Summary of Differential Geometry (January-April 2008)

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- **Review of Differential Calculus**

1. We started with the concepts learnt by you in Analysis 2 of Semester 2.
2. When you mentioned mean value theorem, we started discussing the version of the theorem.
3. The most important trick in calculus of several variables is to reduce to one variable calculus.
4. As an illustration, we discussed the value theorem for functions $f\colon \mathbb{R}^n \to \mathbb{R}$. We proved the following result.

   Theorem. *Let U be an open subset of $\mathbb{R}^n$. Let $f\colon U \to \mathbb{R}$ be differentiable. Assume that $x, y \in U$ be such that the line segment $[x, y] := \{(1-t)x + ty \in U : 0 \le t \le 1\} \subset U$. Then there exists $0 < s < 1$ such that*

   $$f(y) - f(x) = Df((1-s)x + sy)(y-x). \tag{1}$$

   To prove this, we reduced it to one variable calculus by considering the function $g(t) := f((1-t)x + ty)$ for $t \in [0,1]$ and appealed to the mean value theorem of one variable calculus.
5. As a second illustration, we proved the first derivative test for local extrema.

   Theorem. *Let $U \subset \mathbb{R}^n$ be open. Let $f\colon U \to \mathbb{R}$ be differentiable. Assume that $p \in U$ is a local maximum, that is, there exists $r > 0$ such that $B(p, r) \subset U$ and such that for all $x \in B(p, r)$ we have $f(x) \le f(p)$. Then $Df(p) = 0$.*

   Observe that any ball in $\mathbb{R}^n$ (or any normed linear space) is convex. To prove the result, it is enough to prove that $Df(p)(v) = 0$ for any nonzero $v \in \mathbb{R}^n$. To see this, we reduced the problem to one variable calculus by considering the function $g(t) := f(p + tv)$. For $\varepsilon := |t| < r/\|v\|$, $p + tv \in U$ if $|t| < \varepsilon$. We also reviewed the proof in the one variable case to understand where the hypothesis of local extrema (or that the point is interior) is needed.

6. We recalled the concept of directional derivative and proved the following theorem.

   Theorem. *Let $U \subset \mathbb{R}^m$ be open and $f\colon U \to \mathbb{R}^n$ be differentiable. Let $v \in \mathbb{R}^m$ be arbitrary and $a \in U$. Then the directional derivative $D_v f(a)$ exists and it is given by*

$$D_v f(a) = Df(a)(v).$$

7. We proved Riesz representation theorem for $\mathbb{R}^n$: Any linear map $f\colon \mathbb{R}^n \to \mathbb{R}$ is of the form $f(x) = \langle x, v\rangle$ for a unique v. In fact, we established that $v = (f(e_1), \ldots, f(e_n))$.
8. If $f\colon U \to \mathbb{R}$ is differentiable, then $Df(a)$ is represented by a unique vector u. As we saw in Item 7, this vector is given by $u = (Df(a)(e_1), \ldots, Df(a)(e_n))$. But in view of Item 6, $Df(a)(e_i)$ is the directional derivative $D_{e_i} f(a)$, which are traditionally called the partial derivatives. Thus we established

$$Df(a)(v) = \langle v, \operatorname{grad} f(a)\rangle\,, \text{ where } \operatorname{grad} f(a) = \left(\frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a)\right).$$

9. You were asked to review all the items above in *your mind*, discuss with classmates and not to rush to open a book.

Items 1–9 were done on 5 January 2008 (10 A.M. – 11:45 A.M).

10. We started discussing the formulation of the mean value theorem for functions $f\colon \mathbb{R}^m \to \mathbb{R}^n$. While the statement $f(y) - f(x) = Df(z)(y - x)$ for some z in the line segment $[x, y]$ makes sense mathematically, it is in general wrong. Consider $f\colon (-4\pi, 4\pi) \to \mathbb{R}^2$ given by $f(t) = (\cos t, \sin t)$. Take $y = 2\pi$ and $x = 0$.
11. A version which is true is this:

    Theorem. *Let $U \subset \mathbb{R}^m$ be open. Let $f\colon U \to \mathbb{R}^n$ be differentiable. Assume that $x, y \in U$ such that $[x, y] \subset U$. Fix $v \in \mathbb{R}^n$. Then there exists $0 < t < 1$ (which depends on v) such that*

$$\langle f(y) - f(x), v\rangle = \langle Df(x + t(y - x))(y - x), v\rangle\,. \tag{2}$$

    We proved this reducing to one variable calculus by considering the function

$$g(t) := \langle f(x + t(y - x), v\rangle\,.$$

    The function g was the composite of three functions:

$$t \mapsto x + t(y - x), z \mapsto f(z) \text{ and } w \mapsto \langle w, v\rangle\,.$$

12. While computuing the derivative of g as in the last item, we proved that the derivative of a linear map $A\colon \mathbb{R}^m \to \mathbb{R}^n$ at point $p \in \mathbb{R}^n$ is A itself, that is, $DA(p) = A$.

13. A more often used version of the mean value theorem is the mean value inequality:
Theorem. *Keep the notation of the last theorem. We have the following inequality:*

$$\| f(y) - f(x) \| \leq \left(\sup_{t \in [0,1]} \| Df(x + t(y - x)) \| \right) \| y - x \| . \tag{3}$$

Here $\| Df(z) \|$ stands for the operator norm of the linear map $Df(z)$.

14. Applications: Let $U \subset \mathbb{R}^m$ be open, $f \colon U \to \mathbb{R}^n$ be differentiable.
    (a) Let $U \subset \mathbb{R}^m$ be open, $f \colon U \to \mathbb{R}^n$ be differentiable. Assume that there exists M such that $\| Df(z) \| \leq M$ for all $z \in U$. Then f is uniformly continuous on U.
    (b) If $Df(z) = 0$ for all $z \in U$, then f is locally constant. If U is connected, then f is a constant.
    Two difficulties I noticed: 1. Most of you said that f is a constant for *any* open set U. 2. You had difficulty in seeing the differentiability of the function $f \colon U := \mathbb{R}^2 \setminus \{(x, y) : x = 0\} \to \mathbb{R}$ given by $f(x, y) = -1$ if $x < 0$ and $f(x, y) = 1$ if $x > 0$ for $(x, y) \in U$. I had to explain this starting with the example $f \colon \mathbb{R}^* \to \mathbb{R}$ given by $f(x) = -1$ if $x < 0$ and $f(x) = 1$ if $x > 0$.
15. We recalled the definition of C^k and C^∞ (or smooth) functions.
16. We reviewed the inverse function theorem.
17. I introduced the concept of diffeomorphism. Inverse function theorem says that if the derivative of f is nonsingular at a point a, then f maps diffeomorphically an open neighbourhood of a onto an open neighbourhood of $f(a)$.
18. You were asked (i) to review the material covered, (ii) discuss with your friends the statement and the proof of the mean value inequality and (iii) submit a written version of (ii) on a separate sheet of paper with your name and roll-number.

Items 10–18 were done on 8 January 2008 (9:40 A.M. – 11 A.M).

19. We had a detailed look at the concept of diffeomorphism. Compared it with the concept of isomorphisms in algebra and homeomorphisms in topology.
20. Examples: $\exp \colon \mathbb{R} \to (0, \infty)$ is a diffeomorphism. The map $f \colon \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{(0, 0)\}$ given by $f(u, v) := e^u(\cos v, \sin v)$ is with nonsingular derivative at each point. It is a 'local diffeomorphism' which is not a diffeomorphism. The map $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a homeomorphism which is smooth but not a diffeomorphism.

