Structure Theorems for Linear Maps

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A basic course on Linear Algebra is an introduction to the preliminary notions such as basis, linear maps and orthonormal basis in an inner product spaces and orthogonal/unitary linear maps. The second phase of linear algebra is the study of structural results such as the decomposition of the vector space w.r.t. a linear map and investigating the possibility of representing the linear map in simple forms. This article is written with the aim of introducing good students to these aspects of linear algebra, soon after they have had a basic course in linear algebra. The subject matter is developed as a series of exercises with copious hints so as to reach the results as directly and as efficiently as possible.

1 Warm-up

Let V and W be (finite dimensional) vector spaces over a field, say, $\mathbb{R}$ or $\mathbb{C}$. Let $\{v_i : 1 \leq i \leq m\}$ and $\{w_j : 1 \leq j \leq n\}$ be bases of V. Let $A\colon V \to W$ be linear. Let us recall how to write a matrix of a linear map $A\colon V \to W$ with respect to these bases. Write $Av_i := \sum_{j=1}^{n} a_{ji} w_j$. Note the way the indices of the coefficients a_{ji} are written. The i-th column C_i in the matrix of A w.r.t. these bases, is the coefficients of Av_i:

$$C_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}.$$

Let us start with a warm-up exercise.

Ex. 1. Let V and W be vector spaces over a field of dimensions m and n respectively. Let $A\colon V \to W$ be a linear map. Then we can find bases $\{v_i : 1 \leq i \leq m\}$ of V and $\{w_j : 1 \leq j \leq n\}$ of W such that the matrix of A w.r.t. these bases is of the form
(a)

$$\begin{pmatrix} I_{m\times m} \\ 0_{m-n\times m} \end{pmatrix}, \qquad \text{if } A \text{ is one-one.}$$

Hint: If $\{v_i\}$ is a basis of V, then extend the linearly independent set $\{Av_i\}$ to a basis $\{w_j\}$ of W.
(b)

$$\begin{pmatrix} I_{n\times n} & 0_{n\times m-n} \end{pmatrix}, \qquad \text{if } A \text{ is onto.}$$

Hint: Choose a basis $\{u_i : 1 \le i \le m - n\}$ of $\ker A$ and extend it to a basis $\{v_i : 1 \le i \le m\}$ of V so that $v_{n+i} = u_i$. Then $\{w_j := Av_j : 1 \le j \le n\}$ is a basis of W.
(c) Why are these results unsatisfactory? *Hint:* What do these results say when $V = W$?

2 Arbitrary Vector Spaces over $\mathbb{C}$

Let V be a finite dimensional vector space over $\mathbb{C}$ and let $A: V \to V$ be a linear map. We shall assume that $\dim V = n$. All the results in this section are valid if the underlying field is algebraically closed.

Definition 2. Let $A: V \to V$ be a linear map. A scalar $\lambda \in \mathbb{C}$ is said to be an *eigen value* of A if there exists a nonzero vector $v \in V$ such that $Av = \lambda v$. If λ is an eigen value of A, then a vector $w \in V$ is called an eigen vector of A if $Aw = \lambda w$.

Ex. 3. Let λ be an eigen value of A. Then $V_\lambda := \{v \in V : Av = \lambda v\}$, the set of eigen vectors of A, is a vector subspace and is equal to $\ker(A - \lambda I)$.

Ex. 4. V is not necessarily the span of eigenvectors of A. For instance, consider $A: \mathbb{C}^2 \to \mathbb{C}^2$ given by $A(e_1) = 0$ and $A(e_2) = e_1$. Show that the only eigen value of A is zero and $V_0 = \mathbb{C}e_1$. *Hint:* Note that $A^2 = 0$.

Ex. 5. Let $p(X) := \sum_{k=0}^{d} a_k X^k$ be a polynomial in one variable X with complex coefficients a_k. Let $p(A) := \sum_k a_k A^k$. Then $p(A): V \to V$ is linear. It commutes with $q(A)$ where q is any polynomial. Also, if pq denotes the polynomial multiplication, then $pq(A) = p(A) \circ q(A)$.

Ex. 6. Let $A: V \to V$ be linear. Then A has an eigen value. *Hint:* Let $0 \ne v \in V$. Then $\{v, Av, A^2v, \dots, A^n v\}$ must be linearly dependent so that $p(A)v = 0$ for some polynomial p. Let $p(X) = \alpha(X - \lambda_1) \cdots (X - \lambda_n)$. Then $\alpha(A - \lambda_1 I) \cdots (A - \lambda_n I)v = 0$.

