Extension of Uniformy Continuus Functions

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Theorem 1. *Let X be a metric space and Y a complete metric space. Let $A \subset X$ be dense in X. Let $f \colon A \to Y$ be uniformly continuous. Then there exists a unique (uniformly) continuous function $g \colon X \to Y$ such that g extends f, that is, $g(a) = f(a)$ for all $a \in A$.*

Proof. Outline: Given $x \in X$, by density of A in X we can find a sequence (a_n) in A such that $a_n \to x$. Then $(f(a_n))$ is a Cauchy sequence in Y and hence $\lim f(a_n)$ exists. We set $g(x) := \lim_n f(a_n)$. One checks that g is well-defined, that is, if $a_n' \to x$, $a_n' \in A$, then $\lim_n f(a_n) = \lim_n f(a_n')$. For, $d(f(a_n), f(a_n')) \to 0$ as $n \to \infty$. Note that $g(a) = f(a)$ for $a \in A$. To prove the uniform continuity of g, let $\varepsilon > 0$ be given. For $\varepsilon/3$, the uniform continuity of f on A gives us a δ. Let $x_1, x_2 \in X$ be such that $d(x_1, x_2) < \delta/3$. We can find $a_i \in A$ such that $d(x_i, a_i) < \delta/3$ **and** such that $d(g(x_i), g(a_i)) < \varepsilon/3$. (The first is by the density of A and the second by the very definition of g.) Note that

$$d(a_1, a_2) \leq d(a_1, x_1) + d(x_1, x_2) + d(x_2, a_2) < \delta.$$

We have

$$\begin{aligned} d(f(x_1), f(x_2)) &\leq d(f(x_1), f(a_1)) + d(f(a_1), f(a_2)) + d(f(a_2), f(x_2)) \\ &< 3 \times \varepsilon/3. \end{aligned}$$

We now give all the details.

Cliam 1: Given $x \in X$, there exists a sequence (a_n) in A such that $a_n \to x$.

> Since A is dense in A, for any given $n \in \mathbb{N}$, the open ball $B(x, 1/n)$ intersects A. Let $a_n \in B(x, 1/n) \cap A$. Since $d(x, a_n) < 1/n$, it follows that $a_n \to x$.

Claim 2: If $x \in X$ and (a_n) is a sequence in A such that $a_n \to x$ in X, then the sequence $(f(a_n))$ is converegent in Y.

> We show that $(f(a_n))$ is Cauchy in Y. Since Y is complete, there exists $y \in Y$ such that $f(a_n) \to y$ in Y. To show that $(f(a_n))$ is Cauchy, let $\varepsilon > 0$ be given. Since f is uniformly continuous on A, for the ε given above, there exists a $\delta > 0$ such that if $a, a' \in A$ are such that $d(a, a') < \delta$, then $d(f(a), f(a')) < \varepsilon$. Since (a_n) is converegent, it is Cauchy. Hence

there exists n_0 such that for all $n, m \geq n_0$, we have $d(a_n, a_m) < \delta$. By our choice of δ, we see that $d(f(a_n), f(a_m)) < \varepsilon$ for $n, m \geq n_0$. This means that the sequence $(f(a_n))$ is Cauchy in Y.

Claim 3: Let (a_n) and (b_n) be sequences in A such that a_n and b_n converge to the same $x \in X$. Then $\lim_n f(a_n) = \lim_n f(b_n)$.

By Claim 2, we know that the sequences $(f(a_n))$ and $(f(b_n))$ are converegent in Y. Let $y, y' \in Y$ such that $f(a_n) \to y$ and $f(b_n) \to y'$. We need to show that $y = y'$. Let $\varepsilon > 0$ be given. We show that $d(y, y') < \varepsilon$. This will imply that $d(y, y') = 0$ and hence $y = y'$.

For $\varepsilon > 0$ given above, let $\delta > 0$ correspond to $\varepsilon/3$ by the uniform continuity of f on A:

$$a, a' \in A \text{ with } d(a, a') < \delta \implies d(f(a), f(a')) < \varepsilon/3. \tag{1}$$

Since $a_n \to x$ and $b_n \to x$, using $\delta/2$ in the defintion of convergence, there exists $m_1, m_2 \in \mathbb{N}$ such that

$$n \geq m_1 \implies d(a_n, x) < \delta/2 \text{ and } n \geq m_2 \implies d(b_n, x) < \delta/2. \tag{2}$$

Hence it follows from (2) that

$$n \geq m_3 := \max\{m_1, m_2\} \implies d(a_n, b_n) \leq d(a_n, x) + d(x, b_n) < (\delta/2) + (\delta/2) = \delta. \tag{3}$$

From (3) and (1)

$$\text{for all } n \geq m_3 \text{ we have } d(f(a_n), f(b_n)) < \varepsilon/3. \tag{4}$$