- **Differential Geometry of Curves**

1. A line joining $x, y \in V$, a real vector space was defined as the subset

$$\ell(x, y) := \{(1 - t)x + ty : t \in \mathbb{R}\} = \{x + t(y - x) : t \in \mathbb{R}\} = \{y + t(x - y) : t \in \mathbb{R}\}.$$

   We also saw how this defintion of a line joining x and y captures the formulas in 2- and 3-dimensional coordinate geometry.geom of curves

2. Another way of describing a line was to give a point on the line and the direction to which the line is parallel. If p is given and $v \in \mathbb{R}^n$ is a nonzero vector, then the line

$$\ell(p; v) := \{p + tv : t \in \mathbb{R}\} = \{p + t\lambda v : t \in \mathbb{R}\},$$

where λ is any nonzero real number, is the line through p 'going in the direction' of v.

3. A curve in $U \subset \mathbb{R}^n$ was defined as a (smooth) map $c\colon J \to \mathbb{R}^n$ where J is an interval in $\mathbb{R}$. In differential geometry, all curves are thus 'parameterized' curves. For example, in algebraic geometry, the subset $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a curve but it is not a curve according to our definition. We need to 'parameterize' it. See below.

   A physical interpretation is also useful. If $J = [a, b]$, we may think of $c(t)$ as the position vector of the particle as it moves from time $t = a$ to time $t = b$. Thus, the image of c is the trajectory of the particle.

4. Note that the curve is a map and not its image. For example, the curves $c_1(t) := (\cos t, \sin t)$, $t \in [0, 2\pi]$, $c_2(t) := (\cos 2t, \sin 2t)$ $t \in [0, 2\pi]$, $c_3(t) := (\cos 2t, \sin 2t)$, $t \in [0, \pi]$ and $c_4(t) = (\sin t, \cos t)$ for $t \in [0, 2\pi]$ have the same image but are different as curves.

5. If $c\colon J \to \mathbb{R}^n$ is a curve and if we write $c(t) := (x_1(t), \dots, x_n(t))$, then the vector $c'(t) := (x_1'(t), \dots, x_n'(t))$ is known as the tangent vector to c at t. It is also known as the (instantaneous) velocity vector.

6. We explained why $c'(t)$ is called the tangent vector. If c is a standard parameterization of a conic section, then the tangent line to the conic at the point $c(t)$ lying on the conic is the line which passes through $c(t)$ in the direction of $c'(t)$. We verified this in the case of a hyperbola. The verification for the other conic sections was left as home-work.

7. The magnitude, that is the norm, of the vector $c'(t)$ is known as the speed. Using this notion we motivated the definition of the length $\ell(c)$ of a curve $c\colon [a, b] \to \mathbb{R}^n$.

$$\ell(c) := \int_a^b \left\| c'(t) \right\| \, dt.$$

8. We defined a reparameterization of a curve.

9. We motivated and proved that the length of a curve does not change by repramaterization.

Items 1–9 were done on 9 January 2008 (9:40 A.M. – 11 A.M).

10. I pointed out the mistakes in the assignment on the mean value theorem.

11. We again discussed the differentiability of a linear map $T\colon \mathbb{R}^m \to \mathbb{R}^n$ and showed that $DT(a) = T$ for $a \in \mathbb{R}^n$.

12. We showed the differentiability of the map $f\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $f(x, y) := x \cdot y \equiv \langle x, y \rangle$. We proved that $Df(a, b)(h, k) = a \cdot k + b \cdot h$.

13. A regular curve is one for which $c'(t) \neq 0$ for t in the domain. Note that this dependes on the 'parametrization' and not on the trace, track or the image. For instance, the x-axis in $\mathbb{R}^2$ can be parametrized in two ways: $c_1(t) := (t, 0)$ and $c_3(t) = (t^3, 0)$ for $t \in \mathbb{R}$. While c_1 is a regular curve, c_3 is not. Can you think of a non-regular parametrization of the parabola $y = x^2$? (We did this in the class!)

14. The standard parametrizations of a line segment, a circle, and other conic sections are regular.

15. Let $c\colon [a,b] \to \mathbb{R}^n$ be a regular curve. Let $L := \ell(c)$. Then the function

$$h\colon [a,b] \to [0,L], \text{ given by } h(s) := \int_a^s \left\| c'(\tau) \right\| \, d\tau,$$

is an increasing function, is a homeomorphism (and diffeomorphism) of $[a,b]$ onto $[0,L]$. If g is its inverse, we then proved that the reparametrized curve $\sigma := c \circ g$ is of unit speed.

16. We looked at some trivial examples of repametrizing regular curves to unit speed curves. While we can theoretically say that it is possible to repametrize any regular curve to a unit speed curve, in practice this could be quite difficult. For example, look at the case of an ellipse.

Items 10–16 were done on 10 January 2008 (9:40 A.M. – 11 A.M).

17. Exercise: Find the unit speed parameterization of the logarithmic spiral $c(t) = e^t(\cos t, \sin t)$. Ans:

$$\sigma(s) := \left(\left(1 + \frac{s}{\sqrt{2}}\right) \cos\left(\log\left(1 + \frac{s}{\sqrt{2}}\right)\right), \left(1 + \frac{s}{\sqrt{2}}\right) \sin\left(\log\left(1 + \frac{s}{\sqrt{2}}\right)\right)\right).$$

18. The following are the references for our course along with those mentioned in the syllabus.
    1. A. Pressley: Elementary Differential Geometry
    2. A. Gray: Modern Differential Geometry of Curves and Surfaces
    3. do Carmo: Differential Geometry of Curves and Surfaces

19. Find the unit speed parameterization of the helix $\gamma(t) := (a\cos t, a \sin t, bt)$. Ans: $\sigma(s) := (a\cos(s/c), a\sin(s/c), b(s/c))$ where $c := \sqrt{a^2+b^2}$.

20. Exercise: Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a unit speed curve. Let $\sigma\colon [c,d] \to \mathbb{R}^n$ be a reparameterization of γ such that σ is also of unit speed. Show that there exists $t_0 \in \mathbb{R}$ such that $\sigma(t) = \gamma(\pm t + t_0)$. What is the geometric meaning of t_0? (Think in terms of trains and speed.)

21. Consider the set $\{(x,y) \in \mathbb{R}^2 : x^2 - y^3 = 0\}$. Give a parametrization of the curve.

22. Consider curve obtained as the intersection of the sphere $x^2 + y^2 + z^2 = 16$ and the cylinder $x^2 + (y-2)^2 = 4$. Parametrize the curve.

23. We motivated the concept of curvature. We arrived at the following requirements of a notion of curvature:
    - The curvature of a (parametrized) curve must be a real valued function of the parameter.
    - The curves should be parametrized in some standard way. If the speed is higher, it is likely that the curve may appear to be more curved. We decided to consider only the curves with unit-speed parametrization.
    - The curvature is local in the sense that whether the curve is curved at (parameter) point and if so how much depends only on the behaviour of the curve around the point.
    - The straight line must have have curvature zero.
    - The circles are uniformly curved so that the curvature must be a constant function of the parameter.

– Circle with higher radius must have less curvature than the one with smaller radius.

24. We saw that the norm of the vector $\|c''(t)\|$ is a likely candidate for curvature function for a unit speed curve c.

