Vector Analysis

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Good references for this topic are some of the standard American textbooks on Calculus such as Thomas-Finney, Stewart, Howard Anton, Kreyzsig (Advanced Engineering Mathematics) etc. All of these will have good pictures and physical explanations for the concepts and results. These theorem were found by physical reasoning and latter rigorous proofs were given by mathematicians. To have a better feeling for this topic, it is suggested to approach the topic with a physical intuition however weak it is!

1. A vector field F on an open set $U \subset \mathbb{R}^n$ is a C^1-map $F\colon U \to \mathbb{R}^n$. We visualize $F(x)$ by thinking as a directed line segment $F(x)$ emanating/starting from the point x, as we used to do in Physics!

2. The notion of vector fields arises naturally in Physics. If we imagine a fluid flowing through a pipe, then $F(x)$ could the veclocity vector at the pint x of the pipe. If we have two bodies with masses m and M then the gravitational force acting on any object at the point (x, y, z) is given by
$$F(x, y, z) := -\frac{mMG}{\|x\|^3} x,$$
where G is the gravitational constant.

3. An important example both for theory and practice is the gradient field of a function. Let $f\colon U \to \mathbb{R}$ be a C^2 function. Then $F : x \mapsto \operatorname{grad} f(x)$ is a vector field on U, called the gradient vector field of f. f is called the potential of F.

4. A vector field F on $U \subset \mathbb{R}^n$ is said to be conservative if it has a potential, that is, F is the gradient field of a C^2 function f. Is such an f unique?

   Exercise: Show that the gravitational field is conservative on its domain.

5. A path γ in an open set $U \subset \mathbb{R}^n$ is a C^1 or a piecewise C^1 function $\gamma\colon [a, b] \subset \mathbb{R} \to U \subset \mathbb{R}^n$. ($\gamma$ is piecewise smooth iff γ is continuous on $[a, b]$ and is C^1 on partitioning subintervals.) Such paths/curves are said to be parametrized.

   Note that the paths/curves should not be confused with theie images. For example, the paths $\alpha(t) = (t, 0$ and $\beta(t) = (t^3, 0)$ for $t \in [0, 1]$ are distinct though they have the same images. The best analogy would be to think of α and β as trains which travel on the same track but at (possibly) different time intervals and velocities.

6. If γ is C^1, then the tangent vector to γ at $t \in [a.b]$ is deifned by $\gamma'(t)$. We usually write $\gamma(t) = (x_1(t), \ldots, x_n(t))$ in the case of arbitrary n or $\gamma(t) = (x(t), y(t))$ in the case when $n = 2$ etc. Hence $\gamma'(t) = (x_1'(t), \ldots, x_n'(t))$.

7. Let γ be a path in $U \subset \mathbb{R}^n$ and $f \colon U \to \mathbb{R}$ be a continuous function. Then the line integral of f along γ, denoted by $\int_\gamma f$ is defined by

$$\int_\gamma = \int_a^b f(\gamma(t)) \left\|\gamma'(t)\right\| \, dt. = \int_a^b f(\gamma(t)) \sqrt{x_1'(t)^2 + \cdots + x_n'(t)^2} \, dt.$$

   How do we define the line integral $\int_\gamma f$ if γ is piecewise C^1?

8. Given a curve $\alpha \colon [a, b] \to U \subset \mathbb{R}^n$, a reparametrization of α is a C^1 function $h \colon [c, d] \to [a, b]$ such that $h' > 0$ and $h(c) = a$ and $h(d) = b$. The curve $\beta(s) := \alpha(h(s))$ is *also* said to be a reparametrization of α. Note that if we write $t := h(s)$, then $dt = h'(s)$ and the change of variable formular or the substituion theorem of calculus yields

$$\int_\alpha f = \int_\beta f.$$

9. Let $\gamma \colon [a, b] \to U$ be a curve and F be a vector field on U. The the line integral of F over γ is defined as the work done by the force field F by moving a particle along γ. It is mathematically defined by setting

$$\int_\gamma F := \int_a^b F(\gamma(t)) \cdot \gamma'(t) \, dt$$

   Let us derive a classical notation for this when $n = 2, 3$. If we write $\gamma(t) = (x(t), y(t), z(t)$ and $F = (P, Q, R)$, then $F \cdot \gamma'(t) = Px'(t) + Qy'(t) + Rz'(t)$ so that the classical or old fashioned notation for integrand on the right side $Pdx + Qdy + Rdz$. (Note that $x'(t) = \frac{dx}{dt}$ *so that* (?) $x'(t) \, dt = dx$!)

