Vector Fields on Spheres

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

1 Introduction

A vector field on a subset $S \subset \mathbb{R}^n$ is a continuous function $F: S \to \mathbb{R}^n$. Vector fields on $\mathbb{R}^2$ are often represented as the vector $F(x)$, with its tail at x. We look at some examples.

Example 1. If S is any subset of $\mathbb{R}^n$ and $v_0 \in \mathbb{R}^n$ is fixed, then the function $F(x) = v_0$ is a vector field on S. It is called a constant vector field.

Example 2. The function $(x, y) \mapsto (-y, x)$ is a vector field on $\mathbb{R}^2$. This represents the velocity at each point of $\mathbb{R}^2$ when $\mathbb{R}^2$ is rotated in the counterclockwise direction at a uniform rate.

Example 3. Let U be an open set in $\mathbb{R}^n$ and $f: U \to \mathbb{R}$ be C^1, i.e., all partial derivatives of f exist and are continuous. Then the gradient ∇f of f is defined by $\nabla f(x) := (\frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_n}(p))$. Then $p \mapsto \nabla f(p)$ is a vector field on U.

A point $p \in S$ is called a zero of the vector field F on S if $F(p) = 0$. While the vector field F itself behaves well at and around a zero, its direction $F/\|F\|$ can behave quite wildly near a zero of F. For this reason, the zeros of F are called the *singularities* of F in old literature.

There is a notion of index available for vector fields on $\mathbb{R}^2$. Indeed, if F is any nonzero vector field on a subset S of $\mathbb{R}^2$ and if $e^{it} \mapsto \gamma e^{it}$, $0 \le t \le 2\pi$, is a loop in S, then the index of F around γ is defined to be the index of $F \circ \gamma$, regarded as a map from the circle to nonzero complex numbers. (See the article on Winding Numbers of Loops.) The index of F around γ counts the number of times $F(p)$ winds around the origin as p moves around γ.

Lemma 4. *Let F be a vector field on the closed disk $B[0, R]$ that does nor vanish on the boundary circle S^1. Assume that the index of F around the loop $e^{it} \mapsto Re^{it}$ is not zero. Then F has a zero at some point of the open disk $B(0, R)$.*

Proof. If F has no zeros on $B[0, R]$, then we shall show that the index of the map $F_R: e^{it} \mapsto F(Re^{it})$ is zero. To see this, observe that the map $F(r, e^{it}) := F(re^{it})$, $0 \le t \le 2\pi$, $0 \le r \le R$, is a homotopy from the constant map F_0 and F_R. Since the index of a constant map is zero, and homotopic maps have the same index, we conclude that the index of F_R is zero. □

2 Vector Fields on S^2

A vector field F on the unit sphere $S^2 \subset \mathbb{R}^3$ is *tangent* to S^2 if $F(p) \perp p$, i.e., $\langle F(p), p \rangle = 0$. If we write $F = (F_1, F_2, F_3)$, then F is tangent to S^2 iff

$$xF_1(x,y,z) + yF_2(x,y,z) + zF_3(x,y,z) = 0, \qquad \text{for all } (x,y,z) \in S^2.$$

The vector field $F(x,y,z) := (-y, x, 0)$ is tangent to S^2. It points from west to east everywhere except at the poles $(0,0,\pm 1)$. The only constant vector field which is tangent to S^2 is the zero field. Another example is $F(x,y,z) := (xz, yz, z^2 - 1)$, $(x,y,z) \in S^2$. Then F is tangent to the sphere. F points from north to south except at the poles which are zeros of F.

The main result of this article is to prove the following

Theorem 5. *Every tangent vector field on S^2 has a zero.* □

There is a physical interpretation of this result. If we regard S^2 as a realisation of the earth's surface, we obtain a vector field tangent to S^2 by assigning to any point a vector expressing the magnitude and direction of the wind at that point, at some fixed time. The above result says that at any given time there is a point on the earth at which the wind s not blowing.

We give the main ideas behind the formal proof. Suppose that the vector field F does not have a zero at the north pole. Consider a small circle Γ that is centred at the north pole and inside which F is nearly a constant. If an observer stands at the north pole and regards the surface of the earth nearby as flat, then he would compute the index of F around Γ to be zero. However if he were at the south pole and studying the vector field on the map obtained by projecting the sphere stereographically from the north pole into the plane, he would compute the index of the vector field around the circle corresponding to Γ is 2. Hence F vanishes somewhere on the map by Lemma 4.