Ex. 7. Where does the argument in the last exercise break down in the case of vector spaces over $\mathbb{R}$? Try to analyze this in depth. Do not be satisfied with easy answers.

Ex. 8. Nonzero eigen vectors corresponding to distinct eigen values are linearly independent. *Hint:* Let $0 \ne v_j$ be an eigen vector of λ_j and $\lambda_j \ne \lambda_k$ for $1 \le j \ne k \le m$. Assume that $a_1 v_1 + \cdots + a_m v_m = 0$. Apply $(A - \lambda_2 I) \cdots + (A - \lambda_m I)$ on both sides to conclude that $a_1(\lambda_1 - \lambda_2) \cdots (\lambda_1 - \lambda_m)v = 0$.

Ex. 9. Let $A: V \to V$ be a linear map on an n-dimensional vector space. Show that A can have at most n distinct eigen values.

Definition 10. Let λ be an eigen value of A. A vector $v \in V$ is called a *generalized eigenvector* of A if there exists a $k \in \mathbb{N}$ such that $(A - \lambda I)^k v = 0$. (Note that we speak of generalized eigen vectors of a (genuine) eigen value and not of generalized eigen values!)

Ex. 11. Let A be as in Ex. 4. Then 0 is the only eigen value and e_1 is a generalized eigen vector but not an eigen vector.

Ex. 12. Let $V(\lambda)$ denote the set of all generalized eigenvectors corresponding to the eigen value λ. Then $V(\lambda)$ is a vector subspace of V. In fact, $V(\lambda) = \ker(A - \lambda I)^n$. *Hint:* Let $0 \ne v \in V(\lambda)$. Let $k \in \mathbb{N}$ be the least such that $(A - \lambda I)^k v = 0$. Assume that $a_0 v + a_1(A - \lambda I)v + \cdots + a_{k-1}(A - \lambda I)^{k-1}v = 0$. Apply $(A - \lambda I)^{k-1}$ to both sides of the equation to conclude that $a_0 = 0$ and so on. Hence $k \le n$.

Definition 13. Let W_j, $1 \le j \le k$, be vector subspaces of V. Then V is said to be a *direct sum* of W_j if any vector v can be written **uniquely** as a sum of elements from W_j. We then write $V = \oplus_{j=0}^k W_j$. We let $p_i\colon V \to W_i$ denote $p_i(v) \equiv p_i(\sum_j w_j) = w_i$ for $1 \le i \le n$. Then p_i is a linear map.

Ex. 14. (a) Show that if $V = \mathbb{C}^2$ and $W_j = \mathbb{C}e_j$, $j = 1, 2$, then $V = W_1 \oplus W_2$. (b) If $V = \mathbb{C}^3$ and if $W_1 = \operatorname{span}\{e_1, e_2\}$, $W_2 = \operatorname{span}\{e_2, e_3\}$ and $W_3 = \operatorname{span}\{e_1, e_3\}$, then $V = W_1 + W_2 + W_3$ but the sum is not direct.

Ex. 15. The vector space of $n \times n$ matrices is the direct sum of the vector subspaces of symmetric and skew symmetric matrices. Recall that an $n \times n$ matrix $A = (a_{ij})$ is said to be *symmetric* (respectively, *skew-symmetric*) if $a_{ij} = a_{ji}$ (respectively, $a_{ij} = -a_{ji}$) for all i, j.)

Ex. 16. In the notation of the above definition, show that $\dim V = \sum_j \dim W_j$.

Ex. 17. V is the span of generalized eigen vectors of A. *Hint:* Fix an eigen value λ of A. Observe that $V = \ker(A - \lambda I)^n \oplus \operatorname{Im}(A - \lambda I)^n$. For, if v lies in both, then $v = (A - \lambda I)^n u$ so that $(A - \lambda I)^n v = (A - \lambda I)^{2n} u = 0$. Hence $v = (A - \lambda I)^n u = 0$ by Ex. 12. Now complete the proof by induction.

Ex. 18. If 0 is the only eigen value of a linear map T, then T is nilpotent. (A linear map T is said to be *nilpotent* if $T^k = 0$ for some $k \in \mathbb{N}$.)

