Weyl's Integral and Character Formulas for $U(n)$

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Maximal Torus

Let $G = U(n) = \{g \in GL(n, \mathbb{C}),\ gg^* = 1\}$, where $g^* = \overline{g^t}$. G is the group of $n \times n$ unitary matrices. It is compact: $g = (u_{ij}) \in G$ iff $\sum u_{ik}u^*_{kj} = \delta_{ij}$ where $u^* = (u^*_{kj}) = (\overline{u}_{jk})$. Hence $g \in Gl(n, \mathbb{C})$ lies in $U(n)$ iff $\sum u_{ik}\overline{u}_{jk} = \delta_{ij}$. Since in the above equation only continuous functions are involved, G is a closed subset of $\mathbf{C}^{n^2}$. Also by setting $i = j$, we get $\sum_k u_{ik}\overline{u}_{ik} = 1$ or $\sum_i \sum_k |_{ik}|^2 = n$. Hence $U(n)$ is bounded. Thus G is a compact group.

The most basic fact about the structure of G is the following:

Theorem 1. *Given $u \in G$, there exists an $x \in G$ such that $xux^{-1} = \operatorname{diag}(e^{i\theta_1}, \cdots, e^{i\theta_n})$, for $0 \le \theta_i \le 2\pi$.*

Notice first that $T = \{\operatorname{diag}(e^{1\theta_1}, \cdots, e^{i\theta_n}) : 0 \le \theta_j \le 2\pi\} \subseteq G$ and that $T \approx S^1 \times \cdots \times S^1$, an n-torus. How is Theorem 1 proved? Choose $\lambda = \varepsilon$ as a zero of the characteristic equation $\det(\lambda I - u) = 0$. Thus there is a non-zero e such that $u\,e = \varepsilon e$. We may assume that $e = e_1$ is a unit vector. If $W := e_1^{\perp}$, then W is invariant under u:

$$\langle uw, e_1 \rangle = \langle w, u^* e_1 \rangle = \left\langle w, u^{-1} e_1 \right\rangle = \left\langle w, \varepsilon^{-1} e_1 \right\rangle = 0.$$

Thus, $u\,|_W$ is a unitary map. By induction, we have an O.N. basis of W, say $\{e_1, \ldots e_n\}$ consisting of eigenvectors, say, $ue_j = \varepsilon_j e_j$ for $2 \le n$.

Note that the diagonal entries in $t = xux^{-1}$, are the eigenvalues of the given matrix u. They are unique only up to a permutation i.e., by effecting a permutation σ of the eigenvectors (which we assume to be orthonormal) we shall get $t_\sigma = (t_{\sigma(1)}, \ldots, t_{\sigma(n)}) = \sigma xux^{-1}$. To see this more explicitly: for $\sigma \in S_n$, the group of permutations, set $\sigma e_i = e_{\sigma(i)}$. Then σ extends to a linear transformation on $\mathbb{C}^n$; in fact, $\sigma \in U(n)$. (Why?) If we take $y = \sigma x$, then $yuy^{-1}(e_i)$

$$\begin{aligned} yuy^{-1}(e_i) &= \sigma xux^{-1}\sigma^{-1}(e_i) \\ &= \sigma[xux^{-1}(e_{\sigma^{-1}(i)})] \\ &= \sigma\varepsilon_{\sigma^{-1}(i)} e_{\sigma^{-1}(i)} \\ &= \varepsilon_{\sigma^{-1}(i)} \sigma e_{\sigma^{-1}(i)} \\ &= \varepsilon_{\sigma^{-1}(i)} e_i \end{aligned}$$

Thus $yuy^{-1} = \text{diag}\,(\varepsilon_{\sigma^{-1}(1)}, \ldots, \varepsilon_{\sigma^{-1}(n)})$.

Put this in a different way, we have shown that

$$G = \bigcup_{g \in G} gTg^{-1}.$$

T is known as a *maximal torus* of G. Thus every element of G is conjugate to an element of T.

Now given a $u \in T \subseteq G$, when do we have $xux^{-1} = yuy^{-1}$, for $x, y \in G$?

$$xux^{-1} = yuy^{-1} \Leftrightarrow y^{-1}xux^{-1}y = u \Leftrightarrow (y^{-1}x)u(y^{-1}x)^{-1} = u \Leftrightarrow y^{-1}x \in Z_G(u),$$

where $Z_G(u)$ is the centralizer of u in G. If we further assume u has distinct eigenvalues, then $y^{-1}x \in T$. (Note that for $u \in T$, $T \subseteq Z_G(u)$.)

If we set

$$\begin{aligned} T_r &= \{t \in T : t \text{ has distinct eigenvalues }\} \\ &= \{t \in T : t_i = t_j \Rightarrow i = j\}. \end{aligned}$$

then T_r is an open dense subset of T. We set

$$G_r = \bigcup_{g \in G} gT_rg^{-1}.$$

Any element of G_r is said to be *regular*. $G \setminus G_r$ consists of *singular* elements. Note that G_r is open. (Why?)

Weyl Integral Formula

Consider now the map $\psi : G/T \times T_r \to G_r$ given by $(x, t) \mapsto xtx^{-1}$. The map ψ is well-defined (as we have seen earlier). For any $u \in G_r$, there exists precisely $n! = |S_n|$ elements, t_i, $1 \leq 1 \leq n!$ such that $xt_ix^{-1} = u$. (All of them are related via S_n-action as defined earlier). We want to claim that for integration purpose, G_r is a good enough subset of G. i.e., $G_s = G \setminus G_r$ is negligible for measure theoretic reasons.

