = \cup_{\mathcal{F}\in\mathcal{C}}\mathcal{F}$, where $\mathcal{H}$ has the finite intersection property. To see that $\mathcal{H}$ has the finite intersection property, let $A_1, \ldots, A_n \in \mathcal{H}$. Then there exists $\mathcal{F}_j \in \mathcal{C}$ such that $A_j \in \mathcal{F}_j \in \mathcal{C}$. Since $\mathcal{C}$ is totally ordered, and $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are finite in number, there exists k with $1 \leq k \leq n$ such that $\mathcal{F}_j \subseteq \mathcal{F}_k$ for all k. But then $A_1, \ldots, A_n \in \mathcal{F}_k$ and $\mathcal{F}_k$ has the finite intersection property. Hence $A_1 \cap \cdots \cap A_n \neq \emptyset$.

Hence by Zorn's lemma, there exists a maximal family $\mathcal{F} \in \mathcal{C}$, with $\mathcal{F} \supseteq \mathcal{F}_0$. Let $\mathcal{F}^\alpha$ denote $\{E^\alpha := P_\alpha(E), \quad E \in \mathcal{F}\}$. Then $\mathcal{F}_\alpha \subseteq P(X_\alpha)$ has the finite intersection property, (here $P_\alpha \colon X \to X_\alpha$ is the canonical projection map). For otherwise, $E_1^\alpha \cap \cdots \cap E_n^\alpha = \emptyset$ will imply $E_1 \cap \cdots \cap E_n = \emptyset$, where $P_\alpha(E_i) = E_i^\alpha$. Hence, $\overline{\mathcal{F}^\alpha} = \{\overline{E^\alpha}\}$ has finite intersection property.

Since X_α is compact, there exists $x_\alpha \in \cap\overline{E^\alpha}$ where the intersection is over all $E^\alpha \in \mathcal{F}^\alpha$. Let $x \in \prod X_\alpha$ be such that $x(\alpha) := x_\alpha$.

Claim: $x \in \cap_{F\in\mathcal{F}}\overline{F}$.

Since $\mathcal{F} \supseteq \mathcal{F}_0$, the claim completes the proof of the theorem.

Proof of the Claim:

Let U be an open set in X. By definition of product topology, there exists $\alpha_1, \ldots, \alpha_n$ and open sets $U_{\alpha_i} \subseteq X_{\alpha_i}$, $1 \leq i \leq n$ such that $x \in \cap_{i=1}^n P_{\alpha_i}^{-1}(U_{\alpha_i}) \subseteq U$ with $x \in \cap_{i=1}^n P_{\alpha_i}^{-1}(U_{\alpha_i})$. This implies $x_{\alpha_i} \in U_{\alpha_i}$ for all i. By hypothesis on x_α's, $x_{\alpha_i} \in \overline{F}_{\alpha_i}$ for all $F_{\alpha_i} \in \mathcal{F}^{\alpha_i}$. That is, $U_{\alpha_i} \cap \overline{F}_{\alpha_i} \neq \emptyset$, for all $F_{\alpha_i} \in \mathcal{F}^{\alpha_i}$. Hence $P_{\alpha_i}^{-1}(U_{\alpha_i})$ has a non-empty intersection with every $F \in \mathcal{F}$. Therefore $P_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$ (otherwise $\mathcal{F} \cup \{P_{\alpha_i}^{-1}(U_{\alpha_i})\} \supset \mathcal{F}$ and the former has finite intersection property, contradicting the maximality of $\mathcal{F}$). This being true for all i, and $\mathcal{F}$ has finite intersection property, it follows that $\cap_{i=1}^n P_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$. Since $\mathcal{F}$ has the finite intersection property, this basic open set and hence U intersects each member of $\mathcal{F}$ non-trivially. Since U was an arbitrary open neighborhood of x, this means that $x \in \overline{F}$, for all $F \in \mathcal{F}$. Hence the claim. □

H Compact Spaces

H.1 Heine-Borel Theorem

Definition 120. A family $\{U_\alpha : \alpha \in I\}$ of open sets in a topological space X is said to be an *open cover* of a subset $K \subset X$ if $K \subset \cup_\alpha U_\alpha$.

We say that a subset $K \subseteq X$ of topological space X is *compact* if for any given open cover $\{U_\alpha : \alpha \in I\}$, we can find a finite subset $F \subset I$ such that $K \subset \cup_{\alpha\in F} U_\alpha$. We then say the given open cover admits a finite subcover.

Lemma 121. *Any compact subset of a hausdorff space is closed.*

Proof. Let K be a compact subset of a hausdorff space X. Let $z \in X$ be a limit point of K in X. If $z \notin K$, then by Hausdorff property of X, for each $x \in K$, we can find open sets $U_x \ni x$ and $V_x \ni z$ such that $U_x \cap V_x = \emptyset$. The collection $\{U_x : x \in K\}$ is an open covering of the compact set K and hence it admits a finite subcover, say, $U_i := U_{x_i}$, $1 \leq i \leq n$. If we let $V := \cap_{i=1}^n V_{x_i}$, then V is an open set containing z. Also, we observe that

$$V \cap K \subseteq V \cap (U_1 \cup \cdots \cup U_n) = \cup_{i=1}^n (V \cap U_i) \subseteq \cup_{i=1}^n (V_{x_i} \cap U_{x_i}) = \emptyset.$$

This contradicts the fact that z is a limit point of K. □

Definition 122. We say that a subset A of a metric space (X, d) is said to be *bounded* if there exists $x_0 \in X$ and $R > 0$ such that $A \subset B(x_0, R)$.

It is easy to see that this is equivalent to requiring that for any given $x \in X$, there exists $R = R(x)$ such that $A \subset B(x, R)$.

Remark 123. Our definition of bounded subsets uses only the primitive notion of a metric space and is intuitive. By definition the empty set is bounded. The standard definition runs as follows. Given a **nonempty** set $A \subset X$, we define its *diameter*

$$\text{diam}\,(A) := \sup\{d(x, y) : x, y \in A\}, \text{ as an extended real number, possibly } +\infty.$$

We say that A is bounded if A is empty or if A is nonempty and $\text{diam}\,(A) < \infty$. We leave the equivalence of both the definitions as an easy exercise to the reader.

Lemma 124. *Any compact subset of a metric space (X, d) is bounded.*

Proof. Fix $x \in X$. Observe that $K \subset \cup_{n\in\mathbb{N}} B(x, n)$. By compactness of K and by the fact that $B(x, m) \subset B(x, n)$ for $m \leq n$, it follows that $A \subset B(x, N)$ for some N. □

Theorem 125. *Any closed and bounded interval $[a, b] \subset \mathbb{R}$ is compact.*

Proof. Given an open cover $\mathcal{U}$ of $[a, b]$, let

$$E := \{x \in [a, b] \mid [a, x] \text{ is covered by finitely many elements of } \mathcal{U}\}.$$

We note that $E \neq \emptyset$, since $a \in E$: For, $[a, a] = \{a\}$ and since $\mathcal{U}$ is an open cover there exists $U \in \mathcal{U}$ such that $a \in U$. Hence $[a, a]$ is covered by the single element U. In fact, we can say more. Since $a \in [a,b]$, there exists U in the open cover and $\delta > 0$ such that $(a-\delta, a+\delta) \subset U$. Hence $[a, a+\delta/2] \subset U$. In other words, $a + \frac{\delta}{2} \in E$.

E is bounded by b. Hence the supremum of E, say β exists.

We claim that $\beta \in E$ and that $\beta = b$. The claim proves the result. Suppose the claim is false.

Now $\beta \in [a,b]$ since $[a,b]$ is closed. There exists $V \in \mathcal{U}$ such that $\beta \in V$. Hence there exists $\varepsilon > 0$ such that $(\beta - \varepsilon, \beta + \varepsilon) \subseteq V$, as V is open. Assume that $\beta \neq b$. Then we may assume that ε is so small that $(\beta - \varepsilon, \beta + \varepsilon) \subseteq [a,b]$. Since $\beta = \sup E$, $\beta - \varepsilon$ is not an upper bound of E. Thus, there exists $x \in E$, such that $\beta - \varepsilon < x \leq \beta$. Since $x \in E$, there exists finitely many $U_i \in \mathcal{U}$, $1 \leq i \leq n$ such that $[a,x] \subseteq \cup_{i=1}^n U_i$. But then $[a, \beta + \frac{\varepsilon}{2}] \subseteq \cup_{i=1}^n U_i \cup V$. Hence $\beta + \frac{\varepsilon}{2} \in E$, a contradiction since $\beta \geq x$, for all $x \in E$. Hence $\beta = b$. ☐

Ex. 126. Let $a,b,c,d \in \mathbb{R}$ be such that $b-a = d-c$. Let $S := [a,b] \times [c,d]$ be the square in $\mathbb{R}^2$. The vertices of S are (a,c), (b,c), (b,d) and (a,d). We call the point (a,c) as the bottom left vertex of S. The pair of midpoints of its opposite sides are given by $([a+b]/2, c), ([a+b]/2, d)$ and $(a, [c+d]/2), (b, [c+d]/2])$. By joining the midpoints of opposite sides, we get four smaller squares. Observe that if (a_1, c_1) is the bottom left vertex of any of these squares, we have $a \leq a_1$ and $c \leq c_1$.

Theorem 127. *A subset of $\mathbb{R}^2$ is compact iff it is closed and bounded.*

Proof. Any compact subset of a metric space is closed and bounded in view of first two lemmas.

Let K be a closed and bounded set in $\mathbb{R}^2$. Then there exists $R > 0$ such that $K \subset S := [-R, R] \times [-R, R]$. Since a closed subset of a compact set is compact, it suffices to show that S is compact.

Suppose that S is not compact. Then there is an open cover $\{U_i : i \in I\}$ of which there is no finite subcover of S. Let us divide the square S into four smaller squares by joining the pairs of midpoints of opposite sides. (See Exercise 126 above.) One of these square will not have a finite subcover from the given cover. For, otherwise, all these four squares will have finite subcovers so that S itself will admit a finite subcover. Choose one such smaller square and call it S_1. Note that the length of its sides is R and that if (a_1, c_1) is the bottom left vertex of S_1, then $a_1 \geq a_0 = -R$ and $c_1 \geq c_0 = -R$. We repeat the argument by subdividing S_1 into four squares and choosing one of the smaller squares which does not admit a finite subcover of $\{U_i\}$. Call this smaller square as S_2. Note that the length of its sides is $R/2$ and that if (a_2, c_2) is the bottom left vertex of S_2, then $a_1 \leq a_2$ and $c_1 \leq c_2$.

Proceeding recursively, we have a sequence of squares S_n such that S_n dose not admit a finite subcover and the length of sides of S_n is $2^{-n+1}R$ and its bottom left vertex (a_n, c_n) is such that $a_{n-1} \leq a_n$ and $c_{n-1} \leq c_n$. Thus we have two sequences of real numbers (a_n) and (c_n). They are bounded and monotone. Hence there exist real numbers a and c such that $a_n \to a$ and $c_n \to c$. It follows that $(a_n, c_n) \to (a,c) \in \mathbb{R}^2$. Since S is closed, we infer that $(a,c) \in S$. Hence there is U_{i_0} in the open cover such that $(a,c) \in U_{i_0}$. Since U_{i_0} is open there exists an $r > 0$ such that $B((a,c), r) \subset U_{i_0}$.

Choose $n \in \mathbb{N}$ so that (1) $\operatorname{diam} S_n = 2^{-n+1}\sqrt{2}R < r/2$ and (2) $d((a,c),(a_n,c_n)) < r/2$.

We then have, for any $(x, y) \in S_n$,

$$d((a,c),(x,y)) \le d((a,c),(a_n,c_n)) + d((a_n,c_n),(x,y)) < r/2 + 2^{-n+1}\sqrt{2}R < r.$$

Thus $S_n \subset B((a,c),r) \subset U_{i_0}$. But then $\{U_{i_0}\}$ is a finite subcover for S_n, contradicting our choice of S_k's. Therefore, our assumption that S is not compact is not tenable. □

Theorem 128. *A subset of* $\mathbb{R}^n$ *is compact iff it is closed and bounded.*

Proof. One can adapt the proof of Thm. 127 to prove the theorem including the case when $n = 1$. We leave the details to the reader. □

H.2 Characterization of Compact Metric Spaces

Ex. 129. Let A be a compact subset of (X, d). Let $\varepsilon > 0$ be given. Show that there exist finitely many points $x_k \in X$, $1 \le k \le n$ such that $A \subset \cup_{k=1}^n B(x_k, \varepsilon)$. *Hint:* Consider the open cover $\{B(x, \varepsilon) : x \in X\}$ of X.

This exercise motivates the following definition which should be thought of as the back-door entry of compactness!

Definition 130. A subset $A \subset X$ of a metric space is said to be *totally bounded* if for every $\varepsilon > 0$ there exists a finite number of points in X, say, x_j, $1 \le j \le n$, such that $A \subset \cup_k B(x_k, \varepsilon)$. (The number n may depend up on ε.

Ex. 131. Show that any totally bounded subset is bounded. Is the converse true? *Hint:* Consider an infinite set with discrete metric.

Ex. 132. Show that in $\mathbb{R}$ with the usual metric a set is bounded iff it is totally bounded. Extend this result to $\mathbb{R}^n$.

Theorem 133. *For a metric space* (X, d)*, the following are equivalent:*

(1) *X is compact: every open cover has a finite subcover.*
(2) *X is complete and totally bounded.*
(3) *Every infinite set has a cluster point.*
(4) *Every sequence has a convergent subsequence.*

Proof. **(1) $\Longrightarrow$ (2):** Let (X, d) be compact. Given $\varepsilon > 0$, $\{B(x, \varepsilon) \mid x \in X\}$ is an open cover of X. Let $\{B(x_i, \varepsilon) \mid 1 \le i \le n\}$ be a finite subcover. Hence X is totally bounded.

Now let (x_n) be a Cauchy sequence in X. Then for every $k \in \mathbb{N}$ there exits n_k such that $d(x_n, x_{n_k}) < \frac{1}{k}$ for all $n > n_k$. Let $U_k := \left\{x \in X \;\middle|\; d(x, x_{n_k}) > \frac{1}{k}\right\}$. Then U_k is open: If $y \in U_k$ and $\delta := d(x_{n_k}, y) - \frac{1}{k}$ then $B(y, \delta) \subseteq U_k$. Now $x_n \notin U_k$ for $n > n_k$. Hence no finite subcover of U_k's cover X: For, if they did, say, $X = \cup_{i=1}^m U_i$ we take $n > \max\{n_1, \ldots, n_m\}$. Then $x_n \notin U_k$ for any k with $1 \le k \le m$. This implies that $\{U_k\}$ cannot cover X. Thus there exists $x \in X \setminus \cup_{k=1}^\infty U_k$. But then $d(x, x_{n_k}) < \frac{1}{k}$. Hence $x_{n_k} \to x$. Since (x_n) is Cauchy we see that x_n also converges to $\lim_k x_{n_k}$. Thus X is complete. We have thus shown (1) implies (2).

(2) $\implies$ (3): Let E be an infinite subset of X. Let F_n be a finite subset of X such that $X = \cup_{x \in F_n} B(x, \frac{1}{n})$. Then for $n = 1$ there exists $x_1 \in F_1$ such that $E \cap B(x_1, 1)$ is infinite. Inductively choose $x_n \in F_n$ such that $E \cap (\cap_{k=1}^{n} B(x_k, \frac{1}{k}))$ is infinite. Since there is $a \in E \cap B(x_m, \frac{1}{m}) \cap B(x_n, \frac{1}{n})$ we see that $d(x_m, x_n) \leq d(x_m, a) + d(a, x_n) < \frac{1}{m} + \frac{1}{n} < \frac{2}{m}$ if $m < n$. Thus (x_n) is Cauchy. Since X is complete x_n converges to some $x \in X$. Also $d(x, x_n) < \frac{2}{n}$ for all n. Thus $B(x, 3/n)$ includes $B(x_n, \frac{1}{n})$ which includes infinitely many elements of E. Thus x is a cluster point of E. Hence (3) is proved.

(3) $\implies$ (4): If (x_n) is a sequence in X we let $\{x_n \mid n \in \mathbb{N}\}$ be its image. If this set is finite then (4) trivially follows. So assume that $\{x_n \mid n \in \mathbb{N}\}$ is infinite. Let x be a cluster point of this set. Then there exist elements x_{n_k} such that $d(x, x_{n_k}) < \frac{1}{k}$ for all k. Thus $x_{n_k} \to x$ and (4) is thereby proved.

