Tychonoff's Theorem

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Let X be a set. We say that a collection $\mathcal{A}$ of (nonempty) subsets of X has *finite intersection property* (f.i.p., in short) if every finite family $A_1, \ldots, A_n$ of elements in $\mathcal{A}$ has a nonempty intersection.

Ex. 1. A topological space is compact iff every family of closed sets with f.i.p. has nonempty intersection. *Hint:* Start with an open cover $\mathcal{U}$ which does not admit a finite subcover. Look at $\{X \setminus U : U \in \mathcal{U}\}$.

Let us briefly review the product topology. Given a family $\{X_\alpha : \alpha \in I\}$ of topological spaces, the topology on the product set $\prod_{\alpha \in I} X_\alpha$ is the weakest (or the smallest topology) which makes the canonical projection maps $P_\alpha \colon X \to X_\alpha$ continuous. Hence for any open set $U_\alpha \subset X_\alpha$, the set $P_\alpha^{-1}(U_\alpha)$ must be declared open in X. Finite intersections of such sets form a basis for the product topology, that is, $U \subset X$ is open iff for each $x \in U$, there exists a finite subset $F \subset I$ such that $x \in \cap_{\alpha \in F} P_\alpha^{-1}(U_\alpha)$ for some open sets $U_\alpha \subset X_\alpha$.

To bring out the main ideas of the proof clearly, let us list the following two exercises which are set-theoretic in nature. Their solutions can be gleaned from the proof of the theorem below.

Ex. 2. Let $\mathcal{A}$ be a family of subsets of a set X with f.i.p. Then there exists a 'maximal' family $\mathcal{B}$ containing $\mathcal{A}$ with f.i.p., that is, a family $\mathcal{B}$ of subsets such that (i) $\mathcal{A} \subset \mathcal{B}$, (ii) $\mathcal{B}$ has f.i.p. and (iii) if $\mathcal{C}$ is any family with f.i.p. such that $\mathcal{A} \subset \mathcal{B}$, then $\mathcal{C} \subset \mathcal{B}$. *Hint:* Partially order the set of all collections with properties (i) and (ii) by inclusion. Apply Zorn's lemma.

Ex. 3. Let $\mathcal{B}$ be a family of subsets of a set X which is maximal with respect to finite intersection property. Then (i) if $A \subset X$ has nonempty intersection with each member of $\mathcal{B}$, then $A \in \mathcal{B}$ and (ii) the intersection of any finite number of elements of $\mathcal{B}$ again lies in $\mathcal{B}$.

Theorem 4 (Tychonoff). *The product of compact spaces is compact. That is, if X_α is compact for each $\alpha \in I$ and if $X := \prod_{\alpha \in I} X_\alpha$ is endowed with the product topology, then X is compact.*

Proof. We plan to use Ex. 1. Let $\mathcal{F}_0$ be a family of closed sets in X with the finite intersection property (f.i.p). It suffices to show that there is a point common to all the sets $F \in \mathcal{F}_0$. We use Ex. 2 to get maximal family $\mathcal{F} \supseteq \mathcal{F}_0$. The details are in the next paragraph.

Consider the class of all families $\mathcal{F}$ of (not necessarily closed) subsets such that $\mathcal{F}_0 \subset \mathcal{F}$ which have f.i.p. For two families $\mathcal{F}$ and $\mathcal{G}$ in this class we say that $\mathcal{F} \leq \mathcal{G}$ iff $\mathcal{F} \subseteq \mathcal{G}$. Now let $\mathcal{C}$ be any totally ordered chain in this class, that is, if $\mathcal{F}, \mathcal{G} \in \mathcal{C}$, then either $\mathcal{F} \subseteq \mathcal{G}$ or $\mathcal{G} \subseteq \mathcal{F}$. This chain has an upper bound, viz., $\mathcal{H} = \cup_{\mathcal{F} \in \mathcal{C}} \mathcal{F}$. We need only show that $\mathcal{H}$ has f.i.p. Let $A_1, \ldots, A_n \in \mathcal{H}$. Then there exists $\mathcal{F}_j \in \mathcal{C}$ such that $A_j \in \mathcal{F}_j \in \mathcal{C}$. Since $\mathcal{C}$ is totally ordered, and $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are finite in number, there exists k with $1 \leq k \leq n$ such that $\mathcal{F}_j \subseteq \mathcal{F}_k$ for all j. Hence $A_1, \ldots, A_n \in \mathcal{F}_k$. Since $\mathcal{F}_k$ has f.i.p., $A_1 \cap \cdots \cap A_n \neq \emptyset$. Thus $\mathcal{H}$ has f.i.p. and hence is an upper bound for the chain. Therefore, by Zorn's lemma, there exists a maximal family $\mathcal{F} \in \mathcal{C}$, with $\mathcal{F} \supseteq \mathcal{F}_0$.