Since $f(a_n) \to y$, for ε as above, there exists $m_4 \in \mathbb{N}$ such that

$$n \geq m_4 \implies d(f(a_n), y) < \varepsilon/3. \tag{5}$$

Similarly, there exists $m_5 \in \mathbb{N}$ such that

$$n \geq m_5 \implies d(f(b_n), y') < \varepsilon/3. \tag{6}$$

We now estimate $d(y, y)$ using (5), (4) and (6):

$$\begin{aligned} d(y, y') &\leq d(y, f(a_n)) + d(f(a_n), f(b_n)) + d(f(b_n), y') \text{ for any } n \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \text{ if } n \geq m_6 := \max\{m_4, m_3, m_5\}. \\ &= \varepsilon. \end{aligned}$$

Let $g\colon X \to Y$ be defined as follows. For any $x \in X$, by Claim 1, there exists a sequence (a_n) in A such that $a_n \to x$ in X. By Claim 2, there exists $y \in Y$ such that $f(a_n) \to y$. If we set $g(x) := \lim_n f(a_n)$, Claim 3 says that $g(x)$ is well-defined.

Also, if $x \in A$, then we may take $a_n = x$ for all n. Then $a_n \to x$ and $g(x) := \lim f(a_n) = \lim_n f(x) = f(x)$. Hence g is an extension of f

Claim 4: g is uniformly continuous on X.

Let $\varepsilon > 0$ be given. By the uniform continuity of f on A, there exists $\delta > 0$ such that

$$a, a' \in A \text{ with } d(a, a') < \delta \implies d(f(a), f(a')) < \varepsilon/3. \tag{7}$$

We claim that if $x, x' \in X$ are such that $d(x, x') < \delta/3$, then $d(g(x), g(x')) < \varepsilon$, that is, g is uniformly continuous.

Let $x, x \in X$ be such that $d(x, x') < \delta/3$. Let (a_n) and (b_n) be sequences in A such that $a_n \to x$ and $b_n \to x'$. We claim that there exists n_3 such that if $n \geq n_3$, then $d(a_n, b_n) < \delta$. Since $a_n \to x$ and $b_n \to x'$, for $\delta/3$, there exist $n_1, n_2 \in \mathbb{N}$ such that

$$n \geq n_1 \implies d(a_n, x) < \delta/3 \text{ and } n \geq n_2 \implies d(b_n, x') < \delta/3. \tag{8}$$

It follows from (8) that if $d(x, x') < \delta/3$ then for

$$n \geq n_3 := \max\{n_1, n_2\} \implies d(a_n, b_n) \leq d(a_n, x) + d(x, x') + d(x', b_n) < 3(\delta/3) = \delta. \tag{9}$$

In view of (7) and (9), we get

$$d(x, x') < \delta/3 \text{ and } n \geq n_3 \implies d(f(a_n), f(b_n)) < \varepsilon/3. \tag{10}$$

Now since $f(a_n) \to g(x)$ and $f(b_n) \to g(x')$, for $\varepsilon/3$, there exist $k_1, k_2 \in \mathbb{N}$ such that

$$n \geq k_1 \implies d(f(a_n), g(x)) < \varepsilon/3 \text{ and } n \geq k_2 \implies d(f(b_n), g(x')) < \varepsilon/3. \tag{11}$$

It follows from (11), (10) and (11) that for $x, x' \in X$ with $d(x, x') < \delta/3$

$$\begin{aligned} d(g(x), g(x')) &\leq d(g(x), f(a_n)) + d(f(a_n), f(b_n)) + d(f(b_n), g(x')) \text{ for any } n & (12) \\ &< 3(\varepsilon/3) \text{ if } n \geq \max\{k_1, k_2, n_3\}. & (13) \end{aligned}$$

Hence g is uniformly continuous on X.

Claim 5: g is unique.

We are supposed to prove: If $h\colon X \to Y$ is a continuous function such that $h(a) = f(a)$ for all $a \in A$, then $h(x) = g(x)$ for all $x \in X$. Let $x \in X$ be arbitrary. Let (a_n) be a sequence in A such that $a_n \to x$. Recall that $g(x) := \lim_n f(a_n)$. Since h is an extension of f, we have $h(a_n) = f(a_n)$. Since $a_n \to x$ and h is continuous at x, it follows that $h(a_n) \to h(x)$. Since $h(a_n) = f(a_n)$ and since $f(a_n) \to g(x)$, we deduce that $h(x) = g(x)$ thanks to the uniqueness of limits. Since $x \in X$ is arbitrary, we have shown that $h = g$ on X.

□

Ex. 2. Let $f\colon (a, b) \to \mathbb{R}$ be continuous. Show that f extends to $[a, b]$ iff it is uniformly continuous on (a, b).