(4) $\implies$ (1): Let $\{U_\alpha\}$ be an open cover of X. For $x \in X$, let

$$r_x := \sup \{r \in \mathbb{R} \mid B(x, r) \subseteq \text{ for some } U_\alpha\}.$$

We claim that $\varepsilon := \inf \{r_x \mid x \in X\} > 0$. If not there is a sequence (x_n) such that $r_{x_n} \to 0$. But (x_n) has a convergent subsequence, say, $x_{n_k} \to x$. Now $x \in U_\alpha$ for some α and hence there is an $r > 0$ such that $B(x, r) \subset U_\alpha$. For k large enough $d(x, x_{n_k}) < \frac{r}{2}$ so that $r_{x_{n_k}} > \frac{r}{2}$ for all sufficiently large k – a contradiction. Hence the claim is proved.

Let $\varepsilon := \inf \{r_x \mid x \in X\}$. Choose any $x_1 \in X$. Inductively choose x_n such that $x_n \notin \cup_{k=1}^{n-1} B(x_i, \varepsilon/2)$. We cannot do this for all n. For otherwise, (x_n) will not have a convergent subsequence since $d(x_n, x_m) > \frac{\varepsilon}{2}$ for all $m \neq n$. Hence $X = \cup_{k=1}^{N} B(x_k, \frac{\varepsilon}{2})$ for some N. But then for each k there is an α_k such that $B(x_k, \frac{\varepsilon}{2}) \subset U_{\alpha_k}$. Hence $X = \cup_{k=1}^{N} U_{\alpha_k}$. Thus $\{U_\alpha\}$ has a finite subcover or X is compact. $\square$

Ex. 134. Prove Thm. 127 using the fourth characterization (in Thm. 133) of compact metric spaces.

A slight reordering of the equivalences is established in the following result.

Theorem 135. *For a metric space (X, d), the following are equivalent:*
(1) X is compact.
(2) Every infinite set has an accumulation point.
(3) Every sequence has a convergent subsequence.
(4) X is complete and totally bounded.

Proof. (1) $\implies$ (2) Let $E \subseteq X$ be an infinite set. Assume that E has no accumulation point. Then for any $\varepsilon > 0$, and $x \in X$, the ball $B(x, \varepsilon)$ has only finitely many points of E, i.e., $B(x, \varepsilon) \cap E$ is a finite set for every $x \in E$. Since $\{B(x, \varepsilon) \mid x \in X\}$ is an open covering of X and X is compact there exists a finite number of x, say, $x_1, \ldots, x_n$ such that $X \subseteq \cup_{i=1}^{n} B(x_i, \varepsilon)$. But then $E = E \cap X$, so that $|E| \leq \sum |E \cap B(x_i, \varepsilon)|$, a finite number - a contradiction.

(2) $\implies$ (3) Let $n \mapsto x_n$ be a sequence in X. If there exists an $x \in X$ such that for an infinite number of n's we have $x = x_n$, (3) follows. For we take n_1 to be the first n such that $x_n = x$. Let $n_2 > n_1$ be the first n such that $x_n = x$ and so on. Then $k \mapsto n_k \mapsto x_{n_k}$ is a subsequence of the original sequence and it is convergent, as it is a constant sequence.

So, we now assume the set $E := \{x_n \mid n \in \mathbb{N}\}$ is infinite. If it were finite, we are through by the last paragraph. By (2) there exists an $x \in X$ which is an accumulation point of E. But this means that for every $k \in \mathbb{N}$, there exists a n_k such that $x_{n_k} \in B(x, \frac{1}{k})$. $k \mapsto n_k \mapsto x_{n_k}$ is thus a subsequence which converges to x. Thus (3) is proved.

(3) $\implies$ (4) Let $n \mapsto x_n$ be a Cauchy sequence in X. By (3), there exists a convergent subsequence $k \mapsto n_k \mapsto x_{n_k}$ with $\lim_{n\to\infty} x_{n_k} = x$. We claim that $\lim_{n\to\infty} x_n = x$. For, given $\varepsilon > 0$, we choose $N > 0$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ for $n, m \geq N$. Also, we choose $M > 0$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ for $n \geq M$. Let $n_0 = \max\{N, M\}$. Then for all $m \geq n_0$, we have

$$\begin{aligned} d(x, x_n) &\leq d(x, x_{n_k}) + d(x_{n_k}, x_m) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \varepsilon \end{aligned}$$

for any k such that $n_k \geq n_0$. Hence the claim.

We need to prove that X is totally bounded. If X is not totally bounded, then there exists $\varepsilon > 0$ such that no finite number of ε-balls cover X. Let $x_1 \in X$ be arbitrary. Since $B(x_1, \varepsilon) \neq X$, there exists $x_2 \in X \setminus B(x_1, \varepsilon)$. Assume that we have chosen $x_1, \ldots, x_k \in X$ such that $x_j \notin \cup_{i=1}^{j-1} B(x_i, \varepsilon)$. Then there exists $x_{k+1} \in X \setminus \cup_{i=1}^{k} B(x_i, \varepsilon)$. Thus $k \mapsto x_k$ is a sequence in X. It cannot have any convergent subsequence as $d(x_i, x_j) \geq \varepsilon$ for all $i \neq j$. This contradicts our hypothesis that X satisfies (3).

(4) $\implies$ (1) Suppose X is not compact. Then there exists an open cover $\{U_\alpha\}$ of X which admits no finite subcover. Since X is totally bounded, there exists a finite number of y's such that $X = B(y_1, 1) \cup B(y_2, 1) \cup \cdots \cup B(y_n, 1)$. Since X is not covered by finitely many U_α's it follows that there exists y_j such that $B(y_j, 1)$ is not covered by finitely many of U_α's. Let $x_1 := y_j$. Now consider $X_1 := B(x_1, 1)$. Then (X_1, d) is totally bounded. Hence there exists finitely many $y \in X_1$ such that $x_1 = \cup_{j=1}^{n} [B(y_j, Y_2) \cap X_1]$. Note that

$$B_{X_1}(y, \frac{1}{2}) = \{z \in X_1 \mid d(z, y) < \frac{1}{2}\} = B(y_1, \frac{1}{2}) \cap X_1.$$

As above, there exists $y_2 \in X_1$ such that $B_{X_1}(y_j, \frac{1}{2})$ is not covered by finitely many $\{X_1 \cap U_\alpha\}$. Call it X_2. We proceed by induction to choose x_k such that $B_{X_{k-1}}(x_k, \frac{1}{k})$ is not covered by finitely many $\{X_k \cap U_\alpha\}_\alpha$. Now clearly $\{x_k\}$ is a Cauchy sequence in X: $d(x_m, x_n) \leq \frac{1}{m}$ if $m \leq n$. Since X is complete $x_n \to x$ for some $x \in X$. Let $x \in U_\alpha$ for some α. Then, all $x_n \in U_\alpha$ for $n \geq N(\alpha)$. In particular, $B(x_n, \frac{1}{n}) \subseteq U_\alpha$ for n sufficiently large. This contradicts our choice of x_n, viz., $B(x_n, \frac{1}{m})$ is not a subset of finitely many U_α's. □

Theorem 136. *$[a, b]$ is compact.*

Proof. Given an open cover $\mathcal{U}$ of $[a, b]$, let

$$E := \{x \in [a, b] \mid [a, x] \text{ is covered by finitely many elements of } \mathcal{U}\}.$$

We note that $E \neq \emptyset$, since $a \in E$: For, $[a, a] = \{a\}$ and since $\mathcal{U}$ is an open cover there exists $U \in \mathcal{U}$ such that $a \in U$. Hence $[a, a]$ is covered by the single element U.

E is bounded, obviously by b. Hence the supremum of E, say β exists. If $\beta = b$, we are through. Suppose not.

Now $\beta \in [a,b]$ and hence there exists $V \in \mathcal{U}$ such that $\beta \in V$. Hence there exists $\varepsilon > 0$ such that $(\beta - \varepsilon, \beta + \varepsilon) \subseteq V$, as V is open and $(\beta - \varepsilon, \beta + \varepsilon) \subseteq [a,b]$. Since $\beta = \sup E$, $\beta - \varepsilon$ is not an upper bound of E. Thus, there exists $x \in E$, such that $\beta - \varepsilon < x \leq \beta$. Since $x \in E$, there exists finitely many $U_i \in \mathcal{U}$, $1 \leq i \leq n$ such that $[a,x] \subseteq \cup_{i=1}^n U_i$. But then $[a, \beta + \frac{\varepsilon}{2}] \subseteq \cup_{i=1}^n U_i \cup V$. Hence $\beta + \frac{\varepsilon}{2} \in E$, a contradiction since $\beta \geq x$, for all $x \in E$. Hence $\beta = b$. □

H.3 Tychonoff's Theorem

Let X be a set. We say that a collection $\mathcal{A}$ of (nonempty) subsets of X has *finite intersection property* (f.i.p., in short) if every finite family $A_1, \dots, A_n$ of elements in $\mathcal{A}$ has a nonempty intersection.

Ex. 137. A topological space is compact iff every family of closed sets with f.i.p. has nonempty intersection. *Hint:* Start with an open cover $\mathcal{U}$ which does not admit a finite subcover. Look at $\{X \setminus U : U \in \mathcal{U}\}$.

Let us briefly review the product topology. Given a family $\{X_\alpha : \alpha \in I\}$ of topological spaces, the topology on the product set $\prod_{\alpha \in I} X_\alpha$ is the weakest (or the smallest topology) which makes the canonical projection maps $P_\alpha \colon X \to X_\alpha$ continuous. Hence for any open set $U_\alpha \subset X_\alpha$, the set $P_\alpha^{-1}(U_\alpha)$ must be declared open in X. Finite intersections of such sets form a basis for the product topology, that is, $U \subset X$ is open iff for each $x \in U$, there exists a finite subset $F \subset I$ such that $x \in \cap_{\alpha \in F} P_\alpha^{-1}(U_\alpha)$ for some open sets $U_\alpha \subset X_\alpha$.

To bring out the main ideas of the proof clearly, let us list the following two exercises which are set-theoretic in nature. Their solutions can be gleaned from the proof of the theorem below.

Ex. 138. Let $\mathcal{A}$ be a family of subsets of a set X with f.i.p. Then there exists a 'maximal' family $\mathcal{B}$ containing $\mathcal{A}$ with f.i.p., that is, a family $\mathcal{B}$ of subsets such that (i) $\mathcal{A} \subset \mathcal{B}$, (ii) $\mathcal{B}$ has f.i.p. and (iii) if $\mathcal{C}$ is any family with f.i.p. such that $\mathcal{A} \subset \mathcal{C}$, then $\mathcal{C} \subset \mathcal{B}$. *Hint:* Partially order the set of all collections with properties (i) and (ii) by inclusion. Apply Zorn's lemma.

Ex. 139. Let $\mathcal{B}$ be a family of subsets of a set X which is maximal with respect to finite intersection property. Then (i) if $A \subset X$ has nonempty intersection with each member of $\mathcal{B}$, then $A \in \mathcal{B}$ and (ii) the intersection of any finite number of elements of $\mathcal{B}$ again lies in $\mathcal{B}$.

Theorem 140 (Tychonoff)**.** *The product of compact spaces is compact. That is, if X_α is compact for each $\alpha \in I$ and if $X := \prod_{\alpha \in I} X_\alpha$ is endowed with the product topology, then X is compact.*

Proof. We plan to use Ex. 137. Let $\mathcal{F}_0$ be a family of closed sets in X with the finite intersection property (f.i.p.). It suffices to show that there is a point common to all the sets $F \in \mathcal{F}_0$. We use Ex. 138 to get maximal family $\mathcal{F} \supseteq \mathcal{F}_0$. The details are in the next paragraph.

Consider the class of all families $\mathcal{F}$ of (not necessarily closed) subsets such that $\mathcal{F}_0 \subset \mathcal{F}$ which have f.i.p. For two families $\mathcal{F}$ and $\mathcal{G}$ in this class we say that $\mathcal{F} \leq \mathcal{G}$ iff $\mathcal{F} \subseteq \mathcal{G}$. Now let

$\mathcal{C}$ be any totally ordered chain in this class, that is, if $\mathcal{F}, \mathcal{G} \in \mathcal{C}$, then either $\mathcal{F} \subseteq \mathcal{G}$ or $\mathcal{G} \subseteq \mathcal{F}$. This chain has an upper bound, viz., $\mathcal{H} = \cup_{\mathcal{F} \in \mathcal{C}} \mathcal{F}$. We need only show that $\mathcal{H}$ has f.i.p. Let $A_1, \ldots, A_n \in \mathcal{H}$. Then there exists $\mathcal{F}_j \in \mathcal{C}$ such that $A_j \in \mathcal{F}_j \in \mathcal{C}$. Since $\mathcal{C}$ is totally ordered, and $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are finite in number, there exists k with $1 \leq k \leq n$ such that $\mathcal{F}_j \subseteq \mathcal{F}_k$ for all j. Hence $A_1, \ldots, A_n \in \mathcal{F}_k$. Since $\mathcal{F}_k$ has f.i.p., $A_1 \cap \cdots \cap A_n \neq \emptyset$. Thus $\mathcal{H}$ has f.i.p. and hence is an upper bound for the chain. Therefore, by Zorn's lemma, there exists a maximal family $\mathcal{F} \in \mathcal{C}$, with $\mathcal{F} \supseteq \mathcal{F}_0$.

Let $\mathcal{F}^\alpha$ denote $\{E^\alpha := P_\alpha(E), \quad E \in \mathcal{F}\}$, where $P_\alpha \colon X \to X_\alpha$ is the canonical projection map. Then $\mathcal{F}_\alpha \subseteq P(X_\alpha)$, the power set of X_α, has f.i.p. For otherwise, $E_1^\alpha \cap \cdots \cap E_n^\alpha = \emptyset$ will imply $E_1 \cap \cdots \cap E_n = \emptyset$, where $P_\alpha(E_i) = E_i^\alpha$. Hence, $\overline{\mathcal{F}^\alpha} = \{\overline{E^\alpha}\}$ has finite intersection property.

Since X_α is compact, there exists $x_\alpha \in \cap \overline{E^\alpha}$ where the intersection is over all $E^\alpha \in \mathcal{F}^\alpha$. Let $x \in \prod X_\alpha$ be such that $x(\alpha) := x_\alpha$. We claim that $x \in \cap_{F \in \mathcal{F}} \overline{F}$. Since $\mathcal{F} \supseteq \mathcal{F}_0$ and since every element of $\mathcal{F}_0$ is closed, the claim completes the proof of the theorem.

We now prove the claim. Let U be an open set in X. By definition of product topology, there exists $\alpha_1, \ldots, \alpha_n$ and open sets $U_{\alpha_i} \subseteq X_{\alpha_i}$, $1 \leq i \leq n$ such that $x \in \cap_{i=1}^n P_{\alpha_i}^{-1}(U_{\alpha_i}) \subseteq U$. This implies $x_{\alpha_i} \in U_{\alpha_i}$ for all i. By hypothesis on x_α's, x_{α_i} is in the closure of F_{α_i} for all $F_{\alpha_i} \in \mathcal{F}^{\alpha_i}$. Select $y_{\alpha_i} \in U_{\alpha_i} \cap F_{\alpha_i}$ and a $y \in F$ such that $P_{\alpha_i}(y) = y_{\alpha_i}$. Then $y \in P_{\alpha_i}^{-1}(U_{\alpha_i}) \cap F$. Thus $P_{\alpha_i}^{-1}(U_{\alpha_i})$ has a non-empty intersection with every $F \in \mathcal{F}$. Therefore $P_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$ by Ex. 139.

(Reason: Otherwise $\mathcal{F} \cup \{P_{\alpha_i}^{-1}(U_{\alpha_i})\} \supset \mathcal{F}$ and the former has finite intersection property, contradicting the maximality of $\mathcal{F}$.)

Using Ex. 139 again, we infer that $\cap_{i=1}^n P_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$.

(Reason: Since $F, P_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$, we see that $F \cap \left(\cap_i P_{\alpha_i}^{-1}(U_{\alpha_i})\right) \neq \emptyset$. Thus $\cap_i P_{\alpha_i}^{-1}(U_{\alpha_i})$ meets every element of $\mathcal{F}$. Thus $\mathcal{F} \cup \{\cap_i P_{\alpha_i}^{-1}(U_{\alpha_i})\}$ has f.i.p. Since $\mathcal{F}$ is maximal with respect to f.i.p. it follows that $\cap_i P_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$.)

Since $\mathcal{F}$ has f.i.p., this basic open set and hence U intersects each member of $\mathcal{F}$ non-trivially. Since U was an arbitrary open neighborhood of x, this means that $x \in \overline{F}$, for all $F \in \mathcal{F}$. Hence the claim. □

Ex. 141. Prove Thm.128 using Tychonoff's theorem.

Acknowledgement: I thank Vikram Aithal and Rohit Gupta whose discussions with me on this subsection helped me improve its readability.