Let $\mathcal{F}^\alpha$ denote $\{E^\alpha := P_\alpha(E), \quad E \in \mathcal{F}\}$, where $P_\alpha \colon X \to X_\alpha$ is the canonical projection map. Then $\mathcal{F}_\alpha \subseteq P(X_\alpha)$, the power set of X_α, has f.i.p. For otherwise, $E_1^\alpha \cap \cdots \cap E_n^\alpha = \emptyset$ will imply $E_1 \cap \cdots \cap E_n = \emptyset$, where $P_\alpha(E_i) = E_i^\alpha$. Hence, $\overline{\mathcal{F}^\alpha} = \{\overline{E^\alpha}\}$ has finite intersection property.

Since X_α is compact, there exists $x_\alpha \in \cap \overline{E^\alpha}$ where the intersection is over all $E^\alpha \in \mathcal{F}^\alpha$. Let $x \in \prod X_\alpha$ be such that $x(\alpha) := x_\alpha$. We claim that $x \in \cap_{F \in \mathcal{F}} \overline{F}$. Since $\mathcal{F} \supseteq \mathcal{F}_0$ and since every element of $\mathcal{F}_0$ is closed, the claim completes the proof of the theorem.

We now prove the claim. Let U be an open set in X. By definition of product topology, there exists $\alpha_1, \ldots, \alpha_n$ and open sets $U_{\alpha_i} \subseteq X_{\alpha_i}$, $1 \leq i \leq n$ such that $x \in \cap_{i=1}^n P_{\alpha_i}^{-1}(U_{\alpha_i}) \subseteq U$. This implies $x_{\alpha_i} \in U_{\alpha_i}$ for all i. By hypothesis on x_α's, x_{α_i} is in the closure of F_{α_i} for all $F_{\alpha_i} \in \mathcal{F}^{\alpha_i}$. Select $y_{\alpha_i} \in U_{\alpha_i} \cap F_{\alpha_i}$ and a $y \in F$ such that $P_{\alpha_i}(y) = y_{\alpha_i}$. Then $y \in P_{\alpha_i}^{-1}(U_{\alpha_i}) \cap F$. Thus $P_{\alpha_i}^{-1}(U_{\alpha_i})$ has a non-empty intersection with every $F \in \mathcal{F}$. Therefore $P_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$ by Ex. 3.

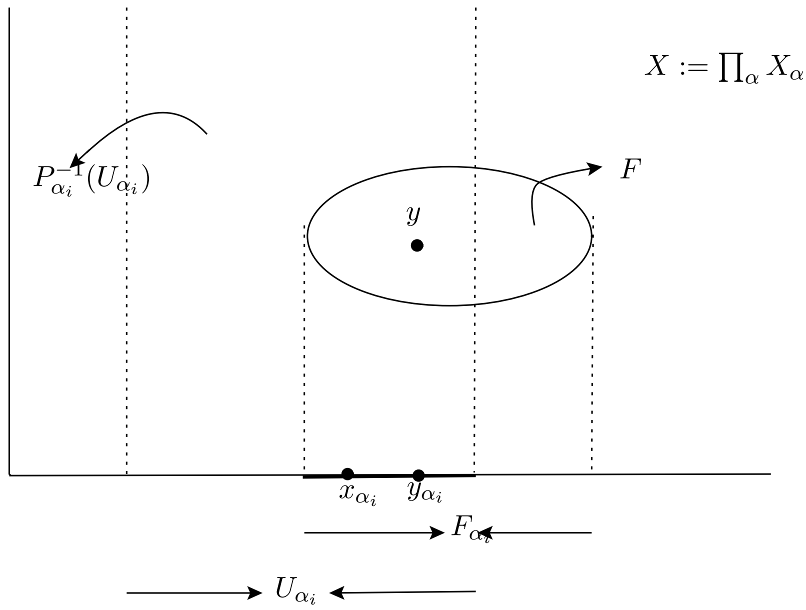

Figure 1: $P_{\alpha_i}^{-1}(U_{\alpha_i}) \cap F \neq \emptyset$

(Reason: Otherwise $\mathcal{F} \cup \{P_{\alpha_i}^{-1}(U_{\alpha_i})\} \supset \mathcal{F}$ and the former has finite intersection property, contradicting the maximality of $\mathcal{F}$.)

Using Ex. 3 again, we infer that $\cap_{i=1}^{n} P_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$.

(Reason: Since $F, P_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$, we see that $F \cap \left(\cap_i P_{\alpha_i}^{-1}(U_{\alpha_i})\right) \neq \emptyset$. Thus $\cap_i P_{\alpha_i}^{-1}(U_{\alpha_i})$ meets every element of $\mathcal{F}$. Thus $\mathcal{F} \cup \{\cap_i P_{\alpha_i}^{-1}(U_{\alpha_i})\}$ has f.i.p. Since $\mathcal{F}$ is maximal with respect to f.i.p. it follows that $\cap_i P_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$.)

Since $\mathcal{F}$ has f.i.p., this basic open set and hence U intersects each member of $\mathcal{F}$ non-trivially. Since U was an arbitrary open neighborhood of x, this means that $x \in \overline{F}$, for all $F \in \mathcal{F}$. Hence the claim. □

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