Ex. 19. Nonzero generalized eigenvectors corresponding to distinct eigenvalues are linearly independent. *Hint:* Assume, in an obvious notation, that $a_1 v_1 + \cdots + a_m v_m = 0$. Let k be the least such that

$$(A - \lambda_1 I)^k v_1 = 0.$$

Apply $(A - \lambda_1 I)^{k-1}(A - \lambda_2 I)^n \cdots (A - \lambda_m I)^n$ to both sides of the first equation to get

$$a_1 (A - \lambda_1 I)^{k-1}(A - \lambda_2 I)^n \cdots (A - \lambda_m I)^n v_1 = 0.$$

Write $(A - \lambda_j I)^n = [(A - \lambda_1 I) + (\lambda_1 - \lambda_j) I]^n$ in the above equation. Conclude that

$$a_1 (\lambda_1 - \lambda_2)^n \cdots (\lambda_1 - \lambda_m)^n (A - \lambda_1 I)^{k-1} v_1 = 0.$$

Ex. 20. Let $V = \oplus_{j=0}^k W_j$. Let $A\colon V \to V$ be a linear map such that $A(W_j) \subset W_j$ for all j. Then if we denote by A_j the restriction of A to W_j, we then can write $A := A_1 \oplus \cdots \oplus A_k$ so that $A(v) = \sum_j A_j w_j$. We can also exhibit A as a block operator:

$$A = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_k \end{pmatrix}$$

State and prove a converse of the above statements.

Ex. 21 (Structure Theorem)**.** Let λ_j, $1 \le j \le m$, be all the distinct eigenvalues of A. Let $V_j := V(\lambda_j)$, $1 \le j \le m$. Then

(a) $V = \oplus_{j=0}^m V_j$.
(b) A maps each V_j to itself.
(c) $(A - \lambda_j I)$ is nilpotent on V_j.
(d) The only eigenvalue of A on V_j is λ_j.

Hint: To prove (d), note that $0 = (A - \lambda_j I)^n v = (\lambda - \lambda_j)^n v$, if $Av = \lambda v$.

Ex. 22. Prove the converse of Ex. 18: If T is nilpotent, then 0 is its only eigen value.

Definition 23. An $n \times n$ matrix $A = (a_{ij})$ is said to be *upper triangular* if $a_{ij} = 0$ if $i > j$.

Ex. 24. Let $A\colon V \to V$ be linear. Then the matrix of A w.r.t. a chosen basis $\{v_j : 1 \leq j \leq n\}$ of V is upper triangular iff $Ae_k \in \operatorname{span}\{e_j : 1 \leq j \leq k\}$.

Ex. 25. Let $A\colon V \to V$ be linear. Then A admits an upper triangular matrix representation iff there exists a *flag* of vector spaces V_k, $0 \leq k \leq n$, such that

$$\{0\} = V_0 \subset V_1 \subset V_2 \cdots \subset V_{n-1} \subset V_n = V$$

and such that $AV_k \subset V_k$, for $0 \leq k \leq n$.

Ex. 26. Let A be nilpotent. Then there exists a basis of V with respect to which the matrix of A is upper triangular. *Hint:* Choose a basis of $\ker A$. Extend it to a basis of $\ker A^2$ and so on.

Ex. 27. Let A be any linear map on V. Then there exists a basis of V with respect to which the matrix of A is of the form

$$\begin{pmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & A_m \end{pmatrix}$$

where

$$A_j := \begin{pmatrix} \lambda_j & \star & \dots & \star \\ 0 & \lambda_j & \star & \star \\ 0 & 0 & \ddots & \star \\ 0 & 0 & 0 & \lambda_j \end{pmatrix}$$

In particular, the matrix of A w.r.t. this basis is triangular. *Hint:* Use the last exercise and Ex. 21.c.

Definition 28. A basis $\{e_k\}$ of V is a said to be a *Jordan basis* of V w.r.t. A if the matrix of A w.r.t. this basis looks like:

$$A := \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix},$$

where each $A_j := \begin{pmatrix} \lambda_j & 1 & & & 0 \\ & \lambda_j & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_j & 1 \\ 0 & & & & \lambda_j \end{pmatrix}$.

Ex. 29 (Jordan canonical Form for Nilpotent Operators)**.** Let $A\colon V \to V$ be nilpotent. Assume that q is the least positive integer such that $A^q = 0$. Assume that $S \subset V$ (possibly empty) be linearly independent and such that $S \cap \ker A^{q-1} = \emptyset$. Then S can be extended to a Jordan basis of V w.r.t. A. *Hint:* Extend S to a linearly independent set S' such that $V = \operatorname{span} S' \oplus \ker A^{q-1}$. Observe that AS' is linearly independent subset of $\ker A^{q-1}$ such that $\operatorname{span} AS' \cap \ker A^{q-2} = \{0\}$. Now induction can be applied. If J' is a Jordan basis of $\ker A^{q-1}$, then the basis $J' \cup S'$, ordered in a way that elements of S' come after elements of J', will be a Jordan basis for V.