Proof. Let $N = (0,0,1)$ be the north pole of the sphere. Define the stereographic projection $\varphi\colon S^2 \setminus \{N\} \to \mathbb{R}^2$ by

$$\varphi(x,y,z) := \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

φ is continuous with a continuous inverse $\psi = \varphi^{-1}$ given by

$$\psi(u,v) := \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right), \qquad (u,v) \in \mathbb{R}^2.$$

Thus φ is a homeomorphism of $S^2 \setminus \{N\}$ onto $\mathbb{R}^2$.

Let $F = (F_1, F_2, F_3)$ be a tangent vector field on S^2. We associate a vector field G on $\mathbb{R}^2$ by

$$G(u,v) = ((1-z)F_1 + xF_3, (1-z)F_2 + yF_3), \qquad \text{where } (x,y,z) = \psi(u,v).$$

Clearly G is continuous. (How is G written down? See remark at the end.)

We now claim that $p \in S^2$ is a zero of F iff $\varphi(p)$ is a zero of G. If $F(x,y,z) = 0$, then it is clear from the definition of G that $G(u,v) = 0$. To see the converse, assume that $G(u,v) = 0$.

Let $\psi(u, v) = (x, y, z)$. We get

$$\begin{aligned}(1-z)F_1 + xF_3 &= 0 \\ (1-z)F_2 + yF_3 &= 0.\end{aligned}$$

Multiply the first equation by x, the second by y, adding and using the relations $xF_1 + yF_2 = -zF_3$ and $x^2 + y^2 = 1 - z^2$, we obtain

$$(1-z)(-zF_3) + (1-z^2)F_3 = 0.$$

This simplifies to $(1-z)F_3 = 0$. Since $z \neq 1$. we have $F_3(x, y, z) = 0$. Eq. 1 and Eq. 1 then yield that $F_i(x, y, z) = 0$ for $i = 1, 2$. Hence $F(x, y, z) = 0$.

Before we go further into the proof let us look at a particular vector field. Let $\tilde{F}(x, y, z) = (z, 0, -x)$ on S^2. Then $\tilde{G}(u, v) = (z - z^2 - x^2, -xy)$. Using the expressions for x, y and z in terms of u and v and using the polar coordinates for u and v we get

$$\tilde{G}(r\cos\theta, r\sin\theta) = \frac{-2r^2}{(r^2+1)^2}\left(\frac{1}{r^2} + \cos 2\theta, \sin 2\theta\right).$$

If $r > 0$, then $\tilde{G}$ does not vanish and we can consider the index of $\tilde{G}$ around the circle $e^{i\theta} \mapsto re^{i\theta}$. This is same as the index of the map $e^{i\theta} \mapsto \left(\frac{1}{r^2} + \cos 2\theta, \sin 2\theta\right)$. The index of this is that of $e^{i\theta} \mapsto (\cos 2\theta, \sin 2\theta)$ which is 2.

Now suppose that F is a vector field on S^2 such that $F(0, 0, 1) = (1, 0, 0) = \tilde{F}(0, 0, 1)$. Let G be the corresponding vector field on $\mathbb{R}^2$. Then $F \cdot \tilde{F} = 1$ at $(0, 0, 1)$. By continuity, $F \cdot \tilde{F} > 0$ in a neighbourhood U of $(0, 0, 1)$. It follows that the convex linear combinations $tF + (1-t)\tilde{F}$ have no zeros in U for $0 \leq t \leq 1$. For, otherwise, we obtain a contradiction by taking the scalar product with $\tilde{F}$. Since $tG + (1-t)\tilde{G}$ corresponds to $tF + (1-t)\tilde{F}$, it follows that $tG + (1-t)\tilde{G}$ has no zeros on $\varphi(U)$. Choose R so large that if $u^2 + v^2 > R$ then $(u, v) \in \varphi(U)$. The family, $tG + (1-t)\tilde{G}$, $0 \leq t \leq 1$, gives a homotopy of the maps $e^{i\theta} \mapsto G(R\cos\theta, R\sin\theta)$ and $e^{i\theta} \mapsto \tilde{G}(R\cos\theta, R\sin\theta)$ in the punctured plane. We have computed the index of the latter to be 2. Consequently, the index of the former is also 2. By Lemma 4, G has no zeros inside $B[0, R]$. Hence F has no zero.