25. When we tried to formulate the analogous concept for surfaces in $\mathbb{R}^3$, we found that the easier notion would be to understand the curvature as the rate of change of a unit normal field.

26. So we settled on the definition: $\kappa(s) := \|N'(s)\|$, where $N(s)$ is a normal field on the unit-speed curve.

Items 21–26 were done on 16 January 2008 (10 A.M. – 11 A.M.).

27. After reviewing the last item, we grappled with two questions: (i) How to choose a unit normal at each point? (ii) How to choose it in a consistent way?

28. We let $N(s)$ denote the unit vector obtained from the unit tangent vector $\gamma'(s)$ by rotation by $\pi/2$ in the anti-clockwise direction. Letting $\gamma'(s) := (x'(s), y'(x))$, and using the rotation matrix, we arrived at $N(s) = (-y'(s), x'(s)$.

29. Since $\langle N(s), N(s)\rangle = 1$ for all s, we saw by differentiation that $N'(s)$ is orthogonal to $N(s)$. Since we are in $\mathbb{R}^2$, this means that

$$N'(s) = k(s)(x'(s), y'(s)), \tag{4}$$

We define the curvature $k(s)$ of a unit speed curve $\gamma(s) = (x(s), y(s))$ by the equation (4).

30. If $c(t) = (x(t), y(t))$ is a curve not-necessarily with unit speed, we arrived at the following formula for the curvature $k(t)$:

$$k(t) = \frac{y''(t)x'(t) - x''(t)y'(t)}{(x'(t)^2 + y'(t)^2)^{3/2}}. \tag{5}$$

31. As an exercise, you were asked to draw pictures of some plane curves such as conics, guess the behaviour of the curvature function and check out your intuition by computing the curvature explicitly.

32. We discussed the geometric significance of the sign of the curvature. You are asked to test out the case of the graph of a function when considered as a curve $t \mapsto (t, f(t))$ for $t \in [a, b]$.

Items 27–32 were done on 17 January 2008 (9:40 A.M. – 11 A.M.).

33. We continued with the last item. To gain intution, we looked at the circle, sine curve and a hyperbola. The sign of the curvature is same as that of $f''(t)$. Since f'' is continuous, if $f''(t_0) > 0$, then it remains so in a neighbourhood of t_0. This means that the curve is 'convex' on this neighbourhood. Similar consideration applies when $f''(t) < 0$.

34. If a smooth (or continuous) function $k\colon [0, L] \to \mathbb{R}$ is given, does there exist a smooth (or C^2) curve $\gamma\colon [0, L] \to \mathbb{R}^2$ with unit speed whose curvature is k?

35. We wanted to tackle the special case when $k = 0$. One of you suggested: solve for x and y in the equation $x''y' = x'y''$. We applied a little geometry to the problem. Zero curvature means that the unit normal field is a constant, which in turn means that the unit tangent field is a constant, say, (u_0, v_0). That is, $\gamma'(t) = (u_0, v_0)$. To get back γ, we integrate the vector valued function $t \mapsto (u_0, v_0)$.

36. We saw how to define the integral of a continuous function $f\colon [a, b] \to H$, where H is a Hilbert space. The value $\int_a^b f(t)\, dt$ is the unique vector v such that the following holds:

$$\left\langle \int_a^b f(t)\, dt, w \right\rangle = \int_a^b \langle f(t), w\rangle \, dt \text{ for all } w \in H. \tag{6}$$

In the case when $H = \mathbb{R}^n$, the integral is seen to be as follows:

$$\int_a^b f(t)\, dt = \left(\int_a^b f_1(t)\, dt, \dots, \int_a^b f_n(t)\, dt \right) \text{ where } f = (f_1, \dots, f_n).$$

37. We derived the following important inequality: Let $f\colon [a, b] \to H$ be continuous. Then

$$\left\| \int_a^b f(t)\, dt \right\| \leq \int_a^b \| f(t) \| \, dt. \tag{7}$$

38. We used the last inequality (7) to prove the following: Let $p, q \in \mathbb{R}^n$. Let $c(t) := p + t(q - p)$ be the straight line curve joining p and q. Then $\ell(c) = \| q - p \|$. If $\gamma\colon [a, b] \to \mathbb{R}^n$ is any (piecewise smooth) curve joining p and q, that is, if $\gamma(a) = p$ and $\gamma(b) = q$, then $\ell(\gamma) \geq \| q - p \|$. *Hint:* $q - p = \gamma(b) - \gamma(a) = \int_a^b \gamma'(t)\, dt$.

39. We returned to the problem in Item 34. Let $\psi(s) := \int_0^s k(\tau)\, d\tau$. Let $\gamma\colon [0, L] \to \mathbb{R}^2$ be defined as follows:

$$\gamma(s) := \left(\int_0^s \cos(\psi(t))\, dt, \int_0^s \sin(\psi(t))\, dt \right).$$

(Do you recall how we arrived at this?)

40. The other problem which we raised was whether there is any uniqueness assertion to the question of Item 34. A closer look at $k = 0$ and $k = 1$ prompted one of you to say that the curves are unique up to a rotation and/or a translation or up to a rigid motion. We decided to take this up later in a more general setting.

    On the way, we wondered why only rotation figured and why not any reflection in the assertion on uniqueness.

41. We started the study the geometry of unit speed curves in $\mathbb{R}^3$. Since $\gamma''(s) \cdot \gamma'(s) = 0$, we decided to call the unit vector $\mathbf{n}(s)$ in the direction of $\gamma''(s)$ as the unit normal *provided* $\gamma''(s) \neq 0$. If we let $\mathbf{t}(s) = \gamma'(s)$, then we wanted to enlarge $\mathbf{t}, \mathbf{n}$ so that we get an O.N. basis of $\mathbb{R}^3$. In fact, one of you suggested that we consider $\mathbf{t}, \mathbf{n}, \mathbf{t} \times \mathbf{n}$. The geometry of space curves is the study of the rate of change of this O.N. basis.

Items 33–41 were done on 19 January 2008 (10:15 A.M. – 11:50 A.M.).

42. We continued the study of space-curves. We defined the curvature (necessarily positive) by the equation: $\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s)$. There were two choices of a unit vector which is perpendicular to both $\mathbf{t}$ and $\mathbf{n}$. Which one to choose? We recalled how we chose $\mathbf{n}$ in the case of a plane curve.

43. Let V be an n-dimensional real vector space. We say that two ordered bases $v := \{v_1, \ldots, v_n\}$ and $w := \{w_1, \ldots, w_n\}$ of V define the *same orientation* if the unique linear endomorphism T of V such that $Tv_i = w_i$ has positive determinant, This defines an equivalence relation on the set of ordered bases of V. There are two equivalence classes. We choose one and say that it defines an *orientaion* of V. Any basis which lies in the orientation is said to be *positively oriented*. Otherwise, it is said to be *negatively oriented*.

44. When $V = \mathbb{R}^n$, the standard orientation of $\mathbb{R}^n$ is the equivalence class to which the standard basis belongs.

    Using this convention, we looked at some exmples of bases in $\mathbb{R}^2$ ad $\mathbb{R}^3$ which are positively orieted and negatively oriented. In general, if v and w are of oppposite orientation, then v and $w := \{w_2, w_1, w_3, \ldots, w_n\}$ are positively oriented. So are v and $\{-w_1, w_2, \ldots, w_n\}$.

45. On the way, we discovered how to write down all the orthogonal 2×2 matrices, their classification into rotations and reflections. Rotations are orientation preserving while reflectons are not. (When do you say a linear map $T \colon \mathbb{R}^n \to \mathbb{R}^n$ is orientation preserving (or reversing)?)