Ex. 30 (Jordan Canonical Form)**.** Let $A\colon V \to V$ be any linear map. Then there exists a Jordan basis of V for A. *Hint:* Ex. 27 and Ex. 21.

Definition 31. Let $A\colon V \to V$ be linear. Let k be the smallest positive integer positive such that I, A, $A^2, \dots, A^k$ are linearly dependent. Then there exist complex numbers c_r, $0 \le r < k$, such that $A^k + c_{k-1}A^{k-1} + \cdots + c_1 A + c_0 I = 0$. The polynomial $p(X) := c_0 + c_1 X + \cdots + c_{k-1}X^{k-1} + X^k$ is called the *minimal polynomial* of A. Thus p is the monic polynomial of smallest degree such that $p(A) = 0$. (A polynomial is monic if the coefficient of the highest power is 1.)

Ex. 32. Let the notation be as in Ex. 21. Let α_j be the smallest positive integer such that $(A - \lambda_j I)^{\alpha_j} = 0$ on $V(\lambda_j)$. Let $p(X) := (X - \lambda_1)^{\alpha_1} \cdots (X - \lambda_m)^{\alpha_m}$. Then

(a) $\deg p \le n$.

(b) If q is a polynomial such that $q(A) = 0$, then p divides q.

(c) p is the minimal polynomial of A.

Hint: To prove (b), if $q(X) = c(X - r_1)^{d_1} \cdots (X - r_t)^{d_t}(X - \lambda_j)^{\delta}$, then $q(A) = 0$ will imply that $(A - \lambda_j)^{\delta} v = 0$, if $v \in V(\lambda_j)$.

Definition 33. If λ is an eigen value of A, then the *multiplicity* of λ is $\dim V(\lambda)$. Keep the notation of Ex. 21. Let $d_k := \dim V(\lambda_k)$. The characteristic polynomial of A is the polynomial $(X - \lambda_1)^{d_1} \cdots (X - \lambda_m)^{d_m}$.

Ex. 34. The characteristic polynomial of A is $\det(XI - A)$. *Hint:* Using the standard notation, for any $\alpha \in \mathbb{C}$, the eigen values of $\alpha I - A$ are $\alpha - \lambda_j$ with multiplicities d_j. Hence the determinant of $\alpha I - A$ is $(\alpha - \lambda_1)^{d_1} \cdots (\alpha - \lambda_m)^{d_m}$.

Ex. 35 (Cayley-Hamilton Theorem)**.** If $q(X)$ is the characteristic polynomial of A, then $q(A) = 0$. *Hint:* Use the Def. 33 and Ex. 32!

Ex. 36. Show that the minimal polynomial of A is the monic generator of the ideal $\{q \in \mathbb{C}[X] : q(A) = 0\}$. (This is the standard definition of the minimal polynomial of the linear map A.)

3 Inner Product Spaces over $\mathbb{C}$

Let us now assume that V is an inner product space with an inner product $\langle , \rangle$. In this section we characterize those operators which produce an orthonormal basis of V consisting entirely of eigenvectors.

Definition 37. Let $A\colon V \to V$ be given. Then $A^*\colon V \to V$ is defined by the equation

$$\langle A^* x, y \rangle = \langle x, Ay \rangle \qquad \text{for all } x, y \in V.$$

Ex. 38. Show that $v \mapsto A^* v$ is a linear map, called the adjoint of A.

Definition 39. A linear map $A\colon V \to V$ is said to be *self-adjoint* or hermitian if $A = A^*$ and normal if $AA^* = A^*A$.

Ex. 40. A is self adjoint iff $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in V$.

Ex. 41. If A is normal, then $\ker A = \ker A^*$. *Hint:* Start with $\langle A^*x, A^*x\rangle$ for $x \in \ker A$.

Ex. 42. If A is normal, then any generalized eigen vector of A is an eigen vector. *Hint:* Prove that $\ker A^k = \ker A$ by induction on k. If $A(A^k x) = 0$, then $AA^*x = 0$.

Ex. 43. If λ is an eigenvalue of A, then $\overline{\lambda}$ is an eigen value of A^*.