   Hence in old texbooks one sees the defintion of the line integral as $\int_\gamma F = \int_\gamma F \cdot d\mathbf{r} = \int_\gamma Pdx + Qdy + Rdz$.

10. Let us compute the line integral of a conservative vector field $F = \operatorname{grad} f$ along a path γ. We find that $\int_\gamma F = f(\gamma(b)) - f(\gamma(a))$. Note that this is analogous to the fundamental theorem of calculus. Note also that the work done, namely the line interal of F, is independent of the path and depends only on the end points of the path.

11. Let $U = \mathbb{R}^2$ minus the origin and $F(x, y) := \frac{1}{x^2+y^2}(-y, x)$. Let $\gamma(t) := (\cos t, \sin t)$ for $t \in [0, 2\pi]$. Then we find that $\int_\gamma F = 2\pi$. Note that here the end points coincide but the line integral is not zero.

12. If $\sigma(t) := \gamma(b + a - t)$, then σ is called the reverse path of γ. We have $\int_\sigma F = -\int_\gamma F$.

13. If $\alpha \colon [a, b] \to U$ and $\beta \colon [b, c] \to U$ are paths such that $\alpha(b) = \beta(b)$, then we have a new path γ with domain $[a, c]$. Also, we have $\int_\gamma F = \int_\alpha F + \int_\beta F$.

14. We say a path $\gamma \colon [a, b] \to U$ is closed if $\gamma(a) = \gamma(b)$.

15. We say that the work done by the vector field F on an open set U is path independent if $\int_\alpha F = \int_\beta F$ for any two paths having the same end points.

16. The work done by the vector field F on an open set U is path independent iff the work done by F along any closed path is zero.

17. Let U be an open convex (or star-shaped) set. If the work done by a vector field F on U is independent of path, then F is a conservative vector field.

    Fix the 'star' $a \in U$. Define a function $f(x)$ as the work done by $F = (F_1, \dots, F_n)$ along the line segment $[a, x]$. To show that the i-th partial derivative of f are F_i proceed as one does in the proof of the fundamental theorem of calculus.

18. Assume that a vector field F on U is conservative. If f is a potential function, then f must be C^2. Since $F_i = \frac{\partial f}{\partial x_i}$ and since $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$, it follows that $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$ for all $1 \leq i, j \leq n$. This set of equations is therefore a set of necessary conditons for the vector field F to be conservative.

    In the case of $n = 2$, if we write $F = (P, Q)$ or $F = (L, M)$, the necessary condtion takes the form $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ etc.

19. The set of necessary conditions is also sufficient if we assume that the domain U is convex. We shall see this after proving Green's theorem.

20. Show that the following vector fileds are conservative by checking the N & S conditions. Find a potential function in each of the cases.

    (a) F on $\mathbb{R}^2$ given by $F(x, y) = (3 + 2xy, x^2 - 3y^2)$.

    (b) F on $\mathbb{R}^3$ given by $F(x, y, z) = (y^2, 2xy + e^{3z}, 3ye^{3z})$.

21. Let $F(x, y) = (e^y, xe^y)$ on $\mathbb{R}^2$. Show that the field is conservative, find a potential function. Use this information to comppute the line integral of F along the semicircular path on the upper half plane $y > 0$ connecting the point $(1, 0)$ to $(-1, 0)$.

22. We now derive the law of conservation of energy by using Newton's second law of motion and computing the work done by a force in two different ways and equating them.

    We have by seond law of motion: $F = mr''(t)$, Using this expression for F, the work done by F turns out $W = \frac{m}{2}\left(\|r'(b)\|^2 - \|r'(a)\|^2\right)$. The quantity $\frac{m}{2}\|r'(b)\|^2$ is called the kinetc energy at b etc. Now if we further assume that F si conservative, say, $F = \operatorname{grad} f$, then the potential energy $P(t)$ at t is defined as $P(t) := -f(\gamma(t))$. Hence the work done by F can be expressed as

    $$W = -P(b) + P(a) = P(a) - P(a).$$

    Equating the expressions for W we get

    $$W = K(b) - K(a) = P(a) - P(b),$$

    or $K(a) + P(a) = K(b) + P(b)$. This is known as the law of conservation of energy. (I used different notations which are clear from the context!)

Students who think that Physics is very 'concrete' as opposed to Mathematics should go through the proof. Observe that the 'law' was obtained by manipulating results obtained mathematically. We define the potential energy (why the negative sign?) and the kintetic energy (why the presence of half?) so that the law if obtained. Is there any physical reason with we could have obtained these expressions?