46. Returning to space curves, we choose the unique (unit) vector $\mathbf{b}(s)$ such that $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ is a positively oriented O.N. basis of $\mathbb{R}^3$. (Why does it exist?) The vector $\mathbf{b}(s)$ is called the *binormal* to the curve at s. The triad $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ is called the Frenet frame to γ at s.

47. What do we mean the study of differential geometry of curves in $\mathbb{R}^2$ and $\mathbb{R}^3$? We (re)interpreted it as the study of the (rate of) change in the Frenet frame ($\{\mathbf{t}(s), \mathbf{n}(s)\}$ in the case of plane curves and $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ in the case of space curves. We know $\mathbf{t}'(s) = \gamma''(s) = \kappa(s)\mathbf{n}(s)$. We need to compute $\mathbf{n}'$ and $\mathbf{b}'$ in terms of the Frenet frame.

48. We recalled that in an i.p.s. V, $x = \sum_{i=1}^n \langle x, v_i \rangle v_i$ where $\{v_1, \ldots, v_n\}$ is an O.N. basis of V.

49. To compute $\mathbf{n}'(s)$ and $\mathbf{b}'(s)$ and in particular to find the 'coordinates', we differentiated identities such as $\mathbf{n} \cdot \mathbf{n} = 1$, $\mathbf{n} \cdot \mathbf{b} = 0$, $\mathbf{n} \cdot \mathbf{t} = 0$ etc. We found that $\mathbf{b}'$ is a multiple of $\mathbf{n}$.

Items 42–49 were done on 22 January 2008 (9:45 A.M. – 11 A.M.).

50. We continued with the last item and derived the (Serret-)Frenet Formulas:

$$\begin{aligned} \mathbf{t}'(s) &= && \kappa(s)\mathbf{n}(s) && \\ \mathbf{n}'(s) &= -\kappa(s)\mathbf{t}(s) && && +\tau(s)\mathbf{b}(s) \\ \mathbf{b}'(s) &= && -\tau(s)\mathbf{n}(s) && \end{aligned} \tag{8}$$

51. If the torsion of a unit speed-curve in $\mathbb{R}^3$ is zero, then it is a plane curve.
52. Since we defined $\mathbf{b}$ geometrically (and algebraically), we need to ensure that $s \mapsto \mathbf{b}()$ is smooth.
53. We defined the cross-product $v \times w$ of 2 vectors in $\mathbb{R}^3$ by the equation

$$\langle u, v \times w \rangle = \det(v, w, u) \text{ for all } u \in \mathbb{R}^3.$$

54. Find formulas for the curvature and torsion of a regular space curve which is not-necessarily parametrized by arc-length.
55. If a unit speed curve lies on a sphere of radius R, then the curvature at any point should be at least $1/R$.
56. Let γ be a unit speed curve. Assume that all the tangent lines to the curve pass through a (fixed) point.Then γ is a straight line. (How many such points are there through which such a curve passes?)

Items 50–56 were done on 23 January 2008 (9:45 A.M. – 11 A.M.).

57. Geometrically we expect that the curvature and the torsion of a helix to be constant. This was verified by computation.
58. If γ is a unit speed curve in the plane with constant curvature k, then γ is a circle. We proved this by solving the system

$$\begin{pmatrix} \mathbf{t}' \\ \mathbf{n}' \end{pmatrix} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \end{pmatrix}.$$

59. If γ is a unit speed curve with constant curvature, the extra condition we need to impose on γ to ensure that it is a circle was found to be $\tau = 0$. This follows easily from Item 51 and Item 58.

    This was also seen in a more geometric way. Hint: Consider $\gamma(s) + (1/k)\mathbf{n}(s)$. (How was this motivated?)
60. We formulated a results on the existence and uniqueness result for space curves.
61. If γ_1 and γ_2 are two unit speed curves (say, with $[0, L]$ as the common domain) in $\mathbb{R}^3$ which have the same curvature and the torsion we wanted to prove that one is got by a rigid motion from the other. What was the translation part? It is $\pm(\gamma_2(0) - \gamma_1(0))$. What was the rotation part? It is the orthogonal (why?) linear transformation, say A, which took the Frenet frame $(\mathbf{t}_1(0), \mathbf{n}_1(0), \mathbf{b}_1(0))$ to $(\mathbf{t}_2(0), \mathbf{n}_2(0), \mathbf{b}_2(0))$. What is the determinant of A? We found that $\det A = 1$. Why is this a rotation and what we mean by a rotation in $\mathbb{R}^3$? We wanted to show that A has 1 as an eigen value.

Items 57–61 were done on 24 January 2008 (10 A.M. – 11 A.M.).

62. We continued with the last item. Since the characteristic polynomial of A is a polynomial of degree 3 with real coefficients, it has a real root by the intermediate value theorem. Since the characteristic values of an orthogonal matrix are of modulus one, this real root must be either 1 or -1. If it is -1, then the other two roots cannot be non-real. (Why?) If they are real, they must be 1 and -1. (Why?) We explained how this leads us to conclude that such an A must be a rotation of a plane.

    In classical mechanics, this known as Euler's theorem: Any such A has an axis of rotation.

63. We may now assume that $\gamma_1(0) = \gamma_2(0)$, $\mathbf{t}_1(0) = A\mathbf{t}_2(0)$, $\mathbf{n}_1(0) = A\mathbf{n}_2(0)$ and $\mathbf{b}_1(0) = A\mathbf{b}_2(0)$ where A is a rotation. We proved that

$$\mathbf{t}_1(s) = A\mathbf{t}_2(s), \mathbf{n}_1(s) = A\mathbf{n}_2(s) \text{ and } \mathbf{b}_1(s) = A\mathbf{b}_2(s) \text{ for } s \in [0, L].$$

    Hint: Show that $\|\mathbf{t}_1(s) - A\mathbf{t}_2(s)\|^2 + \|\mathbf{n}_1(s) - A\mathbf{n}_2(s)\|^2 + \|\mathbf{b}_1(s) - A\mathbf{b}_2(s)\|^2$ is a constant using the Frenet formulas and the orthogonality of A.

    This proves the 'uniqueness' part: if two space curves $\gamma_i\colon [0, L] \to \mathbb{R}^3$ are such that $\kappa_1(s) = \kappa_2(s)$ and $\tau_1(s) = \tau_2(s)$ for all $s \in [0, L]$, then there exists a rigid motion taking one to the other.

64. Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be linear.Let $f\colon [a, b] \to \mathbb{R}^n$ be continuous. We proved

$$\int_a^b (A \circ f)(t)\, dt = A \circ \left(\int_a^b f(t)\, dt \right).$$

    We used the defining equation (6) to prove this.

65. Existence part is proved by invoking the (global) existence theorem for a linear system of ODE. Look at the Frenet formulas (8). (I was not sure whether you were aware of 'global' result!)

    Items 62–65 were done on 29 January 2008 (10 A.M. – 11 A.M.).

- **Tangent Spaces**

1. If $S \subset \mathbb{R}^n$ and $p \in S$, we denote by T_pS, the tangent space at p to S and define it to be the collection of all tangent vectors $c'(0)$ where $c\colon (-\varepsilon, \varepsilon) \to S$ is a smooth curve with $c(0) = p$:

$$T_pS := \{v \in \mathbb{R}^n : \exists c\colon (-\varepsilon, \varepsilon) \to S \text{ with } c(0) = p \text{ and } v = c'(0)\}$$

2. As $c(t) = p$ for all t has 0 as the tangent vector, $T_pS \neq \emptyset$. Also, if $v \in T_p$, then $\lambda v \in T_pS$ for any $\lambda \in \mathbb{R}$. Thus T_pS is a subset of $\mathbb{R}^n$ which is nonempty and closed under scalar multiplication.