H.4 Continuous Functions on Compact Spaces

Theorem 142. *Let $f \colon X \to \mathbb{C}$ be a continuous function from a compact (metric) space to $\mathbb{C}$. Then*
(1) *f is bounded, i.e., there exists a constant $A > 0$ such that $|f(x)| \leq A$ for all $x \in X$.*
(2) *If we further assume that $f(x) > 0$ for all $x \in X$, then there exists $B > 0$ such that $f(x) \geq B$ for all $x \in X$.*

(3) *If f is real-valued, then there exists $p \in X$ and $q \in X$ such that $f(p) \geq f(x)$ for all $x \in X$ and $f(q) \leq f(x)$ for all $x \in X$.*

Proof. To prove (1), consider the open sets $U_n := \{x \in X : |f(x)| < n\}$ for $n \in \mathbb{N}$. Then $U_n \subset U_{n+1}$ and $X = \cup U_n$. By compactness, we conclude that $X = U_N$ for some N. (1) follows.

To prove (2), consider $V_n := \{x \in X : f(x) > 1/n\}$ and argue as in (1).

We now prove (3). By (1), there exits $\alpha \in R$ such that $|f(x)| \leq \alpha$ for all $x \in X$. Let $M := \sup\{f(x) : x \in X\}$. By (1), M exists. If there is no $p \in X$ such that $f(p) = M$, then the sets

$$U_n := \{x \in X ; f(x) < M - \frac{1}{n}\}$$

form an open cover of X. As in (1), we conclude that $X = U_N$ for some N. But this leads to the contradiction: $\sup\{f(x) : x \in X\} \leq M - \frac{1}{N}$!

Note that (2) can be deduced from (3). □

The following is an easy but a most useful result.

Theorem 143. *Let $f \colon X \to Y$ be a bijective continuous map from a compact space X to a hausdorff space Y. Then f is a homeomorphism.*

Proof. It is enough to show that f is a closed map, that is, it maps closed sets of X to closed sets of Y. Let K be a closed subset of X. Then K is compact being a closed subset of a compact space. Since f is continuous, $f(K)$ is compact. Since $f(K)$ is a compact subset of a hausdorff space Y, we deduce that $f(K)$ is closed in Y. Thus, any closed subset of X is mapped to a closed subset of Y. □

H.5 Characterization of Compact Metrizable Spaces via Metrics and Continuous Functions

A topological space X is said to be *metrizable* if there exists a metric on X such that the given topology coincides with the topology defined by the metric.

Theorem 144. *For a metrizable topological space X, the following properties are equivalent:*

1. *X is compact.*
2. *Every metric on X inducing the given topology is bounded.*
3. *Every continuous (real valued) function on X is bounded.*

Proof. We now prove the theorem according to the pattern:

$$(1) \implies (2) \implies (3) \implies (1).$$

(1) $\implies$ (2): If X is compact, so is $X \times X$. Any metric d on X inducing the given topology on X is a continuous function on $X \times X$, whence bounded.

(2) $\implies$ (3): Let f be a continuous function on X. We then push points of X apart at distances bounded from below by f using the following standard technique. Consider the *graph* Z of f:

$$Z := \{(x, f(x)) : x \in X\} \subseteq X \times \mathbb{R}.$$

The map $i\colon X \hookrightarrow Z$ given by $x \mapsto (x, f(x))$ is then a homeomorphism of X onto Z, its inverse being given by the restriction of the first projection $p\colon X \times \mathbb{R} \longrightarrow X$ to Z. The space $X \times \mathbb{R}$ with the product topology is metrizable; e.g. one may take the metric δ defined as

$$\delta((x, s), (y, t)) := d(x, y) + |t - s|.$$

Pulling this metric back to X using the map i therefore equips X with a metric d' inducing the topology given by d, which therefore by assumption is bounded, by a constant $B > 0$, say. Now by construction

$$d'(x, y) = d(x, y) + |f(y) - f(x)|\,,$$

so d' being bounded by B implies

$$|f(y)| \leq |f(x)| + B,$$

for all $x, y \in X$. If we fix $x \in X$, the inequality above shows that f is bounded.

(3) $\implies$ (1): We show that on any noncompact metrizable space X there exists a continuous unbounded function. Let d be any metric on X inducing the given topology and let X' be the completion of X with respect to d. We distinguish the cases X' being compact and being not so.

Case (i): X' compact. Since X is assumed to be noncompact, $X \neq X'$ whence $X' \setminus X$ is not empty. Let x_∞ be a point in $X' \setminus X$. Since X is dense in X', the function f defined by $f(x) := 1/d(x, x_\infty)$ is then a continuous function on X which is not bounded.

Case (ii): X' noncompact. If f is a continuous unbounded function on X', its restriction to X is a continuous unbounded function on X. So we may assume X itself is complete. According to the standard characterization of compactness, X cannot be totally bounded since it is assumed to be noncompact. So there is a real number $\varepsilon > 0$ such that X cannot be covered by finitely many closed ε-balls. Let x_1 be any point in X and put $r_1 := \varepsilon$. Then the closed ball $B[x_1, r_1]$ does not cover X. So there is x_2 in $X \setminus B[x_1, r_1]$. The latter complement being open there is r_2 with $r_1 \geq r_2 > 0$ such that $B[x_2, r_2] \subseteq X \setminus B[x_1, r_1]$. The balls $B[x_1, r_1]$ and $B[x_2, r_2]$ together do not cover X, so there are x_3 and r_3 with $B[x_3, r_3] \subseteq X \setminus B[x_1, r_1] \cup B[x_2, r_2]$ and $r_1 \geq r_2 \geq r_3 > 0$. Continuing this way we obtain sequences $x_1, x_2, \ldots$ and $r_1 \geq r_2 \geq \cdots > 0$. They have the property that the balls $B[x_k, r_k]$ are mutually disjoint. Now we define $f\colon X \to \mathbb{R}$ as follows:

$$f(x) := \sum_{k=1}^{\infty} k \cdot \frac{d(x, X \setminus B(x_k, r_k))}{d(x, x_k) + d(x, X \setminus B(x_k, r_k))}\,.$$

(Visualize f in the case of $X = \mathbb{R}$ and $x_k = k$ and $r_k = 2^{-k}$, say.) If the k-th term of the sum that defines $f(z)$ is nonzero, it means that $z \in B[x_k, r_k]$ and hence all other terms of the series that defines $f(z)$ are zero, since the balls $B[x_k, r_k]$ are mutually disjoint. Hence the series is convergent and $f(x)$ makes sense for any $x \in X$. We thus get a well-defined function f on X. Since $f(x_k) = k$, f is not bounded. It is easily seen to be continuous on X. For, if $x \in U := X \setminus \cup_{k \in \mathbb{N}} B[x_k, r_k]$, then $f(x) = 0$ and since U is open (why?), f is zero in an open set containing x. If $x \in B[x_k, r_k]$, then f is just the k-th term of the series, which is continuous. This finishes the proof. □

Remark 145. The case (ii) of (3) $\implies$ (1) can also be seen as follows. If X is not totally bounded, there is some $\varepsilon > 0$ such that no finite collection of balls of radius ε covers X. So we can pick x_1 in X, $x_2 \in X \setminus B(x_1, \varepsilon)$, $x_3 \in X \setminus (B(x_1, \varepsilon) \cup B(x_2, \varepsilon))$, and so on. Each $d(x_i, x_j) \geq \varepsilon$ for $i \neq j$. Hence there are no nonconstant Cauchy sequences among the x_i. So, the set $\{x_i\}$ is closed in X and also discrete. If we now define $f(x_i) = i$, then f is continuous function on the discrete set. We can extend this by the Tietze theorem to $f \colon X \longrightarrow \mathbb{R}$. The function f is clearly an unbounded continuous function.

I Connected and Path Connected spaces

The aim of this article is to introduce the readers to an easier way of working with connectedness concept. If the reader's background does not include general (abstract) topological spaces, he may assume that the spaces are metric spaces.

I.1 Connectedness

Definition 146. A topological space X is said to be *connected*, if the only subsets of X which are both open and closed are the empty set $\emptyset$ and X. In other words, a topological space is connected whenever a subset A is both open and closed in X, then either $A = \emptyset$ or $A = X$.

A subset A of a topological space X is said to be connected if A is a connected space when considered as a topological space with the induced (or subspace) topology. In the case of metric space (X, d), this amounts to saying that (A, δ) is connected, where δ is the restriction of the metric d on X to A.

Therefore, if a topological space X is not connected, there will be a proper non-empty sub set A of X which is both open and closed in X. If A is a proper non-empty sub set of X and both open and closed, then $B = A^c$, its complement is also a proper non-empty sub set of X which is both open and closed in X. In other words a topological space X is not connected iff there exist two disjoint proper non-empty sub sets A and B such that A and B are both open and closed in X and $X = A \cup B$. In such case we also say that the pair (A, B) is a *disconnection* of X.

Example 147. Let X be a set such that $| X | \geq 2$ with discrete topology (or discrete metric). Then X is not connected.

Example 148. The subset $\{\pm 1\} \subset \mathbb{R}$ with the subspace topology is not connected. (Why?) In the sequel, we consider $\{\pm 1\}$ as a subset of $\mathbb{R}$.

Now we prove a single most important theorem in connectedness which supplies us an abundance of examples of connected and non-connected spaces.

Theorem 149. *A topological space X is connected iff every continuous function $f: X \to \{\pm 1\}$ is a constant function.*

Proof. Let X be a connected space and $f: X \to \{\pm 1\}$ a continuous function. We want to show that f is a constant function. If f is non-constant, then it is on-to. Let $A = f^{-1}(1)$ and $B = f^{-1}(-1)$. Then A and B are disjoint non-empty subsets of X such that A and B are both open and closed subsets of X and $X = A \cup B$.(Why?). This is a contradiction. Therefore f is constant.

Conversely, let us assume that X is not connected. Therefore there exist two disjoint proper non-empty subsets A and B in X such that A and B are both open and closed in X and $X = A \cup B$. Now we define a map $f: X \to \{\pm 1\}$ as

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ -1 & \text{if } x \in B \end{cases}$$

Then $f \colon X \to \{\pm 1\}$ is a continuous non-constant function. (Why?). This completes the proof. $\square$

We shall now use this theorem to get examples of connected spaces.

Example 150. A set $J \subseteq \mathbb{R}$ is connected iff J is an interval.

Proof. Let J be a connected sub set of $\mathbb{R}$. Let us assume that J is not an interval. This means that there exist points $a < b$ in J and $c \in \mathbb{R}$ such that $a < c < b$ but $c \notin J$. Now we define a map $f \colon J \to \{\pm 1\}$ as

$$f(x) = \begin{cases} 1 & \text{if } x < c \\ -1 & \text{if } x > c \end{cases}$$

Now we claim that f is a continuous function. We need only to check that $f^{-1}(1)$ and $f^{-1}(-1)$ are open in J. By our definition $f^{-1}(1) = J \cap (-\infty, c)$ and $f^{-1}(-1) = J \cap (c, \infty)$ which are open, proper and nonempty subsets in J. (Why?) This is a contradiction to the fact that J is connected. Therefore for every pair of points a and b in J such that $a < b$, all the points c such that $a < c < b$ are also in J. This means that J is an interval in $\mathbb{R}$.

Conversely, let us assume that J is an interval in $\mathbb{R}$. Let $f \colon J \to \{\pm 1\}$ be a continuous function. We need to show that f is constant. If not, then there exist $a, b \in J$ such that $f(a) = 1$ and $f(b) = -1$. Since $a, b \in J$ and J is an interval, $[a, b] \subset J$. Hence, by applying the intermediate value theorem to the restriction f to $[a, b]$, there exists $c \in (a, b) \subset J$ such that $f(c) = 0$. This is a contradiction, since the codomain is $\{\pm 1\}$. $\square$

Example 151. Let $\mathrm{M}(2, \mathbb{R})$ denote the set of all 2×2 matrices of real numbers. and $\mathrm{G}L(2, \mathbb{R}) := \{A \in \mathrm{M}(2, \mathbb{R}) : \det(A) \neq 0\}$ is not connected.

Proof. Here we identify $M(2, \mathbb{R})$ with $\mathbb{R}^4$ via the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d) \in \mathbb{R}^4$. Let $f \colon \mathrm{GL}(2, \mathbb{R}) \to \mathbb{R}$ be defined by $f(A) := \det(A)$. Complete the proof. $\square$

Example 152. $\mathrm{O}(2, \mathbb{R}) := \{A \in \mathrm{GL}(2, \mathbb{R}) : AA^t = Id\}$ is not connected.

Proof. The equation $AA^t = Id$ shows that $\det(A) = \pm 1$ for every $A \in \mathrm{O}(2, \mathbb{R})$. This suggests us that we define the map $f \colon \mathrm{O}(2, \mathbb{R}) \to \{\pm 1\}$ by $f(A) := \det(A)$. Complete the proof. $\square$

Proposition 153. *Let X be a topological space. Let A and B be two connected subsets of X such that $A \cap B \neq \emptyset$. Then $A \cup B$ is connected.*

Proof. Let $f \colon A \cup B \to \{\pm 1\}$ be a continuous function. We have to show that f is constant. Let $c \in A \cap B$. Since A is connected, the function $f \mid_A \colon A \to \{\pm 1\}$ is constant so that $f(a) = f(c)$ for all $a \in A$. Similarly, $f(b) = f(c) = 1$ for all $b \in B$. Thus $f(x) = f(c)$ for all $x \in A \cup B$. i.e., f is a constant. $\square$

Proposition 154. *Let A be a connected subset of a space X. Let $A \subset B \subset \overline{A}$. Then B is connected.*

Proof. Let $f\colon B \to \{\pm 1\}$ be a continuous function. Without loss of generality, let us assume that $f = 1$ on A. Let $x \in B$. Since $\{f(x)\}$ is open in $\{\pm 1\}$, the set $U := f^{-1}(f(x))$ is an open containing x. Hence, there exists a point $a \in A \cap U$. Since $a, x \in U$ and $f = f(x)$ on U, it follows that $f(x) = f(a) = 1$. Thus $f = 1$ on B. □

Proposition 155. *Let $\{A_i : i \in I\}$ be a collection of connected subsets of a space X with the property that for all $i, j \in I$ we have $A_i \cap A_j \neq \emptyset$. Then $A := \cup_i A_i$ is connected.*

Proof. Fix A_i. Let $f\colon \cup A_j \to \{\pm 1\}$ be continuous. Since A_i is connected, f is a constant on it, say, $f = 1$ on A_i. Let $x \in A$. Then $x \in A_j$ for some j. Let $y \in A_i \cap A_j$. Then $f(x) = f(y)$ since A_j is connected and $x, y \in A_j$. Since $y \in A_i$, we have $f(y) = 1$. Hence for all $x \in A$, we conclude $f(x) = 1$. Hence A is connected. □

Proposition 156. *Let X be a connected topological space and $g\colon X \to Y$ be a continuous map. Then $g(X)$ is connected.*

Proof. We will show that any continuous map $f\colon g(X) \to \{\pm 1\}$ is constant.

Let $f\colon g(X) \to \{\pm 1\}$ be a continuous map. Then the map $f \circ g\colon X \to \{\pm 1\}$ is continuous. (Why?). Since X is connected, it follows that $g \circ f$ is constant. Hence f is constant. For, otherwise, there exist $y_1, y_2 \in g(X)$ such that $f(y_1) \neq f(y_2)$. Since $y_j \in g(X)$, this implies the existence of $x_j \in X$ such that $g \circ f(x_1) \neq g \circ f(x_2)$. In particular, $g \circ f$ is not a constant. Hence we are forced to conclude that f is constant. Thus $g(X)$ is connected. □

Corollary 157. *In the above proposition, if the map g is onto then Y is connected.*

Ex. 158. Show that the set $GL(2, \mathbb{R})$ is not connected.

Ex. 159. Show that the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected.

Ex. 160. Show that the set $SO(2, \mathbb{R}) := \{A \in O(2, \mathbb{R}) : \det A = 1\}$ is connected. *Hint:* Write down all elements of $SO(2, \mathbb{R})$ explicitly.

Proposition 161. *Let X and Y be connected spaces. Then the product space $X \times Y$ is connected.*

Proof. Let $f\colon X \times Y \to \{\pm 1\}$ be a continuous map. Let $(x_0, y_0) \in X \times Y$ be fixed. Let (x, y) be an arbitrary point in $X \times Y$. If we show that $f((x, y)) = f((x_0, y_0))$, we are through.