23. All the theorems of vector integral calculus are in the spirit of fundamental theorem of calculus: $\int_a^b f'(t)\, dt = f(b) - f(a)$. If we interpret the right side as an integral with a *proper orientation* of the boundeary of domain $[a, b]$ of integration on the left side, we see that the fundamental theorem of calculus expresses the integral of f on the doamin $[a, b]$ as an intergal of f over its orinted boundary $\{a, b\}$. Keep this intution in your mind. As we go along, you may get to see what we mean.

24. Let $F = (P, Q)$ be a vectr field ona domain in $\mathbb{R}^2$. Using the concept of fluid motion we arrived at theconcept of flux density or divergence of a vector field: $\operatorname{Div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$.

25. In a similar way we arrived at the circulation density or curl of a vector field $\operatorname{Curl} F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$.

26. To state Green's theorem precisely, we need the notion of an orientaion of the boundary curve of a domain in $\mathbb{R}^2$. In classical books, one talks of outward normal (which is motivated by our consideration of divergence) and the boundary curve being parametrized in anti-clock wise direction.

27. If $c(t) = (x(t), y(t))$ is the boundary curve, then the tangent vector is (x', y'). Hence the normal is obtained by rotating the tangent vector in the clockwise direction by $\pi/2$ radians. This is same as ssaying that th normal is obtained by rotating the tangent vector in the anticlockwise direction by $-\pi/2$ radians. Hence the 'correct' normal is $(y', -x')$. We denote the unit vector obtained from this vector as $\mathbf{n}$ and is called the unit normal vector of the curve c.

28. Our physical reasoning earlier suggests the following two versions of Green's theorem.

    (a) **Flux-Divergence Form** which usesthe normal component of the vector field:

    $$\int_c F \cdot \mathbf{n} \equiv \int_c (Qdy - Pdx) = \int_U \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dxdy.$$

    (b) **Circulation-Curl Form** which uses the tangent component of the vector field:

    $$\int_c F \cdot \mathbf{T} \equiv \int_c (Pdx + Qdy) = \int_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

    Note that the two forms are equivalent.

29. We prove Green's theorem for domains of some special type. We assume that the domain is bounded by graphs of functions $y = g_1(x)$, $y = g_2(x)$ and $x = a$ and $x = b$. We *also* assume that the domain is bounded by $x = h_1(y)$, $x = h_2(y)$, $y = c$ and $y = d$.

30. The main idea of the proof (of the curl form) is to prove these equations separately:

$$\int_c P\,dx = -\int_U \frac{\partial P}{\partial y}\,dxdy \quad \text{and} \quad \int_c Q\,dy = \int_U \frac{\partial Q}{\partial x}\,dxdy.$$

To prove the first, we use Fubini's theorem to interchange the order and integrate w.r.t. y first, use the paramaterization of the boundary in the first form and the fundamental theorem of calculus in the y-variable. You will arrive at an integral in x. Use the same parametrization to arrive at the expression for the first line integral involving P, which is the same as you got from the double integral. To prove the second, we integrate w.r.t. x first and use the paramaterization of the boundary in the second form.

31. Verify both the forms of Green's theorem where the domain is the disk $\{x^2 + y^2 \leq R^2\}$ and $F(x, y) = (-y, x)$.

32. We derived from Green's theorem the following formulas for the area enclosed by a closed curve:

$$A = \int_c x\,dy = -\int_c ydx = \frac{1}{2}\int_c xdy - ydx.$$

33. Use the formula to obtain the formulas for the areas enclosed by a circle or more generally an ellipse.

34. We used Green's heorem to prove the following theorem. The following statements are equivalent for a vector field $F = (P, Q)$ on an open convex set $U \subset \mathbb{R}^2$:

    (a) F is conservative.

    (b) The work done by F along any path in U depends only on the end points.

    (c) The work done by F along any clsoed path in U is zero.

    (d) $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

    The equivalence of (a)–(c) is seen earlier To prove (d) is equivalent to (c), we employed Green's theorem.

35. We hinted upon how Cauchy proved his integral theorem in Complex Analysis using Green's theorem.

36. Sometimes it is easier to compute a line integral using Green's theorem. Consider $F(x, y) = (y^2, 3xy)$ and domain is the upper half disk $\{(x, y) \in \mathbb{R}^2 : y > 0,\ x^2 + y^2 < 1\}$. Thus the domain is described by $-1 < x < 1$ and $0 < y < \sqrt{1 - x^2}$. The integrand of the double integral, the curl of F truns out to be y which needs to be integrated int he doamin. The answer is $2/3$. To convince yourself, you should try to compute the line integral from its definition.