3. Does T_pS contain nonero vectors? Not necessarily. We have $T_x\mathbb{Q} = \{0\}$ for any $x \in \mathbb{Q}$ and $T_xC = \{0\}$ for any x in the Cantor set C.

4. If $v_1, v_2 \in T_pS$, can we conclude $v_1 + v_2 \in T_pS$? No, we cannot. If $S = \{xy = 0\}$, the union of the axes in $\mathbb{R}^2$, then $e_1, e_2 \in T_{(0,0)}S$, but $e_1 + e_2 \notin T_{(0,0)}S$.

5. Whether T_pS contains nonzero vectors or whether it is closed under (vector) addition (so that it becomes a vector space) depends on some geometric properties of S. We look at some special cases below which are very important for differential geometry and for which the tangent spaces are vector spaces.
6. If $S = U$ is an open subset of $\mathbb{R}^n$, then $T_p(S) = \mathbb{R}^n$.
7. If S is a the vector subspace $H := \{x \in \mathbb{R}^n : x \cdot a = 0\}$ for some nonzero $a \in \mathbb{R}^n$, then $T_pS = H$. If $v \in T_pS$, then $v \in H$ with a correpsonding curve c, then $c(t) \cdot a = 0$ for all t. Differentiating this equation we get $v \in H$.

   More generally, if $W := w + H$ is a plane, then $T_pW = H$.
8. Exercise: Let $W \leq \mathbb{R}^n$ be a vector subspace. Identify T_pW. The same question if S is a coset of W in $\mathbb{R}^n$.
9. Consider $S = S^{n-1} := \{x \in \mathbb{R}^n : x \cdot x = 1\}$. Then $v \in T_pS$ iff $v \perp p$. Enough to show that $v \perp p$ is of unit norm, then $v \in T_pS$. Consider the curve which is the intersection of the sphere S with the two dimensional subspace span $\{p, v\}$. It is parametrized as $t \mapsto \cos tp + \sin tv$.

   Items 1–9 were done on 30 January 2008 (9:45 A.M. – 11 A.M.).
10. We went through the argument of the last item and observed that we can prove a partial generalization. Let $U \subset \mathbb{R}^n$ be open. Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be smooth. Assume that $q \in \mathrm{Im}\,(f)$. Let $p \in S := f^{-1}(q)$. Then $T_pS \subset \ker Df(p)$.

    We did this in two steps. First when $m = 1$ and then the general case. When $m = 1$, we wrote the result in the form $T_pS \subset (\mathrm{grad}\, f(p))^{\perp}$.

    Under an extra assumption on q, we shall later prove that $T_pS - \ker Df(p)$.
11. Let $U \subset \mathbb{R}^n$be open and $f \colon U \to \mathbb{R}$ be smooth. Let S be the "surface" in $\mathbb{R}^{n+1}$ defined as the graph of f: $S = \{(x, f(x)) : x \in U\}$. If γ is a curve in S, and if we write $\gamma(t) = (x(t), f(x(t))$, then $c(t) := x(t)$ is a curve in U. This sets up a 1-1 correspondence between curves in S and those in U.
12. Let $n = 2$. Let

    $$\gamma(t)) = (x(t), y(t), z(t)) = (x(t), y(t), f(x(t), y(t))), \text{ and hence } c(t) = (x(t), y(t)),$$

    then

    $$\begin{aligned} \gamma'(t) &= \left(x'(t), y'(t), \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t)\right) \\ &= x'(t)\left(1, 0, \frac{\partial f}{\partial x}\right) + y'(t)\left(0, 1, \frac{\partial f}{\partial y}\right) \\ &= x'(t)\partial_x + y'(t)\partial_y, \text{ say.} \end{aligned}$$

    In particular,$\gamma'(0) = x'(0)\partial_x + y'(0)\partial_y$. Thus any tangent vector to S is a linear combination of ∂_x and ∂_y. Note that the coefficients are the components of the tangent vector $c'(0)$!

    ∂_x and ∂_y are the tangent vectors to S that correspond to the curves $(x, b, f(x, b))$ and $(a, y, f(a, y))$.
13. Thus, if we wish to show that $w = u\partial_x + v\partial_y \in T_S$, we need only consider a curve c in U passing through (a, b) whose tangent vector is (u, v). This is easy. For example, consider $c(t) = (a, b) + t(u, v)$ so that $\gamma(t) = (c(t), f(c(t))$. We have $\gamma'(0) = w$. Hence T_pS is the two dimensional vector space spanned by ∂_x and ∂_y.

14. ($n = 2$ continued.) Also, note that

$$\partial_x \times \partial_y = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

the 'normal' to the tangent plane.

15. In the general case of the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$, if we let

$$\partial_i = \left(0, \ldots, 1, 0, \ldots, 0, \frac{\partial f}{\partial_{x_i}}\right) \text{ where 1 is at the } i\text{-th place,}$$

then $\gamma'(0) = \sum_{i=1}^n x_i'(0)\partial_i$. Proceeding as in the 2-dimensional case, if $w = \sum_i u_i \partial_i$ s to be shown as a tangent vector in T_pS, we may consider $c(t) = p + tu$ and $\gamma(t) = (c(t), f(c(t)))$.

Items 10–15 were done on 31 January 2008 (9:45 A.M. – 11 A.M.).

16. Exercise: Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be smooth. Let S be the graph of f. Identify T_pS for any $p \in S$ and hence conclude that T_pS is n-dimensional real vector space.

17. Notation as in Item 10. We wanted to impose conditions so that $T_pS = \ker Df(p)$. We arrived at the condition that $Df(p)$ must be of maximal rank. We say that 0 is a regular value of f if for each $p \in f^{-1}(0)$, we have $Df(p)$ is of maximal rank.

18. The idea of the proof of the last item is to recognize S locally as a graph of a smooth function on an n-dimensional domain. This is essentially Implicit Function Theorem.

19. We recalled the implicit function theorem in the following form:
    Let $\Omega \subset \mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$ be open. Let $f : \Omega \to \mathbb{R}^k$ be C^1. Assume that for some $(a, b) \in \Omega$ where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^k$ we have
    1. $f(a, b) = 0$.
    2. $D_2 f(a, b)$ is nonsingular.

    Then there exists a neighbourhood Ω' of (a, b) in $\mathbb{R}^n \times \mathbb{R}^k$, an open set $U \subset \mathbb{R}^n$ containing a and a C^1-map g on U such that
    - i. $D_2 f(x, y)$ is nonsingular for all $(x, y) \in \Omega'$,
    - ii. $\{(x, y) \in \Omega' : f(x, y) = 0\} = \{(x, g(x)) : x \in U\}$.

20. We started with the proof.

Items 16–20 were done on 6 February 2008 (10:00 A.M. – 11 A.M.).

21. We completed the proof of the implicit function theorem.

22. Implicit function theorem says that a level set $f^{-1}(0)$ is locally a graph (with appropriate assumptions on the value 0).