Now let F be an arbitrary vector field on S^2. We claim that F has a zero. If $F(0, 0, 1) = (1, 0, 0)$, we are done. Otherwise, let T be a rotation of $\mathbb{R}^3$ such that $T(F(0, 0, 1)) = \alpha(1, 0, 0)$, $\alpha \in \mathbb{R}$, $\alpha \neq 0$ and $T(0, 0, 1) = (0, 0, 1)$. The vector field $(1/\alpha)T \circ F \circ T^{-1}$ has a zero. Hence so does F. □

Remark 6. We say a vector $v \in \mathbb{R}^n$ is tangent to $S = S^{n-1}$ at $p \in S$ if $v \cdot p = 0$. Thus a tangent vector field is a continuous map F from S to $\mathbb{R}^n$ such that $F() \perp p$. The crucial geometric observation is that any tangent vector at $p \in S$ actually arises as a tangent to a curve c passing through p. That is to say that given $v \in \mathbb{R}^n$ with $v \perp p$, there exists an $\varepsilon > 0$ and a C^1-map $c\colon (-\varepsilon, \varepsilon) \to S$ such that $c(0) = p$ and $c'(0) = v$. Recall that $c'(t) = (x_1'(t), \ldots, x_n'(t))$ if $c(t) = (x_1(t), \ldots, x_n(t))$. For if $v \perp p$ and $\|v\| = 1$, then the curve $c(t) := p + tv$ lies on S for all $t \in \mathbb{R}$ and we have $c(0) = p$ and $c'(0) = v$. (This is the parameterization of the great circle—the intersection of the plane spanned by p and v with S. If $v \perp p$ and v is arbitrary, let

$$\gamma(t) = c(t)/\|c(t)\| = \frac{\cos tp + \sin tv}{(1 + \sin^2 t(\|v\|^2 - 1))^{1/2}}.$$

Then $\gamma(t) \in S$ for all t, $\gamma(0) = p$ and $\gamma'(0) = v$. (Check!)

Now if F is a vector field on S and $p \in S$, let $v := (F_1(p), F_2(p), F_3(p))$. By the last paragraph, there is a curve c in S such that $c(0) = p$ and $c'(0) = v$. We write $c(t) = (x(t), y(t), z(t))$. Then $\gamma(t) := \varphi \circ c(t)$ is curve through $\varphi(p)$. Its tangent vector at $\varphi(p)$ is

$$\begin{aligned}
(\varphi \circ c)'(0) &= \frac{d}{dt}\left(x(t)/(1-z(t)), y(t)/(1-z(t))\right)|_{t=0} \\
&= \left(\frac{(1-z(t))x'(t) + x(t)z'(t)}{(1-z(t))^2}, \frac{(1-z(t))y'(t) + y(t)z'(t)}{(1-z(t))^2}\right) \\
&= (1/(1-z)^2)\left((1-z)F_1 + xF_3, (1-z)F_2 + yF_3\right) \\
&= G(u,v)/(1-z)^2,
\end{aligned}$$

a nonzero multiple of $G(u, v)$.

3 Vector Fields on Spheres

In this section we give Milnor's proof of the following result. The proof is analytic and uses the change of variable formula.

Theorem 7. *There are no continuously differentiable tangent vector field F with $\| F(p) \| = 1$ for $p \in S^{2k}$.*

We need some preliminary lemmas. Recall that $f\colon (X, d) \to (Y, d)$ is lipschitz if there exists a constant L such that $d(f(x), f(x')) \leq Ld(x, x')$ for all $x, x' \in X$. We say f is locally lipschitz if for every $x \in X$ there exists a neighbourhood U_x of x such that the restriction of f to U_x is lipschitz map from U_x to Y.

