Change of Variable Formula for Multiple Integrals

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Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function and $\varphi\colon [c,d] \to [a,b]$ be a C^1-function. Then the change of variable formula for one variable integrals says that

$$\int_c^t f(\varphi(x))\varphi'(x)dx = \int_{\varphi(a)}^{\varphi(t)} f(y)dy. \tag{1}$$

This formula is very easily established via the fundamental theorem of integral calculus. For multiple integrals the change of variable formula is

Theorem 1. *Let U and V be open subsets of $\mathbb{R}^n$ and $\varphi\colon U \to V$ be an orientaion preserving C^1 diffeomorphism. Then, given a continuous function $f\colon V \to \mathbb{R}$ with compact support, we have*

$$\int_U f(\varphi(x))J(\varphi(x))dx = \int_V f(y)dy, \tag{2}$$

where $J(\varphi(x))$ stands for the determinant of the Jacobian matrix of φ.

The usual proof of this theorem is got by approximating the integral by finite sums and this involves carefully estimating the volumes of images of small cubes under the map φ and other small details.

Recently P.D. Lax[1] found a new approach and proved the following variant of the change of variable formula.

Theorem 2. *Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ be smooth such that $\varphi(x) = x$ for all x with sufficiently large norm. Then for any continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ with compact support, we have*

$$\int f(\varphi(x))J(\varphi(x))\, dx = \int f(y)\, dy. \tag{3}$$

The aim of this article is to deduce the standard version of the change of variable formula 1. To make the article self-contained, we include the proof of Theorem 2 towards the end of this article.

The proof of the form of the change of variable formula as in Theorem 2 is simple and highly original. The proof uses only the change of variable formula for one variable. Its only shortcoming is that it is not in the standard form. To remedy this, we deduced the standard version of the formula from the version due to Lax as soon as we learnt of the proof by Lax. Our proof was so simple that we did not think of writing it up. However, in 2002, we came across another paper by Lax [2] in which he uses his version to deduce other variants of the change of variable of formula but not the standard version. The proofs of other versions are also rather long and it spoils the simplicity and originality of the first result, namely 2. It is worth mentioning that Michael Taylor [3] also proves the version due to P.D. Lax in the language of differential forms and he too does not prove the standard version. These facts convinced us that though our proof is simple and appears to be natural, it may not be so. Hence we decided to write it up.

Proof of Theorem 1:

Let $\varphi\colon U \to V$ be a smooth diffeomorphism of an open set $U \subset \mathbb{R}^n$ to V. Let $f \in C^\infty(V)$ be of compact support. Let $S := \varphi^{-1}(\operatorname{Supp} f)$. Then S is compact and hence contained in a cube of sufficiently large side or any ball of sufficiently large radius. Consider the open cover $\{U_1 := U, U_2 := \mathbb{R}^n \setminus S\}$ of $\mathbb{R}^n$. Let $\{f_j\}$ be the partition of unity subordinate to this cover. We define

$$\psi(x) = f_1(x)\varphi(x) + f_2(x)x, \quad x \in \mathbb{R}^n.$$

Since the support of f_2 is a subset of $\mathbb{R}^n \setminus S$ and hence is disjoint from S, we see that $f_2(x) = 0$ for $x \in S$. It follows that for $x \in S$, $f_1(x) = 1$ and hence $\psi(x) = \varphi(x)$. On the other hand, if $x \in U_2$, then $x \notin S$. Since $U_1 \cap U_2 = U_1 \setminus S$, the support of f_1 is contained in a compact neighbourhood of S in $U_1 = U$. Thus, if $x \in U_2$ but not in the support of f_1, then $f_1(x) = 0$ so that $\psi(x) = x$ for x outside a compact set containing S. Therefore, we can now apply the version due to Lax, viz., theorem 2 and get

$$\int f(\psi(x))J(\psi(x))\,dx = \int f(y)\,dy. \tag{4}$$

To deduce the standard version, we need only observe that the LHS of Equation (4) is nothing but $\int f(\varphi(x))J(\varphi(x))\,dx$. ♢

Remark 3. We observe that we needed only the fact that φ is a proper map in our proof.

Now we prove theorem 2. Before we start with the proof of theorem 2, we fix the following notations: We write ∂_{x_i} instead of $\frac{\partial}{\partial x_i}$, ∂_{y_i} instead of $\frac{\partial}{\partial y_i}$ and we write $\varphi(x_1, x_2, \ldots, x_n) = (\varphi_1(x_1, \ldots, x_n), \varphi_2(x_1, \ldots, x_n), \ldots, \varphi_n(x_1, \ldots, x_n))$.

Proof of theorem2:

We prove the result only for differentiable functions.