To prove the above claim, let us first observe that for every point $y \in Y$, the map $i_y\colon X \to X \times Y$ defined by $i_y(x) := (x, y)$ is continuous; similarly the map $i_x\colon Y \to X \times Y$ defined by $i_x(y) := (x, y)$ is continuous for every point x in X. Therefore for every point y in Y, the subset $X \times \{y\} := \{(x, y) : x \in X\}$ is a connected subset of $X \times Y$; similarly, the subset $\{x\} \times Y := \{(x, y) : y \in Y\}$ is a connected subset of $X \times Y$ for every point x in X.

Now the point (x, y_0) lies in both sets $X \times \{y_0\}$ and $\{x\} \times Y$. The restrictions of f to either of these sets are continuous and hence constants. We see that $f(x_0, y_0) = f(x, y_0))$ for all $x \in X$ and similarly, $f(x, y) = f(x, y_0)$ for all $y \in Y$. In particular, $f(x, y) = f(x, y_0) = f(x_0, y_0)$. (See Figure **??**.) □

The following is a **typical way** in which connectedness hypothesis is used.

Theorem 162. *Let X be connected. Let $f: X \to \mathbb{R}$ be a locally constant function, i.e., for each $x \in X$, there exists an open set U_x containing x with the property that f is a constant on U_x. Then f is a constant on X.*

Proof. First of all note that any locally constant function is necessarily continuous.

Fix $x_0 \in X$. We show that $f(x) = f(x_0)$ for all $x \in X$. Consider the set $E := \{x \in X \mid f(x) = f(x_0)\}$. As $x_0 \in E$, we see that E is nonempty. Since $E = f^{-1}(f(x_0))$, E is the inverse image of a closed set under the continuous map f and hence is closed.

If $x \in E$, since f is locally constant, there exists an open set U_x with $x \in U_x$ and f is constant on U_x. Thus for each $y \in U_x$, we have $f(y) = f(x)$. Since $x \in E$, we have $f(x) = f(x_0)$. Hence it follows that $f(x) = f(x_0)$ for all $x \in U_x$. In other words, $U_x \subset E$. Hence E is open. Thus E is nonempty, open and closed subset of the connected space X. Hence we must have $E = X$. □

As an immediate corollary we have

Theorem 163. *Let U be an open connected subset of $\mathbb{R}^n$ and $f: U \to \mathbb{R}$ be a differentiable function such that $Df(p) = 0$ for all $p \in U$. Then f is a constant function.*

Proof. To prove this theorem we will use the following fact which follows from mean value theorem. *Let U be an open convex subset of $\mathbb{R}^n$ and $f: U \to \mathbb{R}$ be a differentiable function such that $Df(p) = 0$ for all $p \in U$. Then f is constant on U.*

Now let f be as in the theorem. Then for each $x \in U$, since U is open, there exists an open ball $B(x, r_x) \subset U$. It is easy to see that any ball in $\mathbb{R}^n$ is convex. Thus an application of the calculus result quoted above shows us that f locally constant. □

I.2 Path Connected spaces

Definition 164.

1. Let X be a topological space. A continuous map $\gamma: [0,1] \to X$ is called a *path* in X. If $\gamma(0) = x$ and $\gamma(1) = y$, then γ is also called a path joining the points x and y or simply a path from x to y.
2. A topological space X is said to be path connected if for all points x and y in X, there exists a path $\gamma: [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

I.3 Examples & Exercises

Example 165. The space $\mathbb{R}^n$ is path connected. Any two points can be joined by a line segment: $\gamma(t) := x + t(y - x)$, for $0 \le t \le 1$. We call this path γ a linear path.

Example 166. For every $r > 0$, the circle $C_r := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$ is path connected.(Why?)

Example 167. The set $\{(x, y) \in \mathbb{R}^2 : x \ge 0 \quad \& \quad x^2 - y^2 = 1\}$ is path connected. Draw the picture and see that it is the "right" hand of the hyperbola $x^2 - y^2 = 1$. Similarly the left hand of a hyperbola is also path connected. However the hyperbola is not path connected. (Why?)

Example 168. The parabola $\{(x, y) \in \mathbb{R}^2 : y^2 = x\}$ is path connected.

Example 169. The union of the two parabolas $\{(x, y) \in \mathbb{R}^2 : y^2 = x\}$ and $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$ is path connected.

Example 170. The union of the parabolas $\{(x, y) \in \mathbb{R}^2 : y^2 = x\}$ and $\{(x, y) \in \mathbb{R}^2 : y^2 = -x\}$ is path connected.

Example 171. The set $S^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ is path connected. Let X and Y be two points in S^2. Then define $\gamma\colon [0,1] \to S^2$ by $\gamma(t) := \frac{X+t(Y-X)}{\|X+t(Y-X)\|}$. Then check that this gives us a path from X to Y. (Does it?).

Proposition 172. *Let X be a topological space. Let $\gamma_1\colon [0,1] \to X$ and $\gamma_2\colon [0,1] \to X$ be two paths such that $\gamma_1(1) = \gamma_2(0)$. Then there exists a path $\gamma_3\colon [0,1] \to X$ such that $\gamma_3(0) = \gamma_1(0)$ and $\gamma_3(1) = \gamma_2(1)$.*

Proof. Define the map $\gamma_3\colon [0,1] \to X$ such that

$$\gamma_3(t)\colon = \begin{cases} \gamma_1(2t) & \text{if} \quad t \leq \frac{1}{2} \\ \gamma_2(2t-1) & \text{if} \quad t \geq \frac{1}{2} \end{cases}$$

Now we leave it as an exercise to verify that γ_3 is a path in X meeting our requirements. (Draw pictures and see geometrically). □

Proposition 173. *Let X be path connected. Then X is connected.*

Proof. Let $f\colon X \to \{\pm 1\}$ be a continuous function. We need to show that f is constant.

Let $x \neq y$ be two points in X. Since X is path connected, there exists a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Now, the map $f \circ \gamma\colon [0,1] \to \{\pm 1\}$ is continuous. Since $[0,1]$ is connected, the map $f \circ \gamma$ is constant. Therefore f is constant. (Why?). This proves that X is connected. □

The converse is not always true. However, in the case of open subsets of $\mathbb{R}^n$, the converse is also true and we prove this in

Theorem 174. *Let U be an open connected subset of $\mathbb{R}^n$. Then U is path connected.*

Proof. Let x_0 be a point in U and let

$$E := \{x \in U : \text{ there exists a path } \gamma \text{ such that } \gamma(0) = x \ \&\ \gamma(1) = x_0\}.$$

We will show that the set E is non-empty, both open and closed in U. Then since U is connected, it will follow that $E = U$ and this will prove the theorem. (Why?)

First we note that the set E is non-empty. The map $\gamma\colon [0,1] \to X$ defined by $\gamma(t) = x_0$ for all t is a path in X. Therefore x_0 is in E. Let x be a point in E. Since U is open there exists $r > 0$ such that $B(x, r) \subseteq U$. Let y be a point in $B(x, r)$. Since $B(x, r)$ is convex, there exists a linear path, say, γ_1, joining the points y and x. Since x is in E there exists a path γ_2 from x to the point x_0. From Proposition 172, it follows that there exists a path γ_3 from y to x_0. This means that $B(x, r) \subseteq E$. Hence E is open.

We will now show that E is also closed in U. Let $x \in U$ be a limit point of E. Therefore there exists a sequence x_n of points in E such that the sequence x_n converge to the point x. Since U is open there exists an $r > 0$ such that the open ball $B(x, r) \subseteq U$. Since the sequence x_n converges to the point x, there exists N in $\mathbb{N}$ such that the points $x_n \in B(x, r)$ for all $n \geq N$. Let γ_1 be the linear path from x to the point x_N and γ_2 be a path from x_N to x_0. From Proposition 172, there exists a path γ_3 from x to x_0. This means that the point x is in E. Hence E is closed and therefore $E = U$. $\square$

J Proper Maps

Definition 175. A map $f\colon X \to Y$ is said to be *proper* if for every compact subset $L \subset Y$, the inverse image $f^{-1}(L)$ is a compact subset of X.

Example 176. Any continuous map from a compact space to any hausdorff space Y is proper.

Example 177. Let p be a nonconstant polynomial with complex coefficients. A most important and typical example of a proper map is the function $z \mapsto p(z)$. Recall the standard estimate: There exists $R > 0$ such that

$$|p(z)| \geq \frac{|a_n|}{2}|z|^n, \text{ for } |z| \geq R, \text{ where } p(z) = \sum_{k=0}^{n} a_k z^k.$$

Let K be a compact subset of $\mathbb{C}$. Since p is continuous, $p^{-1}(K)$ is closed. If $p^{-1}(K)$ is not compact, we conclude that $p^{-1}(K)$ is not bounded. (Why? Heine-Borel theorem!) Hence there exists a sequence $z_n \in p^{-1}(K)$ such that $|z_n| \to \infty$, but $p(z_n) \in K$ for all n. By the estimate quoted above, $p(z_n) \to \infty$. But since $p(z_n) \in K$ and K is compact, $\{p(z_n) : n \in \mathbb{N}\}$ is bounded. This contradiction shows that p is proper.

Ex. 178. The exponential map $\exp\colon \mathbb{R} \to \mathbb{R}$ or $\exp\colon \mathbb{C} \to \mathbb{C}$ is not proper.

Lemma 179. *Let $f\colon X \to Y$ be a closed map. Assume that $f^{-1}(y)$ is compact for each $y \in Y$. Then f is proper.*

Proof. Let $L \subset Y$ be a compact subset. Let $\{U_i : i \in I\}$ is an open cover of $K := f^{-1}(L)$. For each $y \in L$, by hypothesis, $f^{-1}(y)$ is compact. Hence, there exists a finite set $J_y \subset I$ such that $\{U_i : i \in J_y\}$ is a finite subcover of $f^{-1}(y)$. Let $U_y := \cup_{i \in J_y} U_i$. Then U_y is open and so $A_y := X \setminus U_y$ is closed in X. Since f is closed, the set $V_y := Y \setminus f(C_y)$ is open in Y. Note that $f^{-1}(V_y) \subset U_y$. Since $y \in V_y$, the collection $\{V_y : y \in L\}$ is an open cover of the compact set L. Hence there exists a finite number of points y_j, $1 \leq j \leq n$ such that $L \subset V_1 \cup \cdots \cup V_n$ where $V_j := V_{y_j}$. But then

$$\begin{aligned} f^{-1}(L) &\subset f^{-1}(V_1) \cup \cdots \cup f^{-1}(V_n) \\ &\subset U_1 \cup \cdots \cup U_n \\ &= \cup\{U_i : i \in J_{y_i}, 1 \leq i \leq n\}, \end{aligned}$$

a finite subcover. □

Lemma 180. *Let X be compact. Then for any topological space Y, the projection $\pi_Y\colon X \times Y \to Y$ is closed.*

Proof. Let $L \subset X \times Y$ be closed. We have to show that $\pi_Y(L)$ is closed in Y. We show that its complement is open in Y. Let $y \in Y$ but $y \notin \pi_Y(L)$. Note that this means that $(x, y) \in L$ for any $x \in X$. What we plan to do is something similar to the preliminary step, the so-called tube lemma, in the proof of compactness of $X \times Y$: There exists an open set V such that $y \in V$ and $(x, y') \notin L$ for any $x \in X$ and $y' \in V$. From this it follows that such a $V \subset Y \setminus \pi_Y(L)$.

Since L is closed and $(x, y) \notin L$, we can find a basic open set $U_x \times V_x$ such that $(x, y) \in U_x \times V_x \subset (X \times Y) \setminus L$. By the compactness of X, we can find $x_1, \ldots, x_n \in X$ such that $U_i := U_{x_i}$, $1 \leq i \leq n$, cover X. Let $V := V_1 \cap \cdots \cap V_n$, where, as is our standard practice $V_i := V_{x_i}$, $1 \leq i \leq n$. Note that V is an open set containing y. We have

$$(X \times Y) \cap L = [(U_1 \cup \cdots \cup U_n) \times (V_1 \cap \cdots \cap V_n)] = \emptyset.$$

□

Proposition 181. *If X is compact, then $\pi_Y \colon X \times Y \to Y$ is proper.*

Proof. Immediate consequence of the last two lemmas. □

Theorem 182. *If X and Y are compact, then $X \times Y$ is compact.*

Proof. By the last proposition, the projection π_Y is proper and hence $X \times Y = \pi_Y^{-1}(Y)$ is compact. □

The next theorem is the philosophical reason for the introduction of proper maps. Loosely speaking, a continuous map is proper iff it maps points near to infinity to points near to infinity. Compare and contrast the non-constant polynomial maps and the exponential maps.

We have a characterization of proper maps between locally compact hausdorff spaces in terms of their one-point compactifications.

Given a locally compact noncompact hausdorff space X, let $X_\infty := X \cup \{\infty\}$ where $\infty \notin X$. Let $\mathcal{T}$ denote the topology on X. Consider

$$\mathcal{T}_\infty := \mathcal{T} \cup \{V \subset X_\infty : X_\infty \setminus V \text{ is compact}\}.$$

Then
(i) $\mathcal{T}_\infty$ is a hausdorff topology on X_∞.
(ii) The subspace topology on X is $\mathcal{T}$.
(iii) $(X_\infty, \mathcal{T}_\infty)$ is compact.
(iv) X is dense in X_∞.

Theorem 183. *Let X and Y be locally compact hausdorff spaces. Then a continuous map $f \colon X \to Y$ is proper iff it extends to a continuous map of X_∞ to Y_∞ with $f(\infty_X) = \infty_Y$.*

Proof. Let f be proper. Extend f as above. Then we need to check its continuity. Let V be open in Y. The $f^{-1}(V)$ is an open subset of X and hence of X_∞. If $V \ni \infty_Y$, then $L := Y_\infty \setminus V$ is a compact subset of Y and hence $f^{-1}(L)$ is a compact subset of X, since f is proper. Since X is hausdorff, $f^{-1}(L)$ is closed. Hence $X \setminus f^{-1}(L)$ is open. But it is nothing but $f^{-1}(V)$.

Let f, the extension as in the statement, be continuous. Then $f^{-1}(Y) = X$, since $f(\infty_X) = \infty_Y$. If $L \subset Y$ is compact, then L is closed in Y and hence in Y_∞. So $f^{-1}(L)$ is closed in X_∞. Since X_∞ is compact, $f^{-1}(L)$ is compact. It is clearly a subset of X. Hence f is proper. □

Proposition 184. *Let $f\colon X \to Y$ be a proper map (i) either between two locally compact hausdorff spaces or (ii) between two metric spaces. Then f is closed.*

Proof. Assume Case (i). Let g denote the extension of f to X_∞. If F is closed in Y, then $F_\infty := F \cup \{\infty_X\}$ is closed in X_∞ and hence is compact. Hence $g(F_\infty)$ is compact in Y_∞ and hence is closed, since Y_∞ is hausdorff. But then $f(F) = g(F_\infty) \cap Y$ is closed in Y. This proves the result in the first case.

We can also prove this directly without recourse to the one-point compactifcations as follows. Let C be closed in X. Let $q \in Y$ be a limit point of $f(C)$. Let V be an open set such that $q \in V$ and $L := \overline{V}$ is compact. (This is possible since Y is locally compact and hausdorff.) Consider $K := f^{-1}(L)$. Then K is closed, since f is proper. As $K \cap C$ is compact, we have $f(K \cap C) = L \cap f(C)$ (verify!) is compact and hence closed since Y is hausdorff. Since $q \in \overline{f(C)}$, and V is an open neighbourhood of q, we see that

$$q \in \overline{L \cap f(C)} = L \cap f(C) = f(K \cap C) \subset f(C).$$

This shows that any limit point q of $f(C)$ lies in $f(C)$ and hence $f(C)$ is closed.

Assume that X and Y are metric spaces. Let $C \subset X$ be closed. Let w be a limit point of $f(C)$. Then there exists a sequence $w_n \in f(C)$ such that $w_n \to w$. Since $w_n \in f(C)$, there exists $z_n \in C$ such that $w_n = f(z_n)$. Now the subset $L := \{w_n : n \in \mathbb{N}\} \cup \{w\}$ is a compact subset of Y. Since f is proper, its inverse image $K := f^{-1}(L)$ is compact. By our choice, (z_n) is a sequence in the compact set K and hence has a convergent subsequence, say, (z_{n_k}) converging to $z \in K$. Since C is closed, we conclude that $z \in C$. By continuity of f at z, we see that $f(z_{n_k}) \to f(z)$. Since $f(z_n) \to w$, it follows that $f(z) = w$. Hence we have shown that $w \in f(C)$, that is, $f(C)$ is closed. □

K Existence of Continuous Functions

If $x \neq y$ are two distinct points of a space X, is there a continuous function $f\colon X \to \mathbb{R}$ such that $f(x) \neq f(y)$? In general, this may not be true. There may not exist continuous functions on the given space other than the constants. For each pair of distinct points, if there is an $f \in C(X, \mathbb{R})$ with $f(x) \neq f(y)$, we say that the family $C(X, \mathbb{R})$ separates points. This is the reason for defining the completely regular and normal spaces which ensures plenty of continuous functions. One kind of spaces for which existence of an abundance of continuous real valued functions is assured is the class of metric spaces. We shall look at them first.