Lemma 8. *Let (X, d) be a compact metric space. Let $f\colon X \to Y$ be locally Lipschitz from X into another metric space Y. Then f is Lipschitz on X.*

Proof. By local lipschitz condition, for any $x \in X$ there exist $r_x > 0$ and $L_x > 0$ such that $d(f(x_1), f(x_2)) \leq L_x d(x_1, x_2)$ for all $x_1, x_2 \in B(x, r_x)$. By compactness, there exist finitely many points x_i such that $X = \cup B(x_i, r_i)$ where $r_i := r_{x_i}$. We let L_i stand for the lipschitz constant corresponding to x_i and B_i for $B(x_i, r_i)$. Consider the continuous function $h\colon X \times X \setminus \cup_i (B_i \times B_i)$ given by $h(x, y) := d(x, y)$. Then h is a continuous function on a compact set taking values in positive reals. Hence there exists $\varepsilon > 0$ such that $h(x, y) \geq \varepsilon$ for all (x, y) in the domain of the function h. If we take $M \geq \max\{L_i, \operatorname{diam} f(X)/\varepsilon\}$, then M is a lipschitz constant for f on X. □

Lemma 9. *Let $f\colon U \to \mathbb{R}^m$ be a C^1 map from an open set U in $\mathbb{R}^n$. Let K be a compact set in U. Then $f\colon K \to \mathbb{R}^m$ is Lipschitz.*

Proof. This follows easily from the mean value theorem of differential calculus and the last lemma. By the mean value theorem, if $B[x, r_x] \subset U$, we have

$$\| f(x_1) - f(x_2) \| \leq \sup_{0 \leq t \leq 1} \| Df(x_1 + t(x_2 - x_1)) \| \, \| x_1 - x_2 \|, \qquad x_1, x_2 \in B[x, r_x].$$

Since Df is continuous on U and hence on the compact set $B[x, r_x]$, f is lipschitz with the lipschitz constant $L_x = \sup\{\|Df(z)\| : z \in B[x, r_x]\}$. Thus f is locally lipschitz on K and hence lipschitz on K. □

Lemma 10. *Let U be an open connected bounded set in $\mathbb{R}^n$ so that $A = \overline{U}$ is compact and connected. Let F be a continuously differentiable vector field in an open set $V \supset A$. For $t \in \mathbb{R}$, let $F_t(x) := x + tF(x)$, for $x \in A$. If t is sufficiently small, then the mapping F_t is one-to-one and maps A onto $F_t(A)$ whose volume is a polynomial function of t.*

Proof. Since A is compact and F is C^1, F is lipschitz on A, say with lipschitz constant L: $\|F(x) - F(y)\| \leq L\|x - y\|$, for $x, y \in A$. If t is such that F_t is not one-to-one, then $F_t(x) = F_t(y)$ so that $x - y = t(F(x) - F(y)$ and hence $\|x - y\| \leq L|t|\,\|x - y\|$. So, if we choose $|t| < 1/L$, then F_t is one-to-one. The Jacobian matrix of F_t is of the form $I + t(\frac{\partial f_i}{\partial x_j})$, where I is the identity matrix. Hence the determinant of the Jacobian, DF_t is a polynomial function of t of the form $1 + t\alpha_1(x) + \cdots + t^n\alpha_n(x)$ where α_i are continuous functions of x. By change of variable formula, we see that the volume of the image of A under F_t is a polynomial function of t:

$$m(F_t(A)) = a_0 + a_1 t + \cdots + a_n t^n,$$

where a_i is the integral of α_i over A. □

Lemma 11. *Assume that $F \colon S^{n-1} \to \mathbb{R}^n$ be a C^1 tangent vector field on the sphere with $\|F(x)\| = 1$ for all x. If t is sufficiently small, then F_t maps the unit sphere in $\mathbb{R}^n$ onto the sphere of radius $\sqrt{1 + t^2}$.*