Let f be a differentiable function with compact support and $g\colon \mathbb{R}^n \to \mathbb{R}$ be the function defined by

$$g(y_1, y_2, \cdots\cdots, y_n) := \int_{-\infty}^{y_1} f(z, y_2, y_3, \cdots\cdots, y_n)dz. \tag{5}$$

Let us now observe that $\partial_{y_1} g = f$. Since the function f has compact support, there exists a real number c such that Support(f) is contained in the n-cube defined by the equations $|y_i| \le c$ for all $i = 1, 2, \ldots, n$. It follows from equation (5) that $g(y_1, y_2, \cdots\cdots, y_n) = 0$ when $|y_j| \ge c$ for some $j \ge 2$ or when $y_1 \le -c$.

We assume that $\varphi(x) = x$ for all x such that $\|x\| \ge r$. We may take c large so that the c-cube contains the ball of radius r in $\mathbb{R}^n$. Since φ is identity outside the ball of radius r, it follows that $f(\varphi(x)) = 0$ outside the c-cube. So in the integrals in equation (4) we can restrict the integration to the c-cube.

In the left hand side of the equation (4) we substitute $f = \partial_{y_1} g$ and get the expression

$$\int \partial_{y_1} g(\varphi(x))J(\varphi(x))dx. \tag{6}$$

Then $\partial_{x_j}(g \circ \varphi) = \sum_{i=1}^{n}(\partial_{y_i} g)(\partial_{x_j}\varphi_i)$ for $j = 1, 2, \dots, n$ and the gradient

$$\begin{aligned} D(g\circ\varphi) &= (\partial_{x_1}(g\circ\varphi), \dots, \partial_{x_n}(g\circ\varphi)) \\ &= \Big(\sum_{i=1}^{n}(\partial_{y_i} g)(\partial_{x_1}\varphi_i), \dots, \sum_{i=1}^{n}(\partial_{y_i} g)(\partial_{x_n}\varphi_i)\Big). \end{aligned} \tag{7}$$

If we let $D\varphi_j = (\partial_{x_1}\varphi_j, \dots, \partial_{x_n}\varphi_j)$ denote the gradient of the function $\varphi_j : \mathbb{R}^n \to \mathbb{R}$, then we can write the equation (7) as

$$D(g\circ\varphi) = \sum_{i=1}^{n} \partial_{y_i} g \cdot D\varphi_i. \tag{8}$$

Therefore

$$\begin{aligned} \det\begin{pmatrix} D(g\circ\varphi) \\ D\varphi_2 \\ \vdots \\ D\varphi_n \end{pmatrix} &= \det\begin{pmatrix} \sum_{j=1}^{n} \partial_{y_i} g \cdot D\varphi_i \\ D\varphi_2 \\ D\varphi_3 \\ \vdots \\ D\varphi_n \end{pmatrix} \\ &= \sum_{i=1}^{n} \partial_{y_i} g \cdot \det\begin{pmatrix} D\varphi_i \\ D\varphi_2 \\ D\varphi_3 \\ \vdots \\ D\varphi_n \end{pmatrix} \\ &= (\partial_{y_1} g)(J(\varphi(x))). \end{aligned}$$

Let us now expand the determinant $\det(D(g\circ\varphi), \varphi_2, \dots, \varphi_n)$ with respect to the first row to obtain an expression of the form

$$\sum_i M_i \partial_{x_i}(g\circ\varphi), \tag{9}$$

where M_i's are the co-factors of the first row $(\partial_{x_1}(g\circ\varphi), \dots, \partial_{x_n}(g\circ\varphi))$ of the Jacobian matrix $\begin{pmatrix} D(g\circ\varphi) \\ D\varphi_2 \\ \vdots \\ D\varphi_n \end{pmatrix}$. If we now substitute the expression (9) into the integrand in equation (6), we get

$$\int (M_1\partial_{x_1}(g\circ\varphi) + M_2\partial_{x_2}(g\circ\varphi) + \cdots + M_n\partial_{x_n}(g\circ\varphi))dx. \tag{10}$$

Since the function φ is twice differentiable, we can integrate each term by parts over the c-cube and get

$$-\int (g \circ \varphi)(x) \left(\sum_i \partial_{x_i} M_i \right) dx + \text{boundary terms.} \tag{11}$$

Now we prove the identity

$$\sum_i \partial_{x_i} M_i = 0.$$

This is a classical identity. For the sake of completeness we give a proof of this identity below. Another proof can also be found in [4].