Ex. 44. If A is normal, then eigen vectors corresponding to distinct eigen values are orthogonal. *Hint:* Start with $(\lambda - \mu)\langle x, y\rangle$.

Ex. 45 (Spectral Theorem for Normal Operators)**.** A is normal iff V has an orthonormal basis consisting of eigen vectors of A. *Hint:* By induction on the dimension. Observe that if $V(\lambda)$ is an eigen space, then $A\colon V(\lambda)^\perp \to V(\lambda)^\perp$.

Ex. 46. Show that the eigen values of a self-adjoint linear map are real.

Ex. 47. Let A be a linear map on an inner product space V. Then there exists an orthonormal basis of V w.r.t. which the matrix of A is upper triangular. *Hint:* Let $v \in V$ be an eigen vector of A with norm 1. Consider $W := (\mathbb{C}v)^\perp$ and the map $B\colon W \to W$ given by $B = P \circ A$ where $P\colon V \to W$ is the orthogonal projection, given by $Ax := x - \langle x, v\rangle v$. Argue by induction.

Compare this exercise with Ex. 27.

Ex. 48. Let $A\colon V \to V$ be linear. Then the following are equivalent.
(a) $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in V$.
(b) $\| Ax\| = \|x\|$ for all $x \in V$.
(c) $AA^* = A^*A = I$.
(d) A takes an orthonormal basis of V to an orthonormal basis.

Definition 49. A linear map $A\colon V \to V$ is said to be *unitary* if A satisfies any one (and hence all) of the conditions of the last exercise.

Ex. 50. Let A be a unitary operator and λ an eigen value of A. Then $|\lambda| = 1$.

Ex. 51 (Spectral Theorem for Unitary Operators)**.** Let A be a unitary operator. Then there exists an orthonormal basis of V w.r.t. which $A = \operatorname{diag}(e^{it_1}, \dots, e^{it_n})$. *Hint:* Let $v \in V$ be an eigen vector of unit norm. Consider W, the orthogonal complement of v. Then A maps W to itself. Apply induction.

4 Inner Product Spaces over $\mathbb{R}$

Let V be a finite dimensional vector space over $\mathbb{R}$ and $A\colon V \to V$ be linear.

Ex. 52. Let $V = \mathbb{R}^2$ and $A\colon (x, y) \mapsto (-y, x)$. Show that A (the rotation by right angle in anti-clockwise direction) has no real eigen value and hence no eigen vector.

Ex. 53. Let $p(X)$ be a polynomial with real coefficients. Then $p(X)$ is a product of real polynomials of degree 1 or 2. Moreover, if $X^2 + bX + c$ is one such factor, then $b^2 - 4c < 0$. *Hint:* Observe that if $\lambda \in \mathbb{C}$ is a root of p iff $\overline{\lambda}$ is so. Hence $p(X) = (X - \lambda)(X - \overline{\lambda})q(X)$. Prove that q has real coefficients. Now induction applies.

Ex. 54. Let $A\colon V \to V$ be linear. Then there exists an A-invariant subspace of dimension 1 or 2. *Hint:* Argue as in Ex. 6 making use of Ex. 53. If one of the factors $A^2 + bA + c$ has nonzero kernel, so that $(A^2 + bA + cI)w = 0$, for some nonzero w, then $W = \operatorname{span}\{w, Aw\}$ is as required.

In the sequel, V denotes a finite dimensional inner product space over $\mathbb{R}$.

Ex. 55. Let $A\colon V \to V$ be self-adjoint: $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in V$. Let $b, c \in \mathbb{R}$ be such that $b^2 - 4c < 0$. Then $A^2 + bA + c$ is invertible. *Hint:* Show that $\big\langle (A^2 + bA + cI)v, v\big\rangle > 0$ for all $v \neq 0$.

Ex. 56. Let $A\colon V \to V$ be self-adjoint. Then A has an eigen value. *Hint:* Go through the proofs of Ex. 6 and Ex. 54.

Ex. 57. Let A be self-adjoint. The nonzero eigen vectors of A corresponding to distinct eigen values are orthogonal to each other. Hence V is the orthogonal direct sum of eigen spaces of A. *Hint:* $\langle Av_1, v_2\rangle = \langle v_1, Av_2\rangle$, where $Av_j = \lambda_j v_j$.

Ex. 58 (Spectral Theorem for Real Self-Adjoint Operators)**.** $A\colon V \to V$ is self-adjoint iff there exists an orthonormal basis of V consisting of eigenvectors of A. *Hint:* If v is a nonzero eigen vector, then A maps $(\mathbb{R}v)^\perp$ to itself.