K.1 Case of Metric Spaces

The crucial fact here is the simple observation: If (X, d) is a metric space and $x \in X$, then the function $f_x(y) := d(x, y)$ is continuous on X. For, by triangle inequality we have

$$|f_x(y) - f_x(z)| = |d(x, y) - d(x, z)| \leq d(y, z).$$

Thus $\{f_x : x \in X\}$ is a separating family of continuous functions on X. More generally, we have

Lemma 185. *Let A be any nonempty subset of a metric space (X, d). Define $d(x, A) \equiv d_A(x) := \inf\{d(x, a) : a \in A\}$. Then $|d_A(x) - d_A(y)| \leq d(x, y)$ and hence d_A is uniformly continuous on X.*

Proof. Observe from the triangle inequality $d(x, a) \leq d(x, y) + d(y, a)$, we obtain

$$\begin{aligned} \inf_{a \in A} d(x, a) &\leq \inf_{a \in A} (d(x, y) + d(y, a)) \\ &= d(x, y) + \inf_{a \in A} d(y, a), \end{aligned}$$

so that $d_A(x) \leq d(x, y) + d_A(y)$. Thus, $d_a(x) - d_A(y) \leq d(x, y)$. Interchanging x and y yields the result. □

Ex. 186. $d_A(x) = 0$ iff x is a limit point of A. Hence if A is a closed set then $d(x, A) = 0$ iff $x \in A$.

Lemma 187 (Urysohn's Lemma for Metric Spaces)**.** *Let A and B be nonempty disjoint closed subsets of a metric space X. Then there exists an $f \in C(X, \mathbb{R})$ such that $0 \leq f(x) \leq 1$ for $x \in X$ and $f = 0$ on A and $f = 1$ on B.*

Proof. Note that for any $x \in X$, $d(x, A) + d(x, B) \neq 0$. For, if it were so, then $d(x, A) = 0 = d(x, B)$. Since A and B are closed $x \in A$ and $x \in B$ by the last exercise. This contradicts our hypothesis that $A \cap B = \emptyset$.

The function $f(x) := \frac{d(x,A)}{d(x,A)+d(x,B)}$ meets our requirements. □

Theorem 188 (Tietze extension theorem for metric spaces)**.** *Let Y be a closed subspace of a metric space (X, d). Let $f\colon Y \to \mathbb{R}$ be a bounded continuous function. Then there exists a continuous function $g\colon X \to \mathbb{R}$ such that $g(y) = f(y)$ for all $y \in Y$ and*

$$\inf\{g(x) : x \in X\} = \inf\{f(y) : y \in Y\}, \qquad \sup\{g(x) : x \in X\} = \sup\{f(y) : y \in Y\}.$$

Proof. Assume that $f \geq 0$. Consider the function $M_x(r) := \sup\{f(y) : y \in Y \cap B(x,r)\}$. Then, for each $x \in X$, M_x is real valued, bounded and monotonic increasing in r. Hence it is Riemann integrable as a function of r over any finite interval. Let $\delta(x) := d(x,Y)$. Note that $\delta(x) > 0$ iff $x \notin Y$. We define g by $g(x) = f(x)$ if $x \in Y$ and if $x \notin Y$,

$$g(x) := \frac{1}{\delta(x)} \int_{\delta(x)}^{2\delta(x)} M_x(r)\, dr.$$

We claim that g is continuous at any $y \in Y$. If $x \notin Y$, then

$$\min_{Y \cap B(y,3\delta)} f \leq g(x) \leq \max_{Y \cap B(y,3\delta)} f, \qquad \text{where } \delta := d(x,y).$$

Since $3\delta \to 0$ as $x \to y$ (with $x \notin Y$), it follows that, for any $\varepsilon > 0$, $|g(x) - f(y)| < \varepsilon$ if $x \notin Y$ and $\delta(x) < \delta_0$ for δ_0 sufficiently small. On the other hand, $|g(x) - f(y)| = |f(x) - f(y)| < \varepsilon$ if $x \in Y$ and $d(x,y) < \delta_1$ by continuity of f on Y. Thus g is continuous at y.

Consider next the continuity of g at $z \notin Y$. Let x be any point in X with $d(x,z) < d(z,Y)/3$. Let $\alpha := d(x,z)$. Then $2\alpha < d(x,Y)$. For, otherwise, $d(x,Y) \leq 2\alpha$ so that $d(z,Y) \leq d(z,x) + d(x,Y) < 3\alpha$. Hence $\alpha > d(z,X)/3$, contradicting our assumption on x. Since $|\delta(x) - \delta(z)| \leq d(x,z) = \alpha$ (by Lemma 185), $M_z(r) \geq M_x(r-\alpha)$ as $B(x, r-\alpha) \subset B(z,r)$ and $M_z(r) \geq 0$, we have

$$\begin{aligned}
g(x) - g(z) &= \frac{1}{\delta(x)} \int_{\delta(x)}^{2\delta(x)} M_x(r)\, dr - \frac{1}{\delta(z)} \int_{\delta(z)}^{2\delta(z)} M_z(r)\, dr \\
&\leq \frac{1}{\delta(x)} \int_{\delta(x)}^{2\delta(x)} M_x(r)\, dr - \frac{1}{\delta(x)+\alpha} \int_{\delta(x)+\alpha}^{2\delta(x)-2\alpha} M_x(r-\alpha)\, dr \\
&= \frac{1}{\delta(x)} \int_{\delta(x)}^{2\delta(x)-3\alpha} M_x(r)\, dr + \frac{1}{\delta(x)} \int_{2\delta(x)-3\alpha}^{2\delta(x)} M_x(r)\, dr \\
&\quad - \frac{1}{\delta(x)+\alpha} \int_{\delta(x)}^{2\delta(x)-3\alpha} M_x(s)\, ds, \text{ using a change of variable} \\
&= \frac{\alpha}{\delta(x)[\delta(x)+\alpha]} \int_{\delta(x)}^{2\delta(x)-3\alpha} M_x(r)\, dr + \frac{1}{\delta(x)} \int_{2\delta(x)-3\alpha}^{2\delta(x)} M_x(r)\, dr \\
&\leq \frac{4M\alpha}{\delta(x)},
\end{aligned}$$

where $M = \sup_Y f$. A similar inequality holds with x and z interchanged. Hence $g(x) \to g(z)$ as $x \to z$. One easily checks that g is as desired.

To treat the general case, let $m := \inf_Y f$. Consider $F := f - m$. Apply the first case to F to get a continuous extension G. Then $g := G + c$ is as required. □

We shall use Weierstrass approximation theorem to give a proof of Tietze theorem for $\mathbb{R}^n$.

Proof. (of Tietze theorem for $\mathbb{R}^n$.) Let us prove the result when the closed set is compact. So, we assume that $f\colon K \to \mathbb{R}$ is a continuous function on a compact subset of $\mathbb{R}^n$. By

Weierstrass approximation theorem, for each $k \in \mathbb{Z}_+$, there exists a polynomial p_k such that $|f(x) - p(x)| < 2^{-k-2}$ for all $x \in K$. We let $q_0 = p_0$ and $q_k := p_k - p_{k-1}$. Then $p_k = \sum_{i=1}^k q_i$ and $\sum q_k$ converges uniformly to f on K.

Let $M := \sup\{|f(x)| : x \in K\}$. Then $|p_0(x)| \leq 2^{-2} + M$ for $x \in K$. Also, $|q_k(x)| < 2^{-k}$ for $k \geq 1$ and $x \in K$. We let

$$\begin{aligned} h_0 &:= \max\{-2^{-2} - M, \min\{q_0, 2^{-2} + M\}\}, \\ h_k &:= \max\{-2^{-k}, \min\{q_k, 2^{-k}\}\}, \qquad \text{for } k \geq 1. \end{aligned}$$

Then $h_k(x) = q_k(x)$ for $x \in K$, h_k is continuous on $\mathbb{R}^n$ and $|h_k(x)| \leq 2^{-k}$ for $x \in \mathbb{R}^n$ and for all k. Hence $\sum h_k$ converges uniformly on $\mathbb{R}^n$ to a continuous function h. Then h is continuous and $h(x) = f(x)$ for $x \in K$.

We now extend to result if the subset K is any arbitrary closed subset. If K is bounded the result follows from the previous paragraph. So, we assume that K is not bounded. Let $k \in \mathbb{N}$ be such that $B[0, k] \cap K$ is nonempty. Let f_k be the restriction of f to this nonempty compact set. Then there exists a continuous function h_k on $\mathbb{R}^n$ which extends f_k. Define

$$g_k(x) := \begin{cases} h_k(x), & \text{if } x \in B[0, k] \\ f(x), & \text{if } x \in K \cap B[0, k+1]. \end{cases}$$

Then g_k is continuous on the compact set $B[0, k] \cup (K \cap B[0, k+1])$. There is an extension h_{k+1} on $\mathbb{R}^n$. Let

$$g_{k+1}(x) := \begin{cases} h_{k+1}(x), & \text{if } x \in B[0, k+1] \\ f(x), & \text{if } x \in K \cap B[0, k+2]. \end{cases}$$

Continuing in this way, we obtain a sequence (g_m) whose domains are increasing to $\mathbb{R}^n$. Define $g(x) := g_m(x)$ if $x \in B[0, m]$. Then g is an extension of f. □

K.2 Normal Spaces

Lemma 189. *A space X is a normal space iff for each closed set F and an open set V containing F there exists an open set U such that $F \subset U \subset \overline{U} \subset V$.*

Proof. Let X be normal and F, V as above. Then F and $X \setminus V$ are disjoint closed sets. By normality of X there exist open sets U and W such that $F \subset U$ and $X \setminus V \subset W$ and $U \cap W = \emptyset$. Since $U \subset X \setminus W$ and $X \setminus W$ is closed, we see that $\overline{U} \subset X \setminus W \subset V$. Thus U is as required. The converse is left as an exercise. □

Ex. 190. Recall that a dyadic rational is a rational number of the form $p/2^n$, where p and n are integers. Show that the set of dyadic rationals is dense in $\mathbb{R}$.

Lemma 191. *urys2 Let X be a normal space. If A and B are closed subsets of X, for each dyadic rational $r = k2^{-n} \in (0, 1]$, there is an open set U_r with the following properties: (i) $A \subset U_r \subset X \setminus B$, (ii) $\overline{U}_r \subset U_s$ for $r < s$.*

Proof. Let $U_1 := X \setminus B$. By the last lemma, there exist disjoint open sets V and W such that $A \subset V$ and $B \subset W$. Let $U_{1/2} = V$. Then, since $X \setminus W$ is closed, we have

$$A \subset U_{1/2} \subset \overline{U}_{1/2} \subset X \setminus W \subset X \setminus B = U_1.$$

Applying the same lemma once again to the open set $U_{1/2}$ containing A and to the open set U_1 containing $\overline{U}_{1/2}$, we get open sets $U_{1/4}$ and $U_{3/4}$ such that

$$A \subset U_{1/4} \subset \overline{U}_{1/4} \subset U_{1/2} \subset \overline{U}_{1/2} \subset AU_{3/4} \subset \overline{U}_{3/4} \subset V.$$

Continuing this manner, we construct, for each dyadic rational $r \in (0,1)$, an open set U_r with the following properties:
(i) $\overline{U}_r \subset U_s$, $0 < r < s \leq 1$.
(ii) $A \subset U_r$, $0 < r \leq 1$. (iii) $U_r \subset U_1$, $0 < r \leq 1$.

More formally, we proceed as follows. We select U_r for $r = k2^{-n}$ by induction on n. Assume that we have chosen U_r for $r = k2^{-n}$, $0 < k < 2^n$, $1 \leq n \leq N-1$. To find U_r for $r = (2j+1)2^{-N}$, $0 \leq j < 2^{N-1}$, observe that $\overline{U}_{j2^{1-N}}$ and $X \setminus U_{(j+1)2^{1-N}}$ are disjoint closed sets. So once again appealing to the last lemma, we can choose an open set U_r such that

$$\overline{U}_{j2^{1-N}} \subset U_r \subset \overline{U}_r \subset U_{(j+1)2^{1-N}}.$$

These U_r's are as desired. □

Theorem 192. Urysohn's Lemma. *A space X is a normal space iff the following is true: For any two disjoint closed subsets A and B of X there exists a continuous function $f\colon X \to [0,1]$ such that $f = 0$ on A and $f = 1$ on B.*

Proof. Let U_r's be as in the last lemma. We define the function f so that the sets ∂U_r are the level sets of f for the value r. We achieve this by defining

$$f(x) = \begin{cases} 0, & x \in U_r \text{ for all } r \\ \sup\{r : x \notin U_r\}, & \text{otherwise.} \end{cases}$$

Clearly, $0 \leq f \leq 1$, $f = 0$ on A and $f = 1$ on B. We need only establish the continuity of f.

Let $x \in X$ be such that $0 < f(x) < 1$. Let $\varepsilon > 0$. Choose dyadic rationals r and s in $(0,1)$ such that $f(x) - \varepsilon < r < f(x) < s < f(x) + \varepsilon$. Then $x \notin U_t$ for dyadic rationals $t \in (r, f(x))$. By (i), $x \notin \overline{U}_r$. On the other hand $x \in U_s$. Hence $W = U_s \setminus \overline{U}_r$ is an open neighbourhood of x. If $y \in W$, then from the definition of f we see that $r \leq f(y) \leq s$. In particular, $|f(y) - f(x)| < \varepsilon$ for $y \in W$. Thus f is continuous at x. The cases when $f(x) = 0$ or 1 are easier and left to the reader. □

Lemma 193. *Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces. Let $T\colon \mathbf{X} \to \mathbf{Y}$ be a bounded linear map. Assume that for $y_0 \in \mathbf{Y}$ there exist constants M and $r \in (0,1)$ such that there exists $x \in \mathbf{X}$ such that $\|x\| \leq M\|y_0\|$ and $\|y_0 - Tx\| \leq r\|y\|$. Then there exists $z \in \mathbf{X}$ such that $Tz = y_0$ with $\|z\| \leq M/(1-r)$.*

Proof. Let $y \in \mathbf{Y}$ be given. We may assume without loss of generality that $\| y \| = 1$. Given $y \in \mathbf{Y}$ let $z_1 = x$ as given in the lemma. For $y_0 = y - Tz_1$, we can find a $z_2 \in \mathbf{X}$ such that $\| z_2 \| \leq M \| y - Tz_1 \| \leq Mr$ and $\| y - Tz_1 - Tz_2 \| \leq r \| y - Tz_1 \| \leq r^2$. Proceeding by induction, we get a sequence (z_n) in $\mathbf{X}$ such that (i) $\| z_n \| \leq mr^{n-1}$ and (ii) $\| y - \sum_{i=1}^n Tz_i \| \leq r^n$. The series $\sum_{n=1}^\infty z_n$ converges to an element $z \in \mathbf{X}$. We have $Tz = y_0$. □

Theorem 194 (Tietze Extension Theorem)**.** *Let X be a normal space and Y a closed subset of X. Let $f \in \mathbf{Y} := C_b(Y, \mathbb{R})$. Then there exists a $g \in \mathbf{X} := C_b(X, \mathbb{R})$ such that $g(y) = f(y)$ for all $y \in Y$ and* $\sup\{g(x) : x \in X\} = \sup\{f(y) : y \in Y\}$.

Proof. Let $T \colon \mathbf{X} \to \mathbf{Y}$ denote the restriction map $g \mapsto g_{|_Y}$. We show that T satisfies the hypothesis of the previous lemma. Without loss of generality, assume that $|f(y)| \leq 1$ for all $y \in Y$. Let $A := f^{-1}([-1, -1/3])$ and $B := f^{-1}([1/3, 1])$. Then A and B are closed in Y and hence in X. By Urysohn's lemma, there exists a $g \in \mathbf{X}$ such that $|g(x)| \leq 1/3$ for $x \in X$ and $g = -1/3$ on A and $g = 1/3$ on B. One easily checks that $\| Tg - f \|_{\mathbf{X}} \leq 1/3$. If we take $M = 1/3$ and $r = 2/3$, then T satisfies the previous lemma. Note that the assertion about the equality of the norms is also obtained. □

Ex. 195. Let X be a normal space and F a closed subset. Assume that $f \colon F \to (-R, R)$ be a continuous function. Then f can be extended to a continuous function from X to $(-R, R)$. *Hint:* You may need Urysohn's lemma.