23. We completed the proof of the following statement: Let $f : \mathbb{R}^{n+k} \to \mathbb{R}^k$ be smooth. Let $0 \in \mathrm{Im}\,(f)$ be regular. Let $S := f^{-1}(0)$. Then $T_pS = \ker Df(p)$ for any $p \in S$.

24. The tangent plane to S at p is defined as the coset $p + T_pS$, that is, the translate (by p) of the tangent space.

25. Exercise: Let S be the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$. Identify the tangent plane to S at $p = (a, b, f(a, b))$. Do you recognize the significance of your result?

Items 21–25 were done on 7 February 2008 (9:45 A.M. – 11 A.M.).

26. We compute the tangent spaces to some concrete sets using the last result.
27. Let $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ be given by $f(x) := x \cdot x - 1$. Then $f^{-1}(0) = S^n$ is the unit sphere. Since $Df(p)(v) = v \cdot \operatorname{grad} f(p)$, we see that $T_pS^n = (\operatorname{grad} f(p))^{\perp}$, as we have already seen in Item 9.

    Exercise: Identify T_pS where $p \in S := \{x \in \mathbb{R}^{n+1} : x \cdot x = R^2\}$.
28. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ and let $p = (x_0, y_0, z_0) \in S$. We have

    $$T_pS = \{v \in \mathbb{R}^3 : v \cdot (x_0, y_0, 0) = 0\}.$$

    Note that there exist 2 special curves through p whose tangent vectors at p form a basis of T_pS.
29. We now look at a more abstract example. Let $S = O(n, \mathbb{R})$, the group of all $n \times n$ orthogonal matrices. Let $S(n, \mathbb{R})$ be the set of $n \times n$ symmetric matrices. We show that T_IS is the set of all skew-symmetric matrices of size $n \times n$. Consider $F\colon M(n, \mathbb{R}) \to S(n, \mathbb{R})$ given by $F(X) = XX^T - I$. Then $S = F^{-1}(0)$. The only issue here is to compute $DF(I)$.
30. Note that if X is an $n \times n$ skew-symmetric matrix, then $c(t) := e^{tX}$ is a curve in $O(n, \mathbb{R})$ passing through I with $c'(0) = X$.
31. Can you identify the tangent space $T_AO(n, \mathbb{R})$ for $A \in O(n, \mathbb{R})$?

Items 26–31 were done on 8 February 2008 (10 A.M. – 11 A.M.).

32. Let $U(n) := \{A \in M(n, \mathbb{C}) : A \text{ is unitary}\}$. Compute $T_IU(n)$.
33. The notion of tangent occurs even in differential calculus in a subtle way. To start with, if we wish to compute the directional derivative $D_vf(p)$ of a differentiable function, we employ the difference quotient: $\frac{f(p+tv)-f(p)}{t}$. The numerator is the composition of f with the straight line curve $\ell(t) := p + tv$. This curve has the property that $\ell(0) = p$ and $\ell'(0) = v$. Also, we note that the directional derivative is $\frac{d}{dt} f \circ \ell(t)\,|_{t=0}$. If γ is any curve with the same properties $\gamma(0) = p$ and $g'(0) = v$, then again we get (by an application of the chain rule)

    $$D_vf(p) = \frac{d}{dt} f \circ \gamma(t)\,|_{t=0}\ .$$

34. The importance of tangent space in differential geometry is as follows. Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $p \in U$. To know $Df(p)$ it is enough to know $Df(p)(v)$. The latter is the directional derivative $D_vf(p)$ and to compute it, we can use any curve γ such that $\gamma(0) = p$ and $\gamma'(0) = v$. (This follows by a trivial application of the chain rule.) Hence v, which is fed to $Df(p)$, can be thought of a tangent vector to U at p! That is, the domain of $Df(p)$ is T_pU.

    Also, the image $Df(p)(v)$ is the tangent vector to the curve $f \circ \gamma$ at $f(p)$. Thus the codomain of $Df(p)$ is the tangent space $T_{f(p)}V$ if $f\colon U \to V$ is differentiable.

    Thus the derivative may be thought of as a linear map from the tangent space T_pU to the tangent space $T_{f(p)}V$.

    This way of looking at the derivative of a map leads us to the definition of the derivative of a map between two surfaces.

35. We now use our knowledge of T_pS of a level set to bring out the underlying geometry of the method of Lagrange multipliers. First of all look at the following examples in a geometric way.
   (a) Find the extrema of the function $g(x,y) = x$ subject to the constraint $f(x,y) := x^2 + y^2 - 1 = 0$.
   (b) Find the extrema of the function $g(x,y) = x^2 + y^2$ subject to the constraint $f(x,y) := ax + by + c = 0$
   (c) Find the extrema of the function $g(x,y) = x^2 + y^2$ subject to the constraint $f(x,y) := xy - c = 0$.
   (d) Find the extrema of the function $g(x,y) = xy$ subject to the constraint $f(x,y) := x + y - c = 0$.
   (e) Find the extrema of the function $g(x,y) = x^2 + y^2$ subject to the constraint $f(x,y) := (x/a)^2 + (y/b)^2 - 1 = 0$.
   (f) Find the extrema of the function $g(x,y) = \|p - x\|^2$ subject to the constraint $f(x) := x \cdot a - d = 0$, where $a \in \mathbb{R}^n$ is a unit vector, $p \in \mathbb{R}^n$ fixed, and $d \in \mathbb{R}$.

36. In each of the examples of the last item, the (constrained) extrema of the function g occurs at point at which both the level sets f and g meet tangentially. Reformulating this in terms of the normals, this means that at a point of extrema, we have $\operatorname{grad} g$ is a scalar multiple of the gradient of f. This is the essence of the method of Lagrange multipliers.

Items 32–36 were done on 15 February 2008 (10 A.M. – 11 A.M.).

- **Surfaces**

1. We motivated how to assign dimension to some of the easy subsets of $\mathbb{R}^n$.
2. A (nonempty) subset $S \subset \mathbb{R}^n$ is called a *surface* in $\mathbb{R}^n$ if for each $p \in S$, there exists an open neighbourhood U of p in $\mathbb{R}^n$, an open set $V \subset \mathbb{R}^2$ and a homeomorphism $f \colon V \to U \cap S$.
3. Examples of surfaces:
   (a) A nonempty open set $V \subset \mathbb{R}^2$ is a surface in $\mathbb{R}^2$.
   (b) More generally, if $W \subset \mathbb{R}^n$ is a two dimensional vector subspace and V is an open set in W (with the subspace topology on W), then V is a surface in $\mathbb{R}^n$.
   (c) A typical example is the sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. If we let $U_i^{\pm} := \{(x_1, x_2, x_3) \in S : \pm x_i > 0\}$, $i = 1, 2, 3$, then each of these six sets is open in S and is homeomorphic to the open unit disk in $\mathbb{R}^2$. Also, any $p \in S$ will lie in at least one of these open sets.
   (d) The cylinder $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ is a surface. We did this in two ways. On the way, we learnt a bit of real analysis centering around continuous increasing function, their inverese, the domain of inverse functions such as $\cos^{-1}$ and $\sin^{-1}$.
   (e) Let $U \subset \mathbb{R}^2$ be open. Let $f \colon U \to \mathbb{R}^k$ be a (continuous, or better still smooth) map. Then the graph $G(f) := \{x, f(x)) \in \mathbb{R}^2 \times \mathbb{R}^k = \mathbb{R}^{k+2}\}$ is a surface in $\mathbb{R}^{k+2}$.
   On the way, we also saw that if $f \colon X \to Y$ is a continuous function between two topological spaces, then X and the graph of f (with the subspace topology inherited as a subset of $x \times Y$) are homeomorphic.
   (f) Let $f \colon \mathbb{R}^3 \to \mathbb{R}$ be a smooth map with 0 as a regular value. Then the level set $S := f^{-1}(0)$ is 'locally' a graph of a function defined on an open set in $\mathbb{R}^2$. Hence S is a surface in $\mathbb{R}^3$. This was seen earlier, as an application of the implict function theorem, when we discussed the tangent spaces.
   (g) Let $S \subset \mathbb{R}^n$ be a surface and $S' \subset \mathbb{R}^m$ be homomorphic to S. Then S' is a surface.
4. We can mimic the definition of a surface to define k dimensional objects in $\mathbb{R}^n$. They are the ones where each point has a neighbourhood which is homeomorphic to an open set in $\mathbb{R}^k$. They are called k dimensional (topological) submanifolds in $\mathbb{R}^n$. Thus a topological two dimensional manifold S is nothing other than a surface.