Ex. 59. Find the matrix (w.r.t. the standard basis) of an orthogonal map of $\mathbb{R}^2$ with the Euclidean inner product. *Hint:* Note that $\{Ae_1, Ae_2\}$ is an orthonormal basis of $\mathbb{R}^2$ and that any vector of unit norm can be written as $\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$ for some $\theta \in \mathbb{R}$. Hence A is either of the form $k(\theta) := \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$ or of the form $r(\theta) := \begin{pmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{pmatrix}$ for some $\theta \in \mathbb{R}$. $k(\theta)$ (resp. $r(\theta)$) is called a rotation (resp. a reflection).

Ex. 60. If $A\colon V \to V$ is orthogonal and λ is an eigen value of A, then $\lambda = \pm 1$.

Ex. 61. Let $T\colon V \to V$ be orthogonal. Let $A := T + T^{-1} = T + T^*$. Then A is symmetric and let $V := \oplus_i V_i$ be the orthogonal decomposition of A into distinct eigen spaces. Then

(a) T leaves each V_i invariant.

(b) If V_λ is an eigen space of A with eigen value λ, then we have $T^2 - \lambda T + I = 0$ on V_λ.

(c) If $\lambda = \pm 2$, then T acts as $\pm I$ on V_λ.

(d) If $\lambda \neq \pm 2$, then $W := \mathbb{R}v + \mathbb{R}(Tv)$ is a two-dimensional subspace such that $TW \subset W$. Also, if $V_\lambda = W \oplus W^\perp$, then $TW^\perp \subset W^\perp$. Hence V_λ is orthogonal direct sum of two dimensional vector subspaces invariant under T.

(e) If $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is orthogonal and satisfies $T^2 + \lambda T + I = 0$ for some $\lambda \neq \pm 2$, then T is a rotation.

Ex. 62 (Spectral Theorem for Orthogonal Operators)**.** Let T be orthogonal. Then there exists an orthonormal basis of V with respect to which T can be represented as follows:

$$T = \begin{pmatrix} \pm 1 & & & & & \\ & \ddots & & & & \\ & & \pm 1 & & & \\ & & & \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} & & \\ & & & & \ddots & \\ & & & & & \begin{pmatrix} \cos\theta_r & -\sin\theta_r \\ \sin\theta_r & \cos\theta_r \end{pmatrix} \end{pmatrix}.$$

That is, T is the block matrix

$$T = \operatorname{diag}\left(\pm 1, \cdots, \pm 1, k(\theta_1), \cdots, k(\theta_r)\right).$$

Hint: Ex. 61 and Ex. 59.

The last couple of results are valid for inner product spaces over $\mathbb{R}$ or $\mathbb{C}$. Let V be an inner product space over $\mathbb{R}$ or $\mathbb{C}$ and $A\colon V \to V$ be linear.

Definition 63. A is said to be *positive* if (i) A is self-adjoint and (ii) $\langle Ax, x\rangle \geq 0$ for all $x \in V$.

Ex. 64. Show that the eigen values of a positive operator A are nonnegative and that there exists a unique operator S such that S is positive and $S^2 = A$. The operator S is called the positive square root of A.

Ex. 65 (Polar Decomposition for Invertible Maps)**.** Let $A\colon V \to V$ be nonsingular. Then there exists a unique decomposition $A = PU$ where U is unitary (or orthogonal) and P is positive. (This decomposition is called the polar decomposition of A.) *Hint:* Think of complex numbers. The map AA^* is positive and let S be its positive square root. Then $U := S^{-1}A$ may do the job. But why does S^{-1} exist?

Ex. 66 (Polar Decomposition)**.** Let $A\colon V \to V$ be any linear map. Then there exists a unitary (orthogonal) map U and a positive map P such that $A = PU$. *Hint:* Let $S := \sqrt{AA^*}$. Let $W := SV$. Define $U_1\colon W \to V$ by setting $U_1(Sv) := Av$. Observe that $\dim W^{\perp} = \dim(AV)^{\perp}$. Define a unitary map $U_2\colon W^{\perp} \to (AV)^{\perp}$.

A Warning! My emphasis on this article is to give a solid and quick treatment of the theoretical aspects of the results. This cannot be taken as a substitute for a textbook. Even if you could work out all these exercises, I suggest that you take up a book on Linear Algebra and do a lot more concrete and numerical examples.