37. Use Green's theorem to compute the line integral of $F(x, y) = (\sqrt{x} + y^3, x^2 + \sqrt{y})$ where the curve c is given by $y = \sin(x)$ $x \in [0, \pi]$ and the line segment on the x axis from π to 0. Be careful. The curve is not oriented properly!

38. Let $1 \leq k < n$, and $r \geq 1$. We define a k dimensional C^r-submanifold of $\mathbb{R}^n$ is a subset $S \subset \mathbb{R}^n$ with the following property: For each point $p \in S$, there exists an open set $U \subset \mathbb{R}^n$ such that $p \in U$ and an open set $V \subset \mathbb{R}^k$ and a C^r function $\varphi\colon V \to \mathbb{R}^{n-k}$ satisfying the condition $\{(x, \varphi(x)) : x \in V\} = U \cap S$.

    If $k = n$, then an n-dimensional submanifold of $\mathbb{R}^n$ is defined to be an open subset of $\mathbb{R}^n$.

39. Of course a typical simple example is the graph of such a φ. With the notation above, if we let S stand for the graph of φ, then for any $p \in S$, we may take $U = \mathbb{R}^n$.

40. Note that the implicit function theorem gives a lot of examples of k-dimensioanl submanifolds of $\mathbb{R}^n$. If $f\colon \mathbb{R}^{k+n} \to \mathbb{R}^n$ is C^r and if $S = f^{-1}(0) \neq \emptyset$ with the property that for each $p \in S$, the rank of $Df(p)$ is n, then the implict function theorem says that S is a k-dimensional C^r manifold.

41. A particular case of the last item is worht mentioning. If $f\colon \mathbb{R}^{k+1} \to \mathbb{R}$ is C^r and if 0 is a value of f, and if for each $p \in S := f^{-1}(0)$ the gradient $\operatorname{grad} f(p)$ is nonzero, then S is a k dimensional C^r manifold. These submanifolds are called level sets.

42. If S is the graph of a function $f\colon U \to \mathbb{R}$, then the surface is also got as a level set. How do we do this? An obvious chocie will be to define $g(x_,1 \ldots, x_n x_{n+1}) := x_{n+1} - f(x_1, \ldots, x_n)$. What is its domain V? We may take $V := U \times \mathbb{R}$. What is its gradient? It is $\left(-\frac{\partial f}{\partial x_1}, \ldots, -\frac{\partial f}{\partial x_n}, 1\right)$. Now complete the details.

43. While discussing Lagrange multiplier method, we have introduced the notion of a tangent vector to S at $p \in S$ and have established that $T_pS = \ker D\varphi(q)$ where $p = \varphi(q)$. We shall look at it more carefully now.

44. Let $f\colon V \subset \mathbb{R}^2 \to \mathbb{R}$ be C^r and $q \in V$. Since V is open for all sufficiently small $|t|$, the line segments $q + te_i$ lies in V, $i = 1, 2$. If we let $\varphi(x, y) := (x, y, f(x, y))$, then the iamges of these line segments under φ are the curve $t \mapsto (x + t, y, f(x + t, y))$ etc. This passes through $p := \varphi(q) = \varphi(x, y) \in S$ and the tangent vector is $E_1 := (1, 0, f_x)$ where $f_x := \frac{\partial f}{\partial x}$. E_2 is defined similarly. Note that the tangent vectors E_1 and E_2 are linearly independent. Since S also arises as $g^{-1}(0)$ where g is defined as in Item 42, the tangent space T_pS is the 2 dimensional vector space $\ker Dg(q)$. Since $Dg(q)$ is 'represented' by $(-f_x, -f_y, 1)$ the gradient of g, we conclude that T_pS is the linear span of E_1 and E_2. Note that the normal $(-f_x, -f_y, 1)$ to the surface at p is nothing other than the cross-product $E_1 \times E_2$.

45. If the submanifold S arises as a level set of f, then we know that T_pS is $\ker Df(p)$. Note that in this case we cannot write down some'natural' tangent vectors in a concrete fashion. Here $\operatorname{grad} f(p)$ a 'natural' normal to the surface at $p \in S$, if $S = f^{-1}(0)$ for some $f\colon U \subset \mathbb{R}^3 \to \mathbb{R}$.

46. There is a more standard definition of a k-dimensional submanifold in $\mathbb{R}^n$. We say that $S \subset \mathbb{R}^n$ is a k-dimensional C^r submanifold if for each $p \in S$, we have an open set $U \subset \mathbb{R}^n$, an open set $V \subset \mathbb{R}^k$ and a C^r map $\psi\colon V \to \mathbb{R}^n$ such that (i) ψ is a homeomorphism of V onto $U \cap S$ and (ii) the rank of $D\psi(q)$ is k for each $q \in V$.

