Volterra's Proof of Nonexistence of a Function

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Consider the function defined on $(0,1)$ by

$$f(x) = \begin{cases} 1/q, & \text{if } x = \frac{p}{q} \text{ with g.c.d}(p,q) = 1, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

It is easy to show that f is continuous at each irrational point and discontinuous at all rational points of $(0,1)$. One may now want know whether there exists a function on $(0,1)$ which is continuous at all rationals and discontinuous at all irrationals. The nonexistence of such a function is usually proved using Baire Category theorem. Volterra proved this using a very ingenious idea without using Baire's theorem. His proof uses the nested interval theorem, density of rationals and irrationals and the existence of a function which is discontinuous only at rationals! We shall indicate his proof below.

Let us assume that there exists $g\colon (0,1) \to \mathbb{R}$ which is continuous at the rational points and discontinuous at irrationals. Let f be the function defined above. Choose any irrational point $x_0 \in (0,1)$. By continuity of f at x_0, given $\varepsilon = 1/2$, there exists a $\delta > 0$ such that

$$|f(x) - f(x_0)| < 1/2, \text{ whenever } |x - x_0| < \delta.$$

Select points $a_1 < b_1 \in (x_0 - \delta, x_0 + \delta)$. Then for all $x, y \in [a_1, b_1]$, we have

$$|f(x) - f(y)| \leq |f(x) - f(x_0)| + |f(x_0) - f(y)| < 1/2 + 1/2 = 1.$$

We now select a rational point y_0 in the open interval (a_1, b_1). We repeat the above argument using now the continuity of g at y_0 to construct a closed interval $[c_1, d_1] \subset (a_1, b_1)$ such that

$$|g(x) - g(y)| < 1, \text{ for all } x, y \in [c_1, d_1].$$

Note that we have

$$|f(x) - f(y)| < 1 \text{ and } |g(x) - g(y)| < 1 \text{ for all } x, y \in [c_1, d_1].$$

We repeat this argument replacing the open interval $(0,1)$ by the open interval (c_1, d_1) to find a closed interval $[c_2, d_2] \subset (c_1, d_1)$ such that

$$|f(x) - f(y)| < 1/2 \text{ and } |g(x) - g(y)| < 1/2 \text{ for all } x, y \in [c_2, d_2].$$

By induction we construct a sequence of nested intervals $[c_k, d_k] \subset (c_{k-1}, d_{k-1})$ for $k \in \mathbb{N}$ with the property that

$$|f(x) - f(y)| < 2^{-k+1} \text{ and } |g(x) - g(y)| < 2^{-k+1} \text{ for all } x, y \in [c_k, d_k].$$

By the nested interval theorem, there exists a unique point $a \in [c_k, d_k]$ for $k \in \mathbb{N}$. Note that by the fact that $[c_{k+1}, d_{k+1}] \subset (c_k, d_k)$, the point $a \in (c_k, d_k)$ for all k. We now show that f and g are continuous at a. Given $\varepsilon > 0$, choose n such that $2^{-n} < \varepsilon$. Then, for $x \in [c_{n+1}, d_{n+1}]$, we have

$$|f(x) - f(a)| < \varepsilon \text{ and } |g(x) - g(a)| < \varepsilon. \tag{1}$$

Since $a \in (c_{n+1}, d_{n+1})$, we can find $\delta > 0$ such that $(a - \delta, a + \delta) \subset (c_{n+1}, d_{n+1})$. It is clear that if $|x - a| < \delta$, then (1) holds, that is, f and g are continuous at a. It follows then that a must be rational as well as irrational, which is absurd! Thus we conclude no such f exists.

Remark 1. I learnt this proof from Dr. V. Sholapurkar, S.P. College, Pune.