Ex. 196. Let X be a normal space and F a closed subset. Assume that $f \colon F \to \mathbb{R}$ be a continuous function. Then f can be extended to a continuous function from X to $\mathbb{R}$. *Hint:* $\mathbb{R}$ is homeomorphic to $(-1, 1)$.

Ex. 197. Assuming Tietze extension theorem, prove Urysohn's lemma.

Ex. 198. Let A be a closed subset of a normal space X. Let $f \colon A \to S^n$ be continuous. Show that there exists an open set $U \supset A$ (U depends on f) and an extension g of f to U.

Ex. 199. Show that with the notation of Exer. 198 that f may not extend to all of X. *Hint:* What happens (i) if $n = 0$ and X is connected or (ii) if $X := B[0, 1] \subset \mathbb{R}^{n+1}$, $A := S^n$ and f is the identity?

L Topological Groups — via Problems

Definition 200. A *topological group* is a triple $(G, \tau, \cdot)$ such that (G, τ) is a topological space and $(G, \cdot)$ is a group. Both these structures are inter-related in the sense that the group operations are continuous, that is,
(i) the group multiplication $G \times G \to G$ given by $(x, y) \mapsto xy$ and
(ii) inversion map $G \to G$ given by $x \mapsto x^{-1}$ are continuous.

Example 201. $(\mathbb{R}^n, +)$ is a topological group with the usual topology.

Example 202. Let $GL(n, \mathbb{K})$ denote the group of all $n \times n$ invertible matrices with entries in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Then $GL(n, \mathbb{K}) \subset M(n, \mathbb{K}) \simeq \mathbb{K}^{n^2}$ is open. $GL(n, \mathbb{K})$ is a topological group.

Example 203. The group $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is a topological group with the subspace topology.

Example 204. Let G be any group. If G is endowed with the discrete topology, then G is a topological group.

Ex. 205. If we endow an abstract group with the indiscrete topology, does it become a topological group?

Ex. 206. Show that the triple $(G, \tau, \cdot)$ is a topological group iff the map $(x, y) \mapsto xy^{-1}$ is continuous.

Ex. 207. If G_i, $1 \leq i \leq n$, are topological groups, then so is their product with the product topology.

Ex. 208. If H is a subgroup of a topological group, then H is a topological group with the subspace topology.

Ex. 209. How do you define a topological subgroup of a topological group?

Ex. 210. If H is a subgroup of G, then the closure $\overline{H}$ is also a subgroup.

Ex. 211. The following subgroups (of the respective groups) are topological groups with the subspace topology.
(a) Let $SL(n, \mathbb{K})$ denote the subgroup of $GL(n, \mathbb{K})$ with determinant 1.
(b) Let $O(n, \mathbb{R})$ denote the subgroup of $GL(n, \mathbb{R})$ of all orthogonal matrices.
(c) Let $U(n)$ denote the subgroup of all unitary matrices in $GL(n, \mathbb{C})$.
(d) Let $SO(n, \mathbb{R})$ and $SU(n, \mathbb{R})$ denote respectively the subgroups consisting of elements of $O(n, \mathbb{R})$ and $U(n)$ whose determinant is one.
(e) Let $GL^+(n, \mathbb{R})$ denote the subgroup of all elements with positive determinant.
Show that $O(n, \mathbb{R})$, $SO(n, \mathbb{R})$, $U(n)$ and $SU(n)$ are compact.

Ex. 212. The left translations $L_a \colon x \mapsto ax$ are homeomorphisms. So are the right translations R_a.

Ex. 213. If U is an open set, so is $U^{-1} := \{x^{-1} : x \in U\}$. If A is an arbitrary subset of G, then AU and UA are open.

Ex. 214. Let $\mathcal{U}$ denote the set of all neighborhoods of $e \in G$. Show that the topology on G is completely determined by the knowledge of $\mathcal{U}$.

Ex. 215. With the notation of the last exercise, prove that $\mathcal{U}$ has the following properties:
(i) $e \in U$ for all $U \in \mathcal{U}$.
(ii) If $U_1, U_2 \in \mathcal{U}$, then there exists $U \in \mathcal{U}$ such that $U \subset U_1 \cap U_2$. *Hint:* Use the continuity of the group multiplication at (e, e).
(iii) If $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $VV^{-1} \subset U$.
(iv) If $U \in \mathcal{U}$ and $a \in U$, then there exists $V \in \mathcal{U}$ such that $aV \subset U$.
(v) If $U \in \mathcal{U}$ and $a \in G$, there exists $V \in \mathcal{U}$ such that $aVa^{-1} \subset U$.

Ex. 216. Let G be a group. Let $\mathcal{U}$ be a collection of subsets with the properties enumerated in the previous exercise. Show that there exists a topology on G such that G becomes a topological group with this topology and the neighbourhood basis at $e \in G$ is precisely $\mathcal{U}$.

Ex. 217. Let G be a topological group. Let $\mathcal{U}$ be the neighbourhood base at e. Show that G is Hausdorff iff $\cap_{U \in \mathcal{U}} = \{e\}$.

Ex. 218. If H is a normal subgroup of G, then the closure $\overline{H}$ is also a normal subgroup.

Ex. 219. If H is an open subgroup of G, then H is closed. *Hint:* Consider the coset decomposition of G with respect to H. That is, observe that $H = G \setminus \cup_{x \notin H} xH$.

Ex. 220. If H is a closed subgroup of finite index in a topological group, then H is open.

Ex. 221. Let G be a topological group and $E \subset G$. Show that

$$\overline{E} = \cap_{U \in \mathcal{U}} UE = \cap_{U \in \mathcal{U}} EU.$$

Ex. 222. How will you define the uniform continuity of $f : G \to \mathbb{C}$ on any topological group? (G need not be metrizable.)

Ex. 223. Let $f : G \to \mathbb{C}$ be a continuous function with compact support. Show that f is uniformly continuous.

Ex. 224. Let G and H be topological groups. The a group homomorphism $f : G \to H$ is continuous iff it is continuous at e.

Ex. 225. If G is a topological group and H is a subgroup, then the coset space G/H is endowed with the quotient topology. The quotient map $\pi : G \to G/H$ is an open continuous map.

Ex. 226. With the notation of the previous exercise, if we further assume that H is normal in G, then the quotient group G/H becomes a topological group with the quotient topology.

Ex. 227. Show that the quotient group G/H is Hausdorff iff H is closed in G. Is it still true if H is only a subgroup rather than a normal subgroup?

Ex. 228. When is G/H discrete?

Ex. 229. Let G be a connected topological group. Let U be a symmetric neighbourhood of e, that is, $U = U^{-1}$. Then $G = \cup_{n=1}^{\infty} U^n$. *Hint:* Observe that the union is an open subgroup.

Ex. 230. With the notation of the last exercise, assume that G is also compact. Can you sharpen the result in this case?

Ex. 231. Let H be a subgroup of a topological group G. If G/H and H are connected then G is connected.

Ex. 232. Let G be a topological group. Let G_0 denote the connected component of G containing e. Show that G_0 is a closed normal subgroup.

Ex. 233. Show that $GL^+(n, \mathbb{R})$ is connected and that it is the connected component of $GL(n, \mathbb{R})$ containing the identity. *Hint:* By induction. For, $n > 1$, consider the subgroup H consisiting fo elements of the form $g = \begin{pmatrix} 1 & v \\ 0 & h \end{pmatrix}$ where $v \in \mathbb{R}^{n-1}$ and $h \in GL^+(n-1, \mathbb{R})$.

Ex. 234. Show that $GL(n, \mathbb{C})$ is connected.

Ex. 235. How will you define the action of a topological group on a topological space X?

Definition 236. We say that a topological group G acts on a topological space X if the group G acts on X in the algebraic sense and if the group action $G \times X \to X$ given by $(g, x) \mapsto gx$ is continuous. We then say X is a G-space.

Ex. 237. Examples of such actions. $SL(2, \mathbb{R})$ acts on the upper half plane via fractional linear transformations. $O(n, \mathbb{R})$ acts on the unit sphere S^{n-1}. The group of affine transformations $f_{A,v} : x \mapsto Ax + v$ on $\mathbb{K}^n$ where A is a nonsingular linear map and $v \in \mathbb{K}^n$ is fixed. The group law is the composition of maps. This group acts on $\mathbb{K}^n$.

Ex. 238. Let G the a topological group and H a closed subgroup. Let $X := G/H$ be the quotient space. Then G acts on X via $(g, xH) \mapsto gxH$. This action is transitive.

Ex. 239. When do we say two G-spaces X and Y are G-isomorphic?

Ex. 240 (Baire's Theorem)**.** Let X be a locally compact Hausdorff space. Assume that $X = \cup_{n=1}^{\infty} F_n$ where F_n is closed for each n. Show that at least one F_n is open in X. *Hint:* Go through the proof in the case of metric spaces.

Ex. 241. Let X be a locally compact Hausdorff space. Let a locally compact Hausdorff group with a countable basis. Assume that G acts on X transitively. Let $x_0 \in X$ be fixed. Let H be the isotropy of x_0, that is, $H := \{g \in G : gx_0 = x_0\}$. Then X is "isomorphic" to the quotient space G/H as G-spaces.

Ex. 242. What is the isotropy at i when $SL(2, \mathbb{R})$ acts on the upper half plane? Same question when $O(n)$ acts on $S^{n-1} \subset \mathbb{R}^n$.

Ex. 243. Show that $O(n, \mathbb{R})/O(n-1, \mathbb{R})$ is G-isomorphic to S^{n-1}. How does $O(n-1, \mathbb{R})$ sit in $O(n, \mathbb{R})$?

Definition 244. A subgroup $\Gamma \subset G$ is called a *discrete subgroup* if it is a subgroup of G which is both closed and discrete as a subset of G.

Ex. 245. Let Γ be a discrete subgroup of $\mathbb{R}^n$. Then there exist $v_1, \ldots, v_d \in \Gamma$ such that
(i) $v_1, \ldots, v_d$ are linearly independent over $\mathbb{R}$.
(ii) Every element of Γ is uniquely written as an integral linear combination of v_j's.

(iii) d is unique though v_j's need not be.
Hint: Consider the L^1-norm on $\mathbb{R}^n$: $\|x\|_1 := \sum_{i=1}^n |x_i|$. Show that $\inf\{\|\gamma\| : \gamma \in \Gamma, \gamma \neq 0\}$ is positive and attained, say, $v_1 \in \Gamma$. If $\Gamma \neq \mathbb{Z}v_1$, extend it to a basis $\{u_1 = v_1, \ldots, u_n\}$ of $\mathbb{R}^n$. Show that

$$\inf\{\sum_{j\neq 1} |x_j| : \text{ where } \gamma \in \Gamma \setminus \{0\}, \gamma = \sum_{i=1}^n x_i u_i\}$$

is attained at some $v_2 \in \Gamma$.

Ex. 246. If $f\colon G \to H$ is a continuous homomorphism into a locally compact Hausdorff group H, then f is necessarily open.

Ex. 247. Let G be a connected group and H a discrete normal subgroup of G. Then H is contained in the center of G.

M Discrete subgroups of $\mathbb{R}^n$

Proposition 248. *Let Γ be a discrete subgroup of $\mathbb{R}^n$. Then there exists a basis $u_1, u_2, \dots, u_n$ of $\mathbb{R}^n$ such that*

$$\Gamma = \{x \in \mathbb{R}^n : x \text{ is of the form } x = n_1 u_1 + \cdots + n_r u_r, n_j \in \mathbb{Z}\},$$

for some $r \le n$.

Proof. The proof is an inductive construction. Let W be a vector subspace of $\mathbb{R}^n$ such that $\Gamma \cap W = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_k$ for some basis $w_1, \dots, w_k$ of W. Such W's exist, for instance $W = \{0\}$! Suppose that there exists $u \in \Gamma$ that does not lie in W. Consider the set B_W of points

$$a_1 w_1 + \cdots + a_k w_k + bu, \quad 0 \le a_i \le 1, 0 \le b \le 1. \tag{8}$$

This set is bounded in $\mathbb{R}^n$. Since Γ is discrete, this set B_W can contain only finitely many points of Γ. Hence there exists a point $v \in B_W \cap \Gamma$ such that the coefficient b of u in v will be the least positive coefficient, say β. If $a_1 w_1 + \cdots + a_k w_k + bu$ lies in Γ with $a_i, b \in \mathbb{Z}$, then b is a multiple of β. For, otherwise, by division algorithm, we write $b = m\beta + r$ where $0 < r < \beta$. Hence the element

$$a_1 w_1 + \cdots + a_k w_k + ru = (a_1 w_1 + \cdots + a_k w_k + bu) - m\beta u \in \Gamma.$$

Since $w_j \in \Gamma$, by subtracting suitable multiples of w_j, we can assume that $0 \le a_j \le 1$. In other words, $a_1 w_1 + \cdots + a_k w_k + ru \in B_W$. This contradicts our choice of μ. Thus we have established that

$$\Gamma \cap (W + \mathbb{R}u) = (\Gamma \cap W) + \mathbb{Z}v = \mathbb{Z}w_1 + \cdots + \mathbb{Z}w_k + \mathbb{Z}u.$$

Note that the set $\{w_1, \dots, w_k, u\}$ is a basis of $W + \mathbb{R}u$. If there exists $u' \in \Gamma$, we can proceed as above. This process has to stop in a finite number of steps. $\square$

N Non-contractibility of the circle and Brouwer Fixed Point Theorem

The aim of this article is to classify the homotopy classes of maps from a circle to the punctured plane

We prove that the circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ is not contractible and derive its consequences. We start with a lemma from complex analysis which says that it is possible to assign the argument of a complex number in a continuous fashion if we restrict ourselves to $\mathbb{C}$ minus $\{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$, or the complex plane minus any closed half line starting from the origin.

Lemma 249. *There exists a continuous map*

$$\alpha\colon X := C \setminus \{z \in \mathbb{C} : z \in \mathbb{R} \text{ and } \leq 0\} \to (-\pi, \pi)$$

such that $z = |z|e^{i\alpha(z)}$ for all $z \in X$.

Proof. Let us define the following open half-planes whose union is X: $H_1 := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, $H_2 := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $H_3 := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$. We define α_i on H_i which glue together to give the required map.

Let $z \in H_1$. Then $\operatorname{Re} z = |z| \cos\theta$ for some $\theta \in [-\pi, \pi]$ and hence $\cos\theta > 0$. This means that $\theta \in (-\pi/2, \pi/2)$. sin is increasing on $(-\pi/2, \pi/2)$ so that we have the continuous inverse $\sin^{-1}\colon (-1, 1) \to (-\pi/2, \pi/2)$. We define $\alpha_1(z) := \sin^{-1}(\frac{\operatorname{Im} z}{|z|})$. We can similarly define $\alpha_2\colon H_2 \to (0, \pi)$ and $\alpha_3\colon H_3 \to (-\pi, 0)$ by

$$\begin{aligned}\alpha_2(z) &= \cos^{-1}(\frac{\operatorname{Re} z}{|z|})\\ \alpha_3(z) &= \cos^{-1}(\frac{\operatorname{Re} z}{|z|}).\end{aligned}$$

One easily sees that they agree upon their common domains. Thus we get the required function α. ☐

Definition 250. Let f and g be continuous functions from a space X to Y. Then f and g are *homotopic* iff there is a continuous function $H\colon I \times X \to Y$ such that $H(0, x) = f(x)$ and $H(1, x) = g(x)$ for all $x \in X$. H is called a homotopy from f to g. Thus a homotopy enables one to *pass continuously* from one map to another.

Lemma 251. *Assume that $f\colon S^1 \to S^1$ is homotopic to a constant map. Then there is a continuous function $\varphi\colon S^1 \to \mathbb{R}$ such that $f(x) = e^{i\varphi(x)}$ for all $x \in S^1$.*

Proof. Let $H\colon I \times S^1 \to S^1$ be a homotopy with $H(0, x) = c$ and $H(1, x) = f(x)$ for $x \in S^1$. Since H is uniformly continuous, for $\varepsilon = 2$, there is a $\delta > 0$ such that

$$|H(s, x) - H(t, x)| < 2, \qquad \text{for } |s - t| < \delta, \quad x \in S^1.$$

Let $0 = t_0 < t_1 \cdots < t_n = 1$ be a partition of I such that $|t_i - t_{i+1}| < \delta$ for $0 \le i \le n-1$. Note that $H(0,x) = c = e^{i\psi(x)}$ for some constant map $\psi\colon S^1 \to \mathbb{R}$. We show that $H(t_1, x) = e^{i\varphi_1(x)}$ for some φ_1.