Items 1–4 were done on 16 February 2008 (10 A.M. – 11:45 A.M.).

5. A nonempty subset $S \subset \mathbb{R}^n$ is said to be a smooth k-dimensional submanifold of $\mathbb{R}^n$ if for each $p \in S$ there exists a triple (U_p, φ_p, U_p) where U_p is an open set containing p, $V_p \subset \mathbb{R}^k$ is open and $\varphi_p \colon V_p \to U_p$ is a homeomorphism with the following property: The map $\varphi_p \colon V_p \to \mathbb{R}^n$ is smooth o rank k.
6. A topological space M is said to be a smooth k-dimensional manifold if for each p there exists a triple (U_p, φ_p, V_p) where $U_p \ni p$ is open, $V_p \subset \mathbb{R}^k$ is open and $\varphi_p \colon U_p \to V_p$ is a homeomorphism. The triples $\{(U_p, \varphi_p, V_p) : p \in M\}$ satisfy the following compatibility condition (on the transition functions): whenever $U_p \cap U_q \neq$

$\emptyset$, the map $\varphi_q \circ \varphi_p^{-1} \colon \varphi_p(U_p \cap U_q) \to \varphi_q(U_p \cap U_q)$ is a diffeomorphism of the open sets.

7. Let $W \subset \mathbb{R}^n$ be a k-dimensional vector subspace. Then W is a k-dimensional submanifold of $\mathbb{R}^n$. It is also a smooth k-manifold.
8. Any topological k-manifold with an atlas of a single chart is obviously a smooth k-manifold.
9. In particular, the graph of a smooth function $f \colon U \subset \mathbb{R}^k \to \mathbb{R}^n$ is a smooth k-manifold.
10. We saw that the sphere $S^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ admits an atlas with two charts (stereographic projections): $U_\pm := S^n \setminus \{\pm e_{n+1}\}$ with $\varphi_\pm$ as the stereographic projections.
11. Let $U_i^\pm := \{x \in S^n : x_i > 0 \text{ or } x_i < 0\}$. Let B denote the open ball $B(0,1)$ in $\mathbb{R}^n$. Let $\varphi_i^\pm \colon U_I^\pm \to B$ be defined by

$$\varphi(x) := (x_1, \dots, x_{i-1}, \hat{x}_i, x_{i+1}, \dots, x_{n+1})$$

Then the collection $\{(U_i^\pm, \varphi_i^\pm) : 1 \le i \le n+1\}$ is an atlas with $2(n+1)$ charts.
12. There exists no atlas for S^n with a single chart.
13. There exists an atlas consisting of uncountably many charts. The domains of these charts are $U_p := \{x \in S^n : x \cdot p > 0\}$.
14. There exists an atlas for the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$ consisting of a single chart. (Exercise.)
15. The sphere in $\mathbb{R}^{n+1}$ with the atlas of stereographic projections is a smooth n-manifold. We explicitly wrote down the transition function: $\varphi_- \circ \varphi_+^{-1} \colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is given by $x \mapsto x/\|x\|^2$.

Items 5–15 were done on 21, 22, 26 and 27 February 2008 (10 A.M. – 11:00 A.M.).

16. We wish to show that any smooth k-dimensional submanifold of $\mathbb{R}^n$ is a smooth manifold.
17. We proved the following theorem. *Let $S \subset \mathbb{R}^n$ be a smooth k-dimensional submanifold of $\mathbb{R}^n$. Let (V, φ, U) be a chart in S. Let $W \subset \mathbb{R}^m$ be open and $F \colon W \to \mathbb{R}^n$ be smooth. Assume that $F(W) \subset U$. Then the map $\varphi^{-1} \circ F \colon W \to V$ is smooth.*

For a proof, we refer the reader to my article "Four Applications of IFT".
18. Using the last item, we showed that the transition maps of a smooth k-submanifold of $\mathbb{R}^n$ are smooth. Hence Item 16 is proved.

Items 16–18 were done on 17 March 2008 (10 A.M. – 11:00 A.M.).

19. We defined smooth functions on any smooth manifold.
20. Let $S \subset \mathbb{R}^n$ be a smooth submanifold. There are a lot of smooth functions on S.
   (a) Let U be an open set containing S. Assume that $g \colon U \to \mathbb{R}$ be smooth. Let f be the restriction of g to S. Then f is smooth on S.
   (b) Two important classes of functions obtained this way are:

- Let $f\colon S \subset \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) := \|x\|^2$. (This is a constant on the spheres with center at the origin. But we also saw that these are not constant functions if the centre is nonzero vector. Do you recall how we proved this?)
- Let $u \in \mathbb{R}^n$ be of unit norm. Let $h_u\colon S \to \mathbb{R}$ be given by $h_u(x) := x \cdot u$. h_u is called a height function in the u-direction.

Items 19–20b were done on 18 March 2008 (10 A.M. – 11:00 A.M.).

21. **Theorem.** *A function $f\colon S \to \mathbb{R}$ is smooth iff for each $p \in S$, there exists an open set $U \ni p$* in $\mathbb{R}^3$ *and a smooth function $g\colon U \to \mathbb{R}$ such that $g = f$ on $S \cap U$.* $\square$
   See my article 'Four Applications' for a proof.
22. Let $S_i \subset \mathbb{R}^{n_i}$ be k_i-dimensional smooth manifold. A continuous map $F\colon S_1 \to S_2$ is said to be smooth at $p \in S_1$ if for some chart (V_2, φ_2, U_2) containing $F(p)$, and a chart (V_1, φ_1, U_1) containing p with $F(U_1) \subset U_2$, the map $\varphi_2^{-1} \circ F \circ \varphi_1\colon V_1 \to V_2$ is smooth at $p_1 := \varphi_1^{-1}(p)$.
   F is smooth on S_1 if F is smooth at each $p \in S_1$.
   Question: Do we require S_i to be smooth submanifolds of Euclidean spaces?
23. Let S be a k-dimensional smooth manifold. Let (V, φ, U) be a chart. Then $F := \varphi\colon V \to S$ is a smooth map.
24. Keep the notation of the last item. The map $G := \varphi^{-1}\colon U \to \mathbb{R}^k$ is smooth.
25. Let S_1 be the unit sphere in $\mathbb{R}^3$ minus the poles. Consider the cylinder S_2 given by $x^2 + y^2 = 1$ in $\mathbb{R}^3$. Define a map $F\colon S_1 \to S_2$ as follows. Given $p \in S_1$ let $F(p)$ be the point of intersection of the line joining the point on the z-axis closest to p and p with the cylinder. In terms of coordinates we have
$$F(\cos u \cos v, \cos u \sin v, \sin u) = (\cos v, \sin v, \sin u).$$
   (Derive this expression!) Then F is smooth.
26. Think of an *obvious* smooth map from the plane $z = 0$ to the plane given by $ax + by + cz = d$ with $c \neq 0$.
27. Think of an (obvious) smooth map from $S_R(a) := \{x \in \mathbb{R}^{n+1} : \|x - a\| = R\}$ to the unit sphere $S = S_1(0)$. Students gave $x \mapsto (x - a)/R$ and interpreted the map in terms of concepts from Linear Algebra. The (differential) geometric interpretation is that the map takes each point of $S_R(a)$ to the outgoing unit normal at that point.
28. A smooth map from a surface to the unit sphere which assigns to each point of the surface a unit normal at that point is called a normal map or Gauss map. Such a map exists locally always, but may not exist on the entire surface. (This involves the notion of orientability.)
29. For the cylinder $x^2+y^2 = 1$ in $\mathbb{R}^3$, a unit normal map is given by $(x, y, z) \mapsto (x, y, 0)$ and hence is 'of rank' 1.
30. What is the meaning of the phrase that a map $f\colon S_1 \to S_2$ is of rank r?

