Every Real Sequence has a Monotone Subsequence

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Imagine the following scenario. Scientists have predicted the eruption of a dormant volcano in an island in the Indian ocean. There are lots of tourists pouring into the island to watch the spectacular show of Nature. The local business has constructed an infinite number of towers lined in front of the volcano, numbered serially. The n-th tower is of height x_n. See the picture.

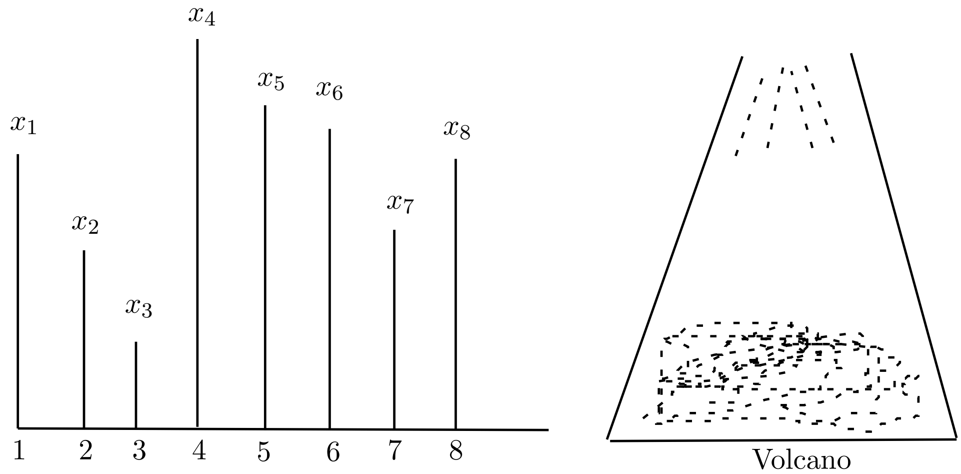

Figure 1: Observation Towers

As tourists, we would like to observe the event from the tallest tower and at the same time farthest from the volcano. If we observe from the n-th tower, we want the heights x_m of the towers which are in front of the n-th tower to satisfy $x_m < x_n$ for $m > n$. This suggests us to consider the set S of the serial numbers of the towers which are suitable for observation. Thus,

$$S := \{n \in \mathbb{N} : x_m < x_n \text{ for } m > n\}.$$

There are two cases: S is finite or infinite.

Case 1. S is finite. Let N be any natural number such that $k \leq N$ for all $k \in S$. Let $n_1 > N$. Then $n_1 \notin S$. Hence there exists $n_2 > n_1$ such that $x_{n_2} \geq x_{n_1}$. Since $n_2 > n_1 > N$, $n_2 \notin S$. Hence we can find an $n_3 > n_2$ such that $x_{n_3} \geq x_{n_2}$. This we way, we can find a monotone nondecreasing (increasing) subsequence, (x_{n_k}).

Case 2. S is infinite. Let n_1 be the least element of S. Let n_2 be the least element of $S \setminus \{n_1\}$ and so on. We thus have a listing of S:

$$n_1 < n_2 < n_3 < \cdots$$

Since n_{k-1} is an element of S and since $n_{k-1} < n_k$, we see that $x_{n_k} < x_{n_{k-1}}$, for all k. We now have a monotone decreasing sequence.

This completes the proof of the statement in the title. $\square$

Applications

Theorem 1 (Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let (x_n) be the given monotone bounded sequence. Let (x_{n_k}) be a monotone subsequence given by the result of the title. Then (x_{n_k}) is a bounded monotone sequence and hence is convergent. $\square$

Theorem 2. *Every Cauchy sequence of real numbers is convergent.*

Proof. Let (x_n) be a Cauchy sequence of real numbers. Then it is bounded. By the result of the title, there exists a monotone subsequence (x_{n_k}). This subsequence is bounded and monotone and hence is convergent. It is well-known that if a Cauchy sequence has a convergent subsequence then the original Cauchy sequence is convergent. The theorem follows. $\square$

The reader may consult the article "The Role of LUB axiom in Real Analysis" for a different treatment of these last two results.