An Elementary Proof of Weierstrass Approximation Theorem

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Theorem 1. *If $f\colon [0,1] \to \mathbb{R}$ is a continuous function and $\varepsilon > 0$ is given, then there exits a polynomial p such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [0,1]$.*

Proof. By subtracting $f(0)$ we may assume that $f(0) = 0$. The idea is to approximate continuous functions by step functions and the latter by polynomials.

By uniform continuity of f, given any ε, we can find δ such that whenever $d(x,y) < \delta$, we have $|f(x) - f(y)| < \varepsilon$. Hence, we can find points x_i, where $x_0 = 0 < x_1 < x_2 .... < x_m < 1 = x_{m+1}$ so that the step function $\sigma(x) := f(x_{i-1})$ if $x \in [x_{i-1}, x_i)$ is such that $|f(x) - \sigma(x)| < \varepsilon$. We may also assume that if $s_i := f(x_i) - f(x_{i-1})$ then $|s_i| < \varepsilon$.

Let

$$H(x) := \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

be the modified Heaviside function. Then we have $\sigma(x) := \sum_j s_j H(x - x_j)$ and hence $|f(x) - \sigma(x)| < \varepsilon$ in [0,1] and $|s_j| < \varepsilon$. Thus the problem reduces to that of approximating H.

We recall the Bernoulli inequality: $(1+h)^n \geq 1 + nh$, for $h \geq -1$, $n \in \mathbb{N}$. (Proved by induction on n).

Consider the polynomial $Q_n(x) := (1 - x^n)^{2^n}$. Q_n decreases from the value 1 at $x = 0$ to 0 at $x = 1$:

$$Q_n'(x) = 2^n (1 - x^n)^{2^n - 1} n x^{n-1} (-1) \leq 0, \text{ for } 0 \leq x \leq 1.$$

In $0 \leq x \leq q < \frac{1}{2}$, Q_n tends to 1 uniformly as $n \to \infty$, for

$$1 \geq Q_n(x) \geq Q_n(q) = (1 - q^n)^{2^n} \geq 1 - 2^n q^n = 1 - (2q)^n \to 1.$$

In $\frac{1}{2} < q \leq x \leq 1$, $Q_n(x) \to 0$ uniformly as $n \longrightarrow \infty$, since $0 \leq Q(x) \leq Q(q)$,

$$\frac{1}{Q(q)} = (1 - q^n)^{-2^n} = [1 + q^n (1 - q^n)^{-1}]^{2^n} \geq 1 + 2^n q^n (1 - q^n)^{-1} \geq (2q)^n \to \infty.$$

We define P_n as follows: $P_n(x) := Q_n(\frac{1-x}{2})$, which lies between 0 and 1 for $-1 \leq x \leq 1$. P_n goes to the Heaviside function H uniformly on $0 < \delta \leq |x| \leq 1$.

We now choose δ such that the intervals $(x_j-\delta, x_j+\delta)$ are pairwise disjoint and then n very large so that if $\sum_j |s_j| := S$ the inequality $|P_n(x) - H(x)| < \varepsilon S^{-1}$ holds in $0 < \delta \leq |x| \leq 1$.

Let $P^*(x) := \sum s_j P_n(x - x_j)$. Then for every $x \in (x_k - \delta, x_k + \delta)$ we have:

$$\begin{aligned}
|\sigma(x) - P^* x| &< |s_j| \cdot |H(x - x_j) - P_n(x - x_j)| + |s_k| \cdot |H(x - x_k) - P_n(x - x_k)| \\
&< \varepsilon S^{-1} S + |s_k| \cdot 1 \\
&< 2\varepsilon.
\end{aligned}$$

Hence we finally obtain

$$|f(x) - P^*(x)| \leq |f(x) - \sigma(x)| + |\sigma(x) - P^*(x)| < 3\varepsilon.$$

□