Items 22–30 were done on 19 March 2008 (10 A.M. – 11:00 A.M.).

31. We recalled the geometric interpretation of the derivative of a smooth map, the domain and codomain of the derivative etc.

32. We showed that if $S \subset \mathbb{R}^n$ is a smooth k-manifold, then T_pS is a k-dimensional vector space. This was based on some observations. Fix a chart (V, φ, U) around p. There is a bijection between the smooth curves in V passing through $p' := \varphi^{-1}(p)$ and those in U through p. The map φ is a diffeomorphism of V onto U.

33. In fact, we proved the following theorem:
Theorem. *Let $S \subset \mathbb{R}^n$ be a smooth k-manifold Let $p \in S$ and (V, φ, U) be a chart containing p. Then $T_pS = D\varphi(\mathbb{R}^k) = D\varphi(q)\,(T_qV)$ where $\varphi(q) = p$.* $\square$

34. What is the significance of the last theorem? The left side, namely, T_pS is defined intrinsically whereas the right side $D\varphi(q)(\mathbb{R}^k)$ depends on the parameterization chosen. In the case of surfaces, in classical books, the tangent vector ∂_u is denoted by X_u. The reason for this is that the patch φ is denoted by $\varphi(u,v) = \mathbf{x}(u,v)$ and so X_u is the partial derivative of $\mathbf{x}$ with respect to u. Also, the condition on the rank of $D\varphi(q)$ is formulated as $X_u \times X_v$ is nonzero!

Items 31–34 were done on 20 March 2008 (10 A.M. – 11:00 A.M.).

35. Let $S_i \subset R^{n_i}$ be a smooth k_i-manifold. We defined the derivative of a smooth map $F\colon S_1 \to S_2$ as follows:
$$DF(p)\colon T_pS_1 \to T_{F(p)}S_2 \text{ given by } DF(p)(v) := (F \circ c)'(0)$$
where c is a smooth curve in S_1 such that $c(0) = p$ and $c'(0) = v$.
Exercise: Show that $DF(p)$ is well-defined and that $DF(p)$ is linear.

36. We computed the derivative of some concrete smooth maps seen earlier in Items 26 and 27.

37. Let $f\colon \{z = 0\} \subset \mathbb{R}^3 \to \mathbb{R}$ be smooth. There exists a smooth $g\colon \mathbb{R}^3 \to \mathbb{R}$ such that $g\,|_{\{z=0\}} = f$.

38. Let $f\colon S := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\} \to \mathbb{R}$ be smooth. Then there exists a smooth function $g\colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ which extends f.

39. The extensions defined above are examples of a set-theoretic principle. If $f\colon X \to Z$ is a given map, and Y is a nonempty set, then we have $g\colon X \times Y \to Z$ given by $g(x,y) = f(x)$. Under g, the entire fibre $\{x\} \times Y$ is mapped to $f(x)$.

40. Let S be the right circular cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$. Let $f\colon S \to \mathbb{R}$ be smooth. Let $U = \mathbb{R}^3 \setminus \{z\text{-axis}\}$. Then we have an extension g of f to U given by $g(x,y,z) = f(x/r, y/r, z)$, where $r^2 = x^2 + y^2$. We interpreted this function in terms of cylindrical coordinates. We also saw how this is another example of the principle enunciated in Item 39: $g(r, \theta, z) = f(\theta, z)$.

41. We proved the following theorem. *Let $S \subset \mathbb{R}^n$ be a smooth k-manifold. Let $f\colon S \to \mathbb{R}$ be smooth. Given $p \in S$, there exists an open set $U \ni p$ in $\mathbb{R}^n$ and a smooth function $g\colon U \to \mathbb{R}$ such that $f = g\,|_S$.*
See Theorem 5 of my article "Four Applications of Inverse Function Theorem".

Items 35–41 were done on 25 March 2008 (10 A.M. – 11:00 A.M.).

42. We proved that any k-dimensional smooth submanifold of $\mathbb{R}^n$ is locally the graph of a smooth function defined on an open subset of $\mathbb{R}^k$ taking values in $\mathbb{R}^{n-k}$.

43. The last item gives another proof of the fact that the tangent space at any point of a smooth k-dimensional submanifold of $\mathbb{R}^n$ is a real vector space of dimension k.

44. Based on our experience with the plane and space curves, we wish to define the curvature at a point p on the surface in $\mathbb{R}^3$ to be the 'rate of change of a unit normal field'. But then we need to ensure the existence (at least locally) of a smooth unit normal field around a given point.

45. Let $S \subset \mathbb{R}^3$ be a smooth surface. Let (V, φ, U) be a chart containing p. The tangent vectors to the u and v parameter curves are denoted by X_u and X_v respectively. Recall that the if $\varphi(u_0, v_0) = p$, and if we write $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$, then the the u-curve through p is

$$c_u(t) := \varphi(u_0 + t, v) = (x(u_0 + t, v), y(u_0 + t, v), z(u_0 + t, v))$$

so that

$$X_u = D\varphi(u_0, v_0)(e_1) = \left(\frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0)\right).$$

Similar considerations apply to X_v. Since the rank of $D\varphi(u_0, v_0)$ is 2, we see that X_u and X_v form a basis of T_pS. We thus have a smooth unit normal field on U:

$$N_q := \frac{X_u \times X_v}{\| X_u \times X_v \|}, \text{ where } \varphi(u,v) = q.$$

46. On any connected neighbourhood of p, there exist at most two smooth unit normal fields.

47. We have two obvious choices of assigning a real number to a given 2×2 real matrix, namely, its determinant and the trace. In our quest to define the curvature at point on the surface at a real number, we are faced with the question: Which one of the two (determinant and trace of $DN(p)$) to choose?

48. Gauss defined the curvature as the limit of $\frac{\text{Area}(N(U))}{\text{Area}(U)}$ as the open neighbourhoods U of p 'converge' to $\{p\}$.

49. We looked at the examples of a plane $ax + by + cz = d$, spheres $S_R(a)$ and the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$. Using Gauss' definition, we found the respective curvatures are 0, $1/R^2$ and 0.

Items 42–49 were done on 27 March 2008 (10 A.M. – 11:00 A.M.).