Proof 1. Assume that A is defined by the inequalities: $1/2 \leq \|x\| \leq 3/2$. We extend the vector field F on A by setting $F(x) := \|x\|\, F(x/\|x\|)$. We also define $F_t(x) = x + tF(x)$ on this set A. Choose t small enough so that $|t| < 1/3$ and $t < L^{-1}$. (L is the lipschitz constant of F.) For each $v_0 \in S^{n-1}$, the map $\varphi \colon x \mapsto v_0 - tF(x)$ maps the complete metric space A into itself. φ is a contraction. Hence by contraction mapping theorem there exists a unique fixed point. Consequently, the equation $F_t(x) = v_0$ has a unique solution. Thus for a given $v_0 \in S^{n-1}$, $F_t(x) = v_0$ has a unique solution in A. Multiplying both x and v_0 by $\sqrt{1 + t^2}$, the lemma follows. (Note that $F_t(rx) = rF_t(x)$.) □

Proof 2. We assume that $n \geq 2$. If t is sufficiently small, then $DF_t(x)$ is nonsingular on all of the compact set A. (This follows from the expression for the determinant of the Jacobian matrix $DF_t(x)$. See the proof of Lemma 10. Or, observe that the set of invertible matrices is an open set, I lies in the open set and for t near to 0, the Jacobian matrices $DF_t(x)$ all lie in a neighbourhood of I for all $x \in A$.) By inverse mapping theorem, F_t is an open map and hence maps the interior of A into an open subset and $F_t(S^{n-1})$ is a relatively open subset of the sphere of radius $\sqrt{1 + t^2}$. But $F_t(S^{n-1})$ is a compact and hence closed subset of the sphere of radius $\sqrt{1 + t^2}$. Since $n \geq 2$, the spheres in $\mathbb{R}^n$ are connected. Hence $F_t(S^{n-1})$ is the sphere of radius $\sqrt{1 + t^2}$. □

Proof of Thm. 7. Given a C^1 field F of unit tangent vectors on S^{n-1}, we consider any annular region $a \leq \|x\| \leq b$ and extend F to this region as in the last lemma. Then F_t maps the sphere of radius r *onto* the sphere of radius $r\sqrt{1 + t^2}$, for t near 0. Hence F_t maps the region

A onto the annular region between the spheres of radii $a\sqrt{1+t^2}$ and $b\sqrt{1+t^2}$. Obviously, the volume of the latter region is given by

$$\text{Volume of } F_t(A) = (\sqrt{1+t^2})^n \text{Volume of } A.$$

If n is odd the volume of $F_t(A)$ is not a polynomial function of t. This contradicts Lemma 10. □

Theorem 12. *An even dimensional sphere does not admit a continuous nowhere vanishing tangent vector field.*

Proof. Suppose F is such vector field. We produce an infinitely differentiable unit tangent vector field. This will contradict Thm. 7.

Let $m := \inf\{\| F(x) \| : x \in S^{n-1}\}$. By (Stone-)Weierstrass theorem there exists a polynomial map $P \colon S^{n-1} \to \mathbb{R}^n$ such that $\| P(x) - F(x) \| < m/2$ for all $x \in S^{n-1}$. We define a differentiable vector field G by setting $G(x) := P(x) - \langle P(x), x \rangle x$ for $x \in S$. Then G is tangent to S. Also, G is nowhere zero. Let, if possible, $G(x_0) = 0$. Then

$$P(x_0) = \langle P(x_0), x_0 \rangle x_0. \tag{1}$$

Since $\| P(x) - F(x) \| < m/2$, by Cauchy-Schwarz inequality

$$|\langle P(x) - F(x), x \rangle| < m/2. \tag{2}$$

But $\langle P(x) - F(x), x \rangle = \langle P(x), x \rangle$, since $\langle F(x), x \rangle = 0$. It the follows from Eq. 2 that

$$|\langle P(x), x \rangle| < m/2. \tag{3}$$

Using this inequality in Eq. eq:vfd1 we get

$$\| P(x_0) \| = |\langle P(x_0), x_0 \rangle| \, \| x_0 \| < m/2. \tag{4}$$

Since $\| F(x) \| \geq m$ and $\| F(x) - P(x) \| < m/2$, by triangle inequality we see that $\| P(x) \| \geq m/2$ for all x. This contradicts Eq. 4. Hence there is no x_0 with $G(x_0) = 0$. The vector field $G(x)/\| G(x) \|$ is then a smooth unit tangent field on S. □

Reference Milnor, J., Analytic Proofs of the “Hairy Ball Theorem” and the Brouwer Fixed Point Theorem, Amer. Math. Monthly, *vol.*85, 1978.