    Note that according to this definiton an n-dimensional submanifold of $\mathbb{R}^n$ is an open subset of $\mathbb{R}^n$.

47. The definitions in Items 38 and Item 46 are equivalent.

Let $S \subset \mathbb{R}^n$ be a $k < n$ dimensional submanifold of $\mathbb{R}^n$ according to Item 38. Using the notation established over there, if we define $\psi(x) := (x, \varphi(x))$, then we see that S is a submanifold according to Item 46.

To see the other way round, we shall restrict ourselves to a surace $S \subset \mathbb{R}^3$ according to Item 46. Keeping the notation of the said item, we write $\psi(u,v) = (x(u,v), y(u,v), z(u,v)$.

Then the jacobian matrix of $D\psi$ is $\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$. If we assume WLOG that the first two rows constitute the nonsingular minor, then the map $F(u,v) := (x(u,v), y(u,v))$ maps an open set V_1 diffeomrphivally onto an open set $W_1 \subset \mathbb{R}^2$. If we let (u_1, v_1) as the coordinates on the codomain of F, then it is easy to see that $\psi(V_1)$ is an open subset of S which is the graph of the function $(u_1, v_1) \mapsto F^{-1}(u_1, v_1) = (u,v) \mapsto \psi(F^{-1}(u_1, v_1)) = (x(u,v), y(u,v), z(u,v)) \mapsto z(u,v)$.

48. There is an important class of surfaces that arise as surfaces of revolution. Let $c(t) = (x(t), 0, z(t))$ be a curve in the (x,z)-plane. Assume that $x(t) > 0$ and that $x'(t)^2 + y'(t)^2 \neq 0$. If we revolve the curve around the z-axis, the z cordinate remains the same while $(x(t), 0)$ in the (x,y) plane is transofrmed by the rotation matrix to yield $(x(t)\cos\theta, x(t)\sin\theta)$ so that a point on the surface of revolution has coordinates

$$(x(t)\cos\theta, x(t)\sin\theta, z(t)).$$

Thus we have a parametrization

$$\psi\colon (a,b) \times (0, 2\pi) \to \mathbb{R}^3 \text{ given by } \psi(t,\theta) = (x(t)\cos\theta, x(t)\sin\theta, z(t).$$

We discuused some special cases of surfaces of revolution such as spheres, cylinder and cones.

49. Given a parametrized surface as in Item 46, we let X_u, X_v stand respectively for the first and second columns of the Jacobian matrix of $D\psi$. Recal that they form the basis of the tangent space. The area element on $\psi(V)$ is defined by $dS := \| X_u \times X_v \| \, dudv$. How do we use this? If $f\colon \psi(V) \subset \mathbb{R} \to \mathbb{R}$ is a continuous function, then its integral on S is defined

$$\int_S f dS := \int_V f \circ \psi \, \| X_u \times X_v \| \, dudv.$$

In particular, if $E \subset \psi(V)$ is compact, then its (surface) area is defined by

$$\text{Area}\,(E) := \int_V \psi^{-1}(E) dS \equiv \int_V \psi^{-1}(E) \, \| X_u \times X_v \| \, dudv.$$

50. We verify that this definition yields the surface area of spheres, cylinder, cones etc. As they all arise as surfaces of revolution (about x-axis)of a graph of a function $f\colon [a,b] \to \mathbb{R}_+$ we shall verify whether we arrive at the formula learnt from Calculus. Note that such a surface is parametrized as $(u,v) \mapsto (u, f(u)\cos v, f(u)\sin v)$. Hence $X_u = (1, f'(u)\cos v, f'(u)\sin v)$ and $X_v = (0, -f(u)\sin v, f(u)\cos v)$ so that

$$dS = f(u)\sqrt{1 + f_u^2} dudv.$$

51.

52. Verify Gauss divergence theorem for $F(x, y, z) = (0, 0, z)$and the surface $S = x^2 + y^2 + z^2 = R^2$.

53. Use divergence theorem to find the outward flow of the vector field $F(x, y, z) = (x^3, y^3, z^2)$ (No typing mistake, the power of z is 2.) across the suarfce of the region bounded bythe cylinder $x^2 + y^2 = 16$ and the planes $z = 0$ and $z = 2$.

54. Evaluate $\int_S F \cdot \mathbf{n} dS$ where $F(x, y, z) = (x, y, z)$ and S is any closed surface bounding a region in $\mathbb{R}^3$.