Proof. Let us recall that

$$M_i = (-1)^{i+1} \sum_{\sigma(i)=i} \operatorname{sgn}(\sigma) \prod_{j \neq i} \partial_{x_j} \varphi_{\sigma(j)}.$$

Hence

$$\begin{aligned} \partial_{x_i} M_i &= (-1)^{i+1} \sum_{\sigma(i)=i} \operatorname{sgn}(\sigma) \partial_{x_i} \left(\prod_{j \neq i} \partial_{x_j} \varphi_{\sigma(j)} \right) \\ &= (-1)^{i+1} \sum_{\sigma(i)=i)} \operatorname{sgn}(\sigma) \Big\{ \partial_{x_i} \partial_{x_1} \varphi_{\sigma(1)} \cdots \partial_{x_1} \varphi_{\sigma(n)} + \cdots \\ &\qquad + \partial_{x_1} \varphi_{\sigma(1)} \cdots \partial_{x_i} \partial_{x_n} \varphi_{\sigma(n)} \Big\} \\ &= (-1)^{i+1} \sum_{k=1}^{n} m_{i,k}, \end{aligned}$$

where $m_{i,k}$ denotes the determinant of the matrix
$\begin{pmatrix} \partial_{x_1}\varphi_2 & \cdots & \partial_{x_i}\partial_{x_k}\varphi_2 & \cdots & \partial_{x_n}\varphi_2 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \partial_{x_1}\varphi_n & \cdots & \partial_{x_i}\partial_{x_k}\varphi_n & \cdots & \partial_{x_n}\varphi_n \end{pmatrix}$ for $k \neq i$ and $m_{i,i} = 0$.

Now we shift the k-th column $\begin{pmatrix} \partial_{x_i}\partial_{x_k}\varphi_2 \\ \vdots \\ \partial_{x_i}\partial_{x_k}\varphi_n \end{pmatrix}$ of the matrix $m_{i,k}$ to the first column and denote the resulting matrix by $M_{i,k}$. Then

$$\partial_{x_i} M_i = (-1)^{i+1} \sum_{k=1}^{n} \sigma(i,k) M_{i,k}$$

where $\sigma(i,k) = (-1)^k$ if $k > i$ and $(-1)^{k-1}$ if $k < i$. Therefore

$$\sum_{i=1}^{n} \partial_{x_i} M_i = \sum_{i,k=1}^{n} (-1)^{i+1} \sigma(i,k) M_{i,k}.$$

Since $M_{i,k} = M_{k,i}$, it follows that

$$\begin{aligned}\sum_{i=1}^{n} \partial_{x_i} M_i &= \sum_{i,k=1}^{n} (-1)^{i+1} \sigma(i,k) M_{i,k} \\ &= \sum_{k>i} (-1)^{i+k+1} M_{i,k} + \sum_{k<i} (-1)^{i+k} M_{i,k} + \\ &= (-1) \sum_{k>i} (-1)^{i+k} M_{i,k} + \sum_{k>i} (-1)^{i+k} M_{k,i} \\ &= (-1) \sum_{k>i} (-1)^{i+k} M_{i,k} + \sum_{k>i} (-1)^{i+k} M_{i,k} \\ &= 0.\end{aligned}$$

This completes the proof of the identity. ♢

Remark 4. For those who know differential forms, the proof of this identity is easy. We need only observe that $d(d\varphi_2 \wedge d\varphi_3 \wedge \ldots \wedge d\varphi_n) = (\sum_{i=1}^{n} \partial_{x_i} M_i) dx_1 \wedge dx_2 \wedge \ldots, \wedge dx_n$ and use the fact $d^2 = 0$.

We now attend to the boundary integral in equation (11). Since $g \circ \varphi(x) = g(x)$, on the boundary of the c-cube, the only non-zero boundary term will come from the side $x_1 = c$. Also, since $M_1 = 1$, when $\varphi(x) = x$, that boundary term is

$$\int g(c, x_2, \ldots\ldots, x_n) dx_2 dx_3 \cdots dx_n. \tag{12}$$

Using the definition of the function g in equation (12) gives

$$\int \int_{-\infty}^{c} f(z, x_2, \ldots\ldots, x_n) dz dx_2 \cdots dx_n.$$

This is nothing but the right hand side of the equation (4). ♢

References

[1] P.D.Lax, *Change of Variables in Multiple Integrals*, Amer. Math. Monthly, pp. 497–501, 1999.

[2] P. D. Lax, *Change of Variables in Multiple Integrals II*, Amer. Math. Monthly, pp.115-119, 2001.

[3] Michael Taylor, *Differential Forms and the Change of Variable Formula for Multiple Integrals*, Vol. 268, pp.378-383, 2002.

[4] N. Dunford and J. Schwartz, *Linear Operators, Part-I*, pp. 476, Wiley-Interscience, New York 1958.

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