Vector Product on $\mathbb{R}^3$

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We define a *cross-product* on a three dimensional real vector space V with an inner product: $(x, y) \mapsto \langle x, y\rangle$. We fix an orthonormal basis $\{e_i\}$ of V such that $\langle e_i, e_j\rangle = \delta_{ij}$. If you wish you may take $V = \mathbb{R}^3$ with the standard basis vectors and the Euclidean inner product $(x, y) \mapsto \langle x, y\rangle := \sum_{i=1}^3 x_i y_i$. For any *ordered* set of three points x_1, x_2, x_3 of V we define the *oriented* volume of the parallelepiped with sides Ox_i by setting:

$$\mathrm{vol}\,(x_1, x_2, x_3) = \det(\alpha_{ji}) \text{ where } x_i = \sum_j \alpha_{ji} e_j.$$

$\mathrm{vol}\,(x_1, x_2, x_3)$ is independent of the choice of the basis as above. We also have the Riesz representation theorem: For any linear map $f : V \to \mathbb{R}$ there exists a unique $u \in V$ such that $f(x) = \langle x, u\rangle$. Hint: With basis vectors e_i we take $u := \sum_i f(e_i)e_i$. We now define the *cross product* or *vector product* on V as follows:

For $x,\ y \in V$, the map $z \mapsto \mathrm{vol}\,(x, y, z)$ is linear map of V to $\mathbb{R}$ and hence by Riesz representation theorem there exists a unique vector $x \times y$ such that

$$\langle x \times y, z\rangle = \mathrm{vol}\,(x, y, z), \text{ for all } z \in V.$$

It is easy to see that if $w := x \times y = \sum_i w_i e_i$ is the unique vector given by Riesz, then $w_j = \langle \sum_i w_i e_i, e_j\rangle = \det(x, y, e_j)$. From this we find that

$$x \times y = (x_2y_3 - x_3y_2)e_1 - (x_3y_1 - x_1y_3)e_2 + (x_1y_2 - x_2y_1)e_3.$$

This product has the following properties which are immediate consequences of well-known properties of determinants:

1. $\lambda x \times y = \lambda(x \times y) = x \times \lambda y$, for $\lambda \in \mathbb{R}$.
2. $y \times x = -x \times y$.
3. $\langle x \times y, z\rangle = \langle y \times z, x\rangle = \langle z \times x, y\rangle$.
4. $\langle x, y \times z\rangle = \langle y, z \times x\rangle = \langle z, x \times y\rangle$.

Proposition 1. *For any three vectors $x, y, z \in V$, we have*

$$x \times (y \times z) = \{(\langle x, z\rangle)y - (\langle x, y\rangle)z\}. \tag{1}$$

Proof. To show that these two vectors are equal, it is enough to show that their inner product with any vector of V (in fact, any vector in an orthonormal basis) are the same:

$$\langle v, x \times (y \times z) \rangle = \langle v, (\langle x, z \rangle) y - (\langle x, y \rangle) z \rangle \, .$$

In view of (4), it is enough to verify for an arbitrary vector v,

$$\langle v \times x, y \times z \rangle = \{ \langle v, y \rangle \langle x, z \rangle - \langle x, y \rangle \langle v, z \rangle \}. \tag{2}$$

We first observe that both sides are linear in each of the variables. Hence it is enough to verify it on $\{e_i\}$. Due to symmetry we may take $y = e_1$, $z = e_2$ so that $y \times z = e_3$. Now it is easily checked that both sides of Eq. 2 are equal to $(v_1 x_2 - v_2 x_1)$. $\square$

The geometric meaning of the vector or cross product $x \times y$ is that it is the vector orthogonal to x and y with the property that $\{x, y, x \times y\}$ is a basis with the same orientation as $\{e_1, e_2, e_3\}$ and is of length $\|x\| \, \|y\| \sin\theta$. This follows for example from Eq. 2. It may be noted that the latter quantity $\|x\| \, \|y\| \sin\theta$ is the area of the parallelogram spanned by x and y.

We often write $x \wedge y$ for $x \times y$.