Since $|H(t_1,x) - H(0,x)| < 2$, we see that $H(t_1,x) \neq -H(0,x)$ and hence that $\frac{H(t_1,x)}{H(0,x)} \neq -1$ for $x \in S^1$. We define a continuous function $\alpha\colon S^1 \to \mathbb{R}$ by setting $\alpha(x)$ to be the argument of x taking values in $(-\pi, \pi)$. (This is possible by Lemma 263.) Thus $\frac{H(t_1,x)}{H(0,x)} = e^{i\alpha(x)}$ and consequently

$$H(t_1,x) = e^{i\alpha(x)} H(0,x) = e^{i(\psi(x)+\alpha(x))} = e^{i\varphi_1(x)},$$

where $\varphi_1(x) = \psi(x) + \alpha(x)$. Continuing this way proves the lemma. □

Definition 252. A space is said to be *contractible* if there is a homotopy between the identity map and a constant map.

Ex. 253. Any convex subset of $\mathbb{R}^n$ is contractible.

Theorem 254. *The circle S^1 is not contractible.*

Proof. If it were, then by Lemma 251 there is a function $\varphi\colon S^1 \to \mathbb{R}$ such that $Id(x) \equiv x = e^{i\varphi(x)}$ for all $x \in S^1$. Hence φ is 1-1 and in particular $\varphi(x) \neq \varphi(-x)$. Define $g\colon S^1 \to \{\pm 1\}$ by

$$g(x) := \frac{\varphi(x) - \varphi(-x)}{|\varphi(x) - \varphi(-x)|}.$$

Then g maps S^1 continuously *onto* $\{\pm 1\}$. This contradicts the connectedness of S^1. □

Definition 255. A subset A of a space X is a *retract* of X if there is a continuous function $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$. r is called a retraction of X onto A.

Corollary 256. *There is no retraction of $\mathbb{R}^2$ onto S^1.*

Proof. Let $r\colon \mathbb{R}^2 \to S^1$ be retraction. Let $p = (0,0)$. Define a homotopy $H\colon I \times S^1 \to \mathbb{R}^2$ by $H(t,x) = tp + (1-t)x$. Then $r \circ H\colon I \times S^1 \to S^1$ is a contraction — contradicting Thm. 279. □

Corollary 257 (Brouwer Fixed Point Theorem)**.** *Let $f\colon B[0,1] \to B[0,1]$ be a continuous map. Then f has a fixed point, i.e., there is an $x \in B[0,1]$ such that $f(x) = x$.*

Proof. If there is no point x such that $f(x) = x$, then the two distinct points $f(x)$ and x determine a line joining $f(x)$ and x. We let $g(x)$ be the point on the boundary at which the line starting from $f(x)$ and going to x meets S^1. Then g is a retraction of $B[0,1]$ onto S^1—a contradiction to Corollary 281. In analytical terms, we have $g(x) = x + tv$, where $v = \frac{x - f(x)}{\|x - f(x)\|}$ and $t = -\langle x, v\rangle + \sqrt{1 - \|x\|^2 + (\langle x, v\rangle)^2}$. □

Corollary 258 (Generalised Brouwer Fixed Point Theorem)**.** *Let $f\colon B[0,1] \to \mathbb{R}^2$ be continuous such that $f(S^1) \subset B[0,1]$. Then f has a fixed point.*

Proof. Define $r\colon \mathbb{R}^2 \setminus \{(0,0)\} \to S^1$ by $r(x) = x/|x|$. If $f(x) \neq x$ for all $x \in B(0,1)$ then S^1 can be contracted via the homotopy

$$H(t,x) = \begin{cases} r(x - 2tf(x)), & 0 \leq t \leq 1/2, \\ r((2-2t)x - f((2-2t)x)), & 1/2 \leq t \leq 1. \end{cases}$$

This contradicts Thm. 279. □

O Maps into Punctured Plane

There is a lot of overlap between this and the last sections. Need to edit this carefully to avoid meaningless repetition. Details!

CARE! The aim of this article is to classify the homotopy classes of maps from a circle to the punctured plane $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Such a classification can be obtained from the knowledge of the fundamental group $\pi_1(S^1)$ of the circle. Our approach will be more analytic and will yield an alternative proof of the isomorphism $\pi_1(S^1) \simeq \mathbb{Z}$.

Definition 259. Let f and g be continuous functions from a space X to Y. Then f and g are *homotopic* iff there is a continuous function $H\colon I \times X \to Y$ such that $H(0, x) = f(x)$ and $H(1, x) = g(x)$ for all $x \in X$. H is called a homotopy from f to g. Thus a homotopy enables one to *pass continuously* from one map to another.

Let X be a topological space. We consider maps from X into $\mathbb{C}^*$. Such functions form a group under pointwise multiplication.

Definition 260. A map $f\colon X \to \mathbb{C}^*$ is an *exponential* if $f = \exp(g) = e^g$ for some continuous map $g\colon X \to \mathbb{C}$.

Ex. 261. The exponential maps form a subgroup of the group of maps from X to $\mathbb{C}^*$.

Theorem 262. *It is impossible to make a continuous choice $\theta(z) \in \arg(z)$ on $\mathbb{C}^*$. That is, there is no continuous map $\theta\colon \mathbb{C}^* \to \mathbb{R}$ such that $z = |z| \exp(\theta(z))$ for $z \in \mathbb{C}^*$.*

Proof. Assuming such a θ exists, consider $f\colon [0, 2\pi] \to \mathbb{R}$ by setting $f(t) := [\theta(e^{it}) + \theta(e^{-it})]/2\pi$. Then f is a real valued continuous function on $[0, 2\pi]$. Then $2\pi f(t)$ is a choice of $\arg(e^{it}e^{-it})$ and hence of $\arg(1)$. Thus it is integer valued continuous function on the interval $[0, 2\pi]$. By intermediate value theorem, it is a constant. In particular, $f(0) = f(\pi)$. This implies that $[\theta(1) + \theta(1)]/2\pi = [\theta(-1) + \theta(-1)]/2\pi$. Or, $\theta(1) = \theta(-1)$, which is impossible as the $\arg(1)$ and $\arg(-1)$ are disjoint. □

However, the following lemma says that it is possible to assign the argument of a complex number in a continuous fashion if we restrict ourselves to $\mathbb{C}$ minus $\{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$, or the complex plane minus any closed half line starting from the origin.

Lemma 263. *There exists a continuous map*

$$\alpha\colon X := C \setminus \{z \in \mathbb{C} : z \in \mathbb{R} \;\&\; z \leq 0\} \to (-\pi, \pi)$$

such that $z = |z|e^{i\alpha(z)}$ for all $z \in X$.

Proof. Let us define the following open half-planes whose union is X: $H_1 := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, $H_2 := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $H_3 := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$. We define α_i on H_i which glue together to give the required map.

Let $z \in H_1$. Then $\operatorname{Re} z = |z| \cos\theta$ for some $\theta \in [-\pi, \pi]$ and hence $\cos\theta > 0$. This means that $\theta \in (-\pi/2, \pi/2)$. sin is increasing on $(-\pi/2, \pi/2)$ so that we have the continuous inverse $\sin^{-1}\colon (-1,1) \to (-\pi/2, \pi/2)$. We define $\alpha_1(z) := \sin^{-1}(\frac{\operatorname{Im} z}{|z|})$. We can similarly define $\alpha_2\colon H_2 \to (0,\pi)$ and $\alpha_3\colon H_3 \to (-\pi, 0)$ by

$$\begin{aligned} \alpha_2(z) &= \cos^{-1}(\frac{\operatorname{Re} z}{|z|}) \\ \alpha_3(z) &= \cos^{-1}(\frac{\operatorname{Re} z}{|z|}). \end{aligned}$$

One easily sees that they agree upon their common domains. Thus we get the required function α. ☐

Every continuous function f from X to positive reals is an exponential. In this case $f = e^g$ where $g := \log f$. More generally we have

Lemma 264. *Suppose $f\colon X \to \mathbb{C}^*$ is a map that omits the negative real axis (that is, $f(X) \cap (-\infty, 0] = \emptyset$). Then f is an exponential.*

Proof. We use the previous lemma. Recall that the principal logarithm Log is defined on the given open subset of $\mathbb{C}$ by $\operatorname{Log} z = |z|e^{i\theta}$, where $\theta \in (-\pi, \pi)$. Thus Log depends continuously on z. If we set $g(z) := \operatorname{Log} f$, then we have $f = e^g$. ☐

The following result is related to Rouche's theorem in Complex Analysis.

Theorem 265. *Let f and g be functions from X to $\mathbb{C}$ satisfying*

$$|f(x) - g(x)| < |f(x)| + |g(x)|, \qquad x \in X. \tag{9}$$

Then f/g and g/f are exponential. In particular f is an exponential iff g is.

Proof. Observe that the strict inequality in Eq. 9 implies that neither f nor g can vanish on X. Dividing Eq. 9 by $f(x)$ we obtain

$$|1 - g(x)/f(x)| < 1 + |g(x)/f(x)|, \qquad x \in X.$$

It follows that g/f cannot assume negative real values, for, then the RHS will equal the LHS. Hence by Lemma 264, g/f is an exponential. As Eq. 9 is symmetric in f and g this means that f/g is also an exponential. The last statement is a consequence of the fact that the product of exponentials is an exponential. ☐

Theorem 266. *Let X be a compact metric space and $f, g\colon X \to \mathbb{C}^*$. Then f and g are homotopic iff f/g is an exponential.*

Proof. Suppose that f/g is an exponential, say, $f/g = e^h$. Then $F(t,x) := g(x)e^{th(x)}$ defines a homotopy from f to g.

Conversely, suppose that f and g are homotopic. Let F be a homotopy from f to g. Since $[0,1] \times X$ is compact, the continuous positive function $|F|$ attains its minimum. The minimum $m := \inf\{|F(t,x| : t \in [0,1], x \in X\}$ is positive. F is uniformly continuous on the compact metric space $I \times X$. Thus, for $\varepsilon := m$ there exists a $\delta > 0$ such that

$$|s - t| < \delta \Longrightarrow |F(s,x) - F(t,x)| < m, \qquad \forall x \in X. \tag{10}$$

We now choose an integer $N > 1/\delta$ and consider the maps $f_j : X \to \mathbb{C}^*$, defined by $f_j(x) := F(j/N, x)$. Now $f_0 = f$ and $f_N = g$. We see from Eq. 10 that

$$|f_j(x) - f_{j-1}(x)| < m \leq |f_j(x)|, \qquad x \in X, 1 \leq j \leq N.$$

By Thm. 265, each f_j/f_{j-1} is an exponential. As $f/g = (f_0/f_1)(f_1/f_2)\cdots(f_{N-1}/f_N)$, we see that f/g is an exponential. □

Corollary 267. *Let X be a compact metric space and $f : X \to \mathbb{C}^*$. Then f is an exponential iff f is homotopic to a constant map.* □

Definition 268. A space is said to be *contractible* if there is a homotopy between the identity map and a constant map.

Ex. 269. Any convex subset of $\mathbb{R}^n$ is contractible.

Corollary 270. *Let X be a compact contractible metric space. Then every map f from X to $\mathbb{C}^*$ is an exponential.*

Proof. Let $F : [0,1] \times X \to X$ be the homotopy of the identity map of X and a constant map x_0. Then $f \circ F$ is a homotopy of f to the constant map $f(x_0)$. By Cor. 267, f is an exponential. □

Now we restrict our attention to maps of S^1 to $\mathbb{C}^*$. We wish to assign to any such map an index that corresponds to the number of times the functions wraps around the origin.

Definition 271. Let $f : S^1 \to \mathbb{C}^*$ be a map. Consider the map $\theta \mapsto f(e^{i\theta})$ of $[0, 2\pi]$ into $\mathbb{C}^*$. Since the interval is contractible, by Corollary 270,

$$f(e^{i\theta}) = e^{g(\theta)} \text{ for some } g : [0, 2\pi] \to \mathbb{C}^*. \tag{11}$$

Let $g_1 : [0, 2\pi] \to \mathbb{C}^*$ be another map which satisfies Eq. 11. Then $e^{g(\theta)-g_1(\theta)} = 1$. Hence $g(\theta) - g_1(\theta)$ must assume values from the discrete set $2\pi i\mathbb{Z}$. Since $g - g_1$ is continuous, it follows that $g - g_1$ is a constant. Thus the number $g(2\pi) - g(0)$ is independent of the choice of g satisfying Eq. 11. Consequently the number

$$\text{ind}\,(f) := [g(2\pi) - g(0)]/2\pi i$$

is well defined. This integer is called the index of the map f.

Ex. 272. Let $f_n(z) = z^n$ for $n \in \mathbb{Z}$. Then $\text{ind}\,(f) = n$.

Theorem 273. *The index function, defined on the maps from S^1 to $\mathbb{C}^*$ has the following properties:*

(i) $\text{ind}\,(fg) = \text{ind}\,(f) + \text{ind}\,(g)$.

(ii) $\text{ind}\,(f) = 0$ *iff f is an exponential.*

(iii) $\text{ind}\,(f/|f|) = \text{ind}\,(f)$.

(iv) If $f : S^1 \to S^1$ is a map such that $f(1) = 1$, then $\text{ind}\,(f)$ *coincides with that of the loop α defined by $\alpha(s) = f(e^{2\pi i s})$, $0 \leq s \leq 1$.*

Proof. (i) is easy and left to the reader.

Suppose $f(e^{i\theta}) = e^{h(e^{i\theta})}$. If we set $g(t) = h(e^{it})$ then g satisfies Eq. 11. Since $g(2\pi) = g(0)$, $\operatorname{ind}(f) = 0$. Conversely, assume that $\operatorname{ind}(f) = 0$. Write $f(e^{it}) = e^{ig(t)}$ for $0 \leq t \leq 2\pi$. Then $g(0) = g(2\pi)$ so that the function $h\colon S^1 \to \mathbb{C}$ defined by setting $h(e^{it}) = g(t)$, $0 \leq t \leq 2\pi$, is well defined and continuous. Since $f = e^h$ f is an exponential. This proves (ii).

Since $|f|$ is an exponential, $\operatorname{ind}(|f|) = 0$ by (ii). By (i), $\operatorname{ind}(f) = \operatorname{ind}(f/|f|) + \operatorname{ind}(|f|)$. (iii) follows.

Let f and α be as in (iv). Choose $h\colon [0,1] \to \mathbb{R}$ such that $h(0)$ and $\alpha(s) = e^{2\pi i h(s)}$, for $0 \leq s \leq 1$. Thus h is a lift of α and hence index $\operatorname{ind}(\alpha) = h(1)$. Define $g\colon [0, 2\pi] \to \mathbb{C}$ by $g(t) := 2\pi i h(t/2\pi)$. Then g satisfies Eq. 11 so that

$$\operatorname{ind}(f) = [g(2\pi) - g(0)]/2\pi i = h(1) = \operatorname{ind}(\alpha).$$

This proves (iv). □

Theorem 274. *Let $f, g\colon S^1 \to \mathbb{C}^*$ be maps. Then the following are equivalent:*
(i) f is homotopic to g.
(ii) $\operatorname{ind}(f) = \operatorname{ind}(g)$.
(iii) f/g is an exponential.

Proof. The equivalence of (i) and (iii) is a special case of Thm. 266.

If f/g is an exponential, then by (ii) of Thm. 273, $\operatorname{ind}(f/g) = 0$. Write $f = g\cdot(f/g)$. By (i) of Thm. 273, we obtain $\operatorname{ind}(f) = \operatorname{ind}(f/g) + \operatorname{ind}(g) = \operatorname{ind}(g)$. Conversely, if $\operatorname{ind}(f) = \operatorname{ind}(g)$, then $\operatorname{ind}(f/g) = 0$. So, by (ii) of Thm. 273, f/g is an exponential. □

Corollary 275. *Each map $f\colon S^1 \to \mathbb{C}^*$ is homotopic to precisely one of the maps $f_m\colon z \mapsto z^n$ where $n = \operatorname{ind}(f)$.* □

Corollary 276. *We have $\pi_1(S^1, 1) \equiv \mathbb{Z}$.*

Proof. Since the maps z^n are not homotopic, the loops $\alpha_n\colon [0,1] \to S^1$ defined by $\alpha_n(t) = e^{2\pi i n t}$ cannot be homotopic with end points fixed. On the other hand, let $\alpha\colon [0,1] \to S^1$ be an arbitrary loop based at 1. Define $f\colon S^1 \to S^1$ by $f(e^{2\pi i s}) = \alpha(s)$. Let $n := \operatorname{ind}(f)$. By Thm. 274, f/α_n is an exponential, say, $f(e^{2\pi i s})/e^{2\pi i n s} = e^{h(e^{2\pi i s})}$ for $0 \leq s \leq 1$. Then $F(t,s) := e^{th(e^{2\pi i s})}e^{2\pi i n s}$ for $0 \leq s, t \leq 1$ is a homotopy from α_n and the loop α with end points fixed. Thus the correspondence $\varphi\colon [\alpha_n] \mapsto n$ is a bijection between $\pi_1(S^1, 1)$ and $\mathbb{Z}$. One easily checks that the product path $\alpha_m \alpha_n$ corresponds to a map from S^1 to itself of index $m + n$, so that $\alpha_m \alpha_n$ is homotopic to a_{m+n}. Thus φ is a group homomorphism. □

Theorem 277 (Fundamental Theorem of Algebra)**.** *A polynomial $p(z) := z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ of degree $n \geq 1$ and with complex coefficients has a zero in $\mathbb{C}$.*

Proof. Choose R so large that

$$|\frac{a_{n-1}}{R} w^{n-1} + \cdots + \frac{a_1}{R^{n-1}} + \frac{a_0}{R^n}| < 1, \qquad |w| \leq 1.$$

This can be done, if, for instance, we take $R > |a_{n-1}| + \cdots + |a_0| + 1$. Define a map $g\colon B[0,1] \to \mathbb{C}$ by setting

$$g(w) := \frac{p(Rw)}{R^n} = w^n + \frac{a_{n-1}}{R} w^{n-1} + \cdots + \frac{a_0}{R^n}, \qquad |w| \leq 1.$$

The estimate above shows that $|g(w) - w^n| < 1$ for $w = 1$. By Thm. 265, the restriction of w^n/g to the unit circle is an exponential. Now w^n is not an exponential since its index is $n \geq 1$ (Theorem 273). Hence $g = w^n \cdot (g/w^n)$ is not an exponential. Corollary 270 shows that g must have a zero on $B[0,1]$, and hence p has a zero in $\mathbb{C}$. □

We now apply some of our earlier results to arrive at some standard theorems of the topology of the plane.

Corollary 278. *Assume that $f\colon S^1 \to S^1$ is homotopic to a constant map. Then there is a continuous function $\varphi\colon S^1 \to \mathbb{R}$ such that $f(x) = e^{i\varphi(x)}$ for all $x \in S^1$.*

Proof. A special case of Corollary 267. □

Theorem 279. *The circle S^1 is not contractible.*

Proof. If it were, then by Corollary 278 there is a function $\varphi\colon S^1 \to \mathbb{R}$ such that $Id(x) \equiv x = e^{i\varphi(x)}$ for all $x \in S^1$. Thus, there is a continuous argument on S^1 and hence on $\mathbb{C}^*$, contradicting Theorem 262.

Or, more directly, such a φ is 1-1 and in particular $\varphi(x) \neq \varphi(-x)$. Define $g\colon S^1 \to \{\pm 1\}$ by

$$g(x) := \frac{\varphi(x) - \varphi(-x)}{|\varphi(x) - \varphi(-x)|}.$$

Then g maps S^1 continuously *onto* $\{\pm 1\}$. This contradicts the connectedness of S^1. □

Definition 280. A subset A of a space X is a *retract* of X if there is a continuous function $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$. r is called a retraction of X onto A.

Corollary 281. *There is no retraction of $\mathbb{R}^2$ onto S^1.*

Proof. Let $r\colon \mathbb{R}^2 \to S^1$ be retraction. Let $p = (0,0)$. Define a homotopy $H\colon I \times S^1 \to \mathbb{R}^2$ by $H(t,x) = tp + (1-t)x$. Then $r \circ H\colon I \times S^1 \to S^1$ is a contraction — contradicting Thm. 279. □

Corollary 282 (Brouwer Fixed Point Theorem)**.** *Let $f\colon B[0,1] \to B[0,1]$ be a continuous map. Then f has a fixed point, i.e., there is an $x \in B[0,1]$ such that $f(x) = x$.*

Proof. If there is no point x such that $f(x) = x$, then the two distinct points $f(x)$ and x determine a line joining $f(x)$ and x. We let $g(x)$ be the point on the boundary at which the line starting from $f(x)$ and going to x meets S^1. Then g is a retraction of $B[0,1]$ onto S^1—a contradiction to Corollary 281. In analytical terms, we have $g(x) = x + tv$, where $v = \frac{x - f(x)}{\|x - f(x)\|}$ and $t = -\langle x, v\rangle + \sqrt{1 - \|x\|^2 + (\langle x, v\rangle)^2}$. □

We end this article with some exercises.

Ex. 283. Let X be compact and $f\colon X \to \mathbb{C}^*$ be an exponential. Show that there exists $\varepsilon > 0$ such that any map $g\colon X \to \mathbb{C}^*$ which satisfies $|f(x) - g(x)| < \varepsilon$ is an exponential.

Ex. 284. Let X be locally compact hausdorff space. Show that two maps f and g from X to $\mathbb{C}^*$ are homotopic iff f/g is an exponential. *Hint:* Consider first the case when X is compact.

Ex. 285. Let X be a locally compact contractible metric space. Show that any map $f\colon X \to \mathbb{C}^*$ is an exponential.

Ex. 286. Let $f\colon S^1 \to \mathbb{C}^*$ be given. Show that there exists a $\varepsilon > 0$ such that any map $g\colon S^1 \to \mathbb{C}^*$ with $|f(z) - g(z)| < \varepsilon$ for $z \in S^1$ has the same index as f.

Ex. 287. Assume that $f, g\colon S^1 \to S^1$ be maps such that f and g do not assume antipodal values at any point of S^1. Show that $\operatorname{ind}(f) = \operatorname{ind}(g)$.

Ex. 288. Show that any map from S^n $(n \geq 2)$ to $\mathbb{C}^*$ is an exponential.

Ex. 289. Show that any map from $\mathbb{P}^n(\mathbb{R})$ $(n \geq 2)$ to $\mathbb{C}^*$ is an exponential. (Note that $\mathbb{P}^n$ is not simply connected. Can you explain what is happening here?)

Ex. 290. Classify the maps from the figure eight to $\mathbb{C}^*$.

P $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ if $m \neq n$

The aim of this article is to prove the theorem of the title. As a rule no first course in topology proves this result. Even if they raise the question of homeomorphism between $\mathbb{R}^m$ and $\mathbb{R}^n$ they refer to Brouwer's theorem on the invariance of domain which is proved as an application of Homology Theory. We wish to make the following elementary proof more widely known. It could be taught in any first course on Topology. There is nothing original in the following proof except the organization of the material available in the literature.

Outline of the Proof

Definition of a simplex, standard simplex; barycentric subdivision.
Dimension of a compact metric space. $\dim s^n \leq n$ where s^n is the standard n-simplex in $\mathbb{R}^n$.
Sperner's lemma — restricted version (applicable only to triangulation arising from barycentric subdivision).
Nagata's Lemma (restricted version): Let $\mathbb{D}^k(s)$ be the k-th barycentric subdivision of an n-simplex s. Let $\{U_i\}_0^n$ be an open cover of s such that $U_i \subseteq s \setminus F_i$ where F_i is the face opposite to the vertex e_i. Then there exists an r and an n-simplex in $\mathbb{D}^r(s)$ which intersects each of U_i.
$\dim s^n = n$.

Definition 291. A topological space X is said to have $\dim X \leq n$ if given any open cover $\mathcal{A}$ of X there exists an open cover $\mathcal{B}$ with the following properties:

For each $B \in \mathcal{B}$ there is an $A \in \mathcal{A}$ such that $B \subset A$.

There exists an element of X which lies in $n+1$ members of $\mathcal{B}$ and no element of X lies in more than $n+1$ members of $\mathcal{B}$.

We say that X is of (covering) dimension n if n is the least integer m such that $\dim X \leq m$. If no such n exists then we write $\dim X = \infty$.

Ex. 292. Homeomorphic spaces have the same dimension.

Ex. 293. Let X be a compact metric space. We say that $\dim X \leq n$ if for every $\varepsilon > 0$ there is a finite open cover $\mathcal{A}$ of X by sets of diameter $< \varepsilon$ such that some point of X lies in $n+1$ members of $\mathcal{A}$ and no point of X lies in more than $n+1$ members of $\mathcal{A}$. *Hint:* Use Lebesgue covering lemma.

Ex. 294. Let X be a compact metric space of dimension n. Let K be a closed subset of X. Then $\dim K \leq n$.

Definition 295. A set of vectors $\{v_0, v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is said to be geometrically (or affinely) independent if the set of vectors $\{v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0\}$ is linearly independent. A singleton set is geometrically dependent by definition.

Example 296. A set $\{v_0, v_1\}$ is geometrically independent iff they are not multiples of each other. A set $\{v_0, v_1, v_2\}$ is geometrically independent iff they are not collinear. A set $\{v_0, v_1, v_2, v_3\}$ is geometrically independent iff they are not coplanar.

Ex. 297. A set of vectors $\{v_0, v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is said to be geometrically (or affinely) independent iff for any set of real numbers λ_i with $\sum_{i=0}^k \lambda_i = 0$ and $\sum_{i=0}^k \lambda_i v_i = 0$ we have $\lambda_i = 0$ for $0 \le i \le k$.

Definition 298. Let $\{v_0, v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ be geometrically independent. The (open) *simplex* s^k is the set

$$s^k := \{x \in \mathbb{R}^n : x = \sum_{i=0}^k \lambda_i v_i, \quad \lambda_i > 0 \text{ and } \sum_i \lambda_i = 1\}.$$

We refer to k as the *algebraic dimension* of s. s is also called a k-simplex. We denote the simplex s^k by $(v_0 v_1 \ldots v_k)$. v_i are called the *vertices* of s^k. The simplex σ_i spanned by $\{v_0, \ldots, \hat{v_i}, \ldots, v_k\}$ is called the i-th face opposite to the vertex v_i. ($\hat{v_i}$ means v_i is omitted.) More generally, one defines a r-face of the simplex $(v_0 \ldots v_k)$ as the r-simplex $(v_{i_1} \ldots v_{i_r})$ for $0 \le i_1 < i_2 \cdots i_r \le k$. If σ is an r-face of s we write $\sigma \prec s$.

Example 299. Any one simplex with vertices v_0 and v_1 is the open line segment joining v_0 and v_1. Any two simplex spanned by three noncollinear vectors is the open interior of the triangle of which they are vertices. Any three simplex spanned by four coplanar vectors is the open tetrahedron. The faces are respectively the endpoints of the line segment, sides of the triangle and the faces of the tetrahedron.

The proof of the theorem of the title depends on the following

Theorem 300. *Let T^n be the closure of any n-simplex in $\mathbb{R}^n$. Then* $\dim T^n = n$.

Assuming the theorem let us complete the proof of the main result. Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be a homeomorphism. Let us assume that, if possible, that $m \ne n$. Let $m < n$ without loss of generality. Let T^n be as above. Then $f(T^n)$ is a compact subset of $\mathbb{R}^m$. Hence there exists an m-simplex, say, s such that $f(T^n) \subset s \subset \overline{s}$. Since $\dim \overline{s} \le m$ and $\dim T^n = n$, we infer that $n = \dim f(T^n) \le m < n$ in view of Exercises 292 and 294. This contradiction shows that $m = n$.

We break the proof of Theorem 300 into two statements: $\dim T^n \le n$ and $\dim T^n \ge n$. In the next section we establish the first and the second in the last section.

P.1 Barycentric Subdivision

Definition 301. A *complex* K is a collection of simplices $\{s\}$ with the following property: If $s \in K$ and $\sigma \prec s$ then $\sigma \in K$. The set $|K| = \cup_{s \in K} s$ is called the geometric realization of the complex K and K is called a *triangulation* of $|K|$.

Definition 302. Let $s := (v_0 \ldots v_k)$ be any simplex. Then the *barycenter* of s is the vector $(v_0 + \cdots + v_k)/(k+1)$. We denote this vector by $b(s)$.

Example 303. The barycenter of any 0-simplex is itself. The barycenter of $(v_0 v_1)$ is the midpoint of the line segment, that of $(v_0 v_1 v_2)$ is the centroid of the triangle and that of $(v_0 \ldots v_3)$ is the centre of gravity of the tetrahedron.

Definition 304. Let a complex K be given. The first barycentric subdivision of K is the complex $\mathbb{D}^1(K)$ whose vertices are $b(s)$ where s runs through all the faces of all simplices of K. The other simplices in the new complex are of the form $(b(s_1)b(s_2)\dots b(s_r))$ where $s_1 \prec s_2 \cdots \prec s_r$. We leave it to the reader that $\mathbb{D}^1 K$ is indeed a complex. Recursively we define $D^r(K) := D^1(D^{r-1}(K))$, called the r-th barycentric subdivision of K.

The following simple exercise will be repeatedly used in the sequel.

Ex. 305. Let $\mu(K) := \max\{\operatorname{diam} \overline{s} : s \in K\}$ be *mesh* of the complex. Let s be any simplex. Show that $\mu(s)$ is the length of the longest side, i.e.,1-dimensional face and $\mu(\mathbb{D}^r(s)) \to 0$ as $r \to \infty$.

Theorem 306. *Let s be an n-simplex. Then* $\dim s \leq n$.

Proof. Given $\varepsilon > 0$ let us choose r sufficiently large so that the mesh of $\mathbb{D}^r(s)$ is less than $\varepsilon/2$. Then the closed simplices in $\mathbb{D}^{r-1}(s)$ is an ε-covering such that each vertex $v_i \in s$ lies in $\cap_{i=0}^n star(v_i)$. This does the job. (Details are to be given.) □

P.2 Sperner's Lemma and its Corollaries

We shall give a very special version of Sperner's lemma which will be sufficient for our purpose. For more general versions, see references.

Theorem 307. *Let $s = (v_0 \dots v_n)$ be a simplex. Let $\mathbb{D}^r(s)$ be the r-th barycentric subdivision of s. Let a map $f \colon V(\mathbb{D}^r(s)) \to V(s)$ be given such that $f(v) = v_i$ where v_i is a vertex of the carrier of v. (Such maps will be called Sperner maps.) Then there exists a simplex $\sigma \in D^r(s)$ such that $f(\sigma) = \{v_0, \dots, v_n\}$.*

Proof. We shall prove this for $r = 1$ by induction on n. For $n = 1$ this is clear. Let us assume that the result is true for $n - 1$. Assume without loss of generality that $f(z) = v_n$ where $z = b(s) \in \mathbb{D}^1(s)$. Then f induces a Sperner map on the $n - 1$-simplex $(v_0 \dots v_{n-1})$. By induction hypothesis, we there is an $n - 1$-simplex, say, $\sigma^{n-1} = (b_0 \dots b_{n-1})$ such that $f(V(\sigma)) = \{v_0, \dots, v_{n-1}\}$. Clearly the simplex $\tau = (b_0 \dots b_{n-1} v_n)$ is of the required type. Thus the result is true for $\mathbb{D}^1(s)$ for any n-simplex s.

To complete the proof we now use induction on r and the previous paragraph. Let a Sperner map $f \colon V(\mathbb{D}^r(s)) \to V(s)$ be given. Then it induces a Sperner map on $\mathbb{D}^{r-1}(s)$ by restriction. By induction there exists a $\sigma \in \mathbb{D}^{r-1}(s)$ such that $f(\sigma) = \{0, 1, \dots, n\}$. By the first part of the proof there exists a simplex $\tau \in \mathbb{D}^1(\sigma)$ such that τ is completely labeled. But then $\tau \in \mathbb{D}^r(s)$ and meets our requirement. □

P.3 $\dim T^n = n$

Theorem 308. *Let $s := (v_0 \dots v_n)$ be an n-simplex and $T^n := \overline{s}$. Let $\{U_i : 0 \leq i \leq n\}$ be an open cover of T such that U_i does not intersect the i-th face. Then there a sufficiently large positive integer r such that there is an n-simplex σ in $\mathbb{D}^r s$ such that $\sigma \cap U_i \neq \emptyset$.*

Proof. Let ε be the Lebesgue number of the covering $\{U_i\}$. We choose r so that the mesh of $\mathbb{D}^r(s)$ is less than $\varepsilon/2$. We define a Sperner map f: $f(v) = i$ if $v \in U_i$ and v_i is a vertex of the carrier of v. This is possible by our assumption on the cover. Sperner lemma gives a simplex of the required kind. □

Theorem 309. *Let the notation be as above. Then* $\dim s^n \geq n$.

Proof. To prove this we need to exhibit an open cover $\mathcal{A}$ such that any $\mathcal{B}$ is in Definition 291 will be of order greater than or equal to $n + 1$. Let us take $A_i = T^n \setminus F_i$, the complement of the i-th face. Let $\mathcal{B}$ be any open cover such that $B \leq \mathcal{A}$. After doing a little jugglery we may assume that $\mathcal{B}$ has $n + 1$ members. By the last result the order of $\mathcal{B}$ is $n + 1$. □

1. Alexandrov, *Combinatorial Topology*,vol. 1.
2. Munkres, *Topology, A First Course.*
3. Nagata, *Dimension Theory.*