If this can be done, then G_r can be 'parametrized' by the product $(G/T) \times T_r$ and we can compute the |jacobian| of the map ψ, call it $w(x, t)$ so that the Haar measure on G can be written as:

$$\int_{G_r} f(g)dg = \int \left(\int (f(xtx^{-1})w(x, t)dt \right) dx,$$

in an obvious notation.

We show that G_s is negligible by showing that the dimension of G_s is $n^2 - 3$. Note that $\dim G = n^2$. (See below.)

The 'best' singular element in T can be written in the form $(\varepsilon, \varepsilon, \varepsilon_3, \cdots \varepsilon_n)$ $\quad \varepsilon_i \neq \varepsilon_j$ for $i \neq j$, $3 \leq 1, j \leq n$. Then any element that commutes with a singular element of the above form can be represented by

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} & & \\ \alpha_{21} & \alpha_{22} & & \\ & & \alpha_3 & \\ & & & \alpha_n \end{pmatrix}, \text{ where } \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in U(2)$$

Thus the set C of all elements that commute with the given singular element has $(n-2)+4 = n+2$ dimensions. Hence $\dim G/C = n^2 - (n+2) = n^2 - n - 2$. The set S_b of the best singular elements in T has dimension $(n-1)$. Since $\psi: G/C \times S_b \to G$ is surjective on the best singular elements in G, the set of such elements has dimension $n^2 - n - 2 + (n-1) = n^2 - 3$.

Now if we take any other singular element, say, of the form $(\varepsilon, \varepsilon, \varepsilon, \varepsilon_4, \ldots, \varepsilon_n)$ or $(\varepsilon_1 = \varepsilon_2, \varepsilon_3 = \varepsilon_1, \ldots, \varepsilon_n)$, then the C corresponding to such element will have dimension $> n+2$ so that G/C will have dimension $< n^2 - (n+2)$ and the set of such elements in T has dimension $\leq n-2 < n-1$ and hence, under ψ, elements which are worse than the best singular element will have dimension $< n^2 - 3$.

In any case, thus, dim $G - G_r < n^2$, and hence G_r is a good set for integration.

Thus we have: G_r is an open dense subset of G, with dim $G \setminus G_r <$ dim G and

$$\psi: G/T \times T_r \to G_r \text{ is a } |S_n| - \text{fold covering.}$$

Computation of the Jacobian of ψ:

We want to identity the tangent space $\mathfrak{g}$ of G at $1 \in G$. Let $t \to \chi(t)$ be a curve in G such that $\chi(0) = 1$. Since $\chi(t)\chi(t)^* = 1$, we have

$$\begin{aligned} \frac{d}{dt}(\chi(t)\chi(t)^* \mid_{t=0} &= 0 \\ \text{i.e. } \chi'(t)^* + \chi'(t)\chi(t)^* \mid_{t=0} &= 0 \\ \text{or } \chi'(0)^* + \chi'(0) &= 0. \end{aligned}$$

That is, $\chi'(0)$ is a skew-hermitian matrix.

Thus $\mathfrak{g}$ consists of all skew-hermitian matrices and $\dim_{\mathbb{R}} \mathfrak{g} = n^2$. $\mathfrak{t} = T_1(T)$, the tangent space of T at 1, can be identified with the set of diagonal matrices in $\mathfrak{g}$, i.e., $\mathfrak{t} = \{(i\theta_1, \cdots, i\theta_n) : \theta_j \in \mathbb{R}\}$. Also, if $\mathfrak{p} = T_1(G/T)$ is the tangent space to G/T at the identity coset then $\mathfrak{p}$ is the span of

$$\{X : (* \quad z - \overline{z} \quad *) : z \in \mathbb{C}\},$$

where z is in the (i,k)-th place, $i < k$. Note that for $t \in T$, and X as above in $\mathfrak{p}$, we have $tXt^{-1} = (t_i/t_k)X$.

We are now ready to compute the Jacobian of ψ: For $h \in \mathfrak{t}$, we have

$$\begin{aligned} \frac{d}{ds}\psi(x, t \exp sh) \mid_{s=0} &= \frac{d}{ds}(xt \exp shx^{-1}) \mid_{s=0} \\ &= \frac{d}{ds}(xtx^{-1}x \exp shx^{-1} \mid_{s=0} \\ &= xtx^{-1}\frac{d}{ds}(x \exp shx^{-1}) \mid_{s=0} \\ &= xtx^{-1}. \end{aligned}$$

For $X \in \mathfrak{p}$, we can compute, for $X = X_{ik}$, a basic element of $\mathfrak{p}$,

$$\frac{d}{ds}\psi(x \exp sX, t) \mid_{s=0} = \frac{d}{ds}(x \exp sXt \exp(-sX).x^{-1}) \mid_{s=0}$$

$$
\begin{aligned}
&= x\chi t x^{-1} - x t \chi x^{-1} \\
&= x t t^{-1} X t x^{-1} - x t x^{-1} x \chi x^{-1} \\
&= x t x^{-1} x t^{-1} \chi t x^{-1} - x t x^{-1} x \chi x^{-1} \\
&= x t x^{-1} (x t^{-1} \chi (x t^{-1)^{-1}} - x \chi x^{-1}).
\end{aligned}
$$

At $x = e$, we have

$$\frac{d}{ds}\psi(\exp sX, t) = t(t^{-1}Xt - X) = t\left((t_k/t_i - 1)X\right).$$

But note that $\mathbb{C}X_{ik}$ is a complex 1-dimensional space and hence on this 1-dimensional $\mathbb{C}$ space, when we consider it as a 2-dimensional real vector space, the linear map $d\psi$ is of the form

$$
\begin{pmatrix} (t_k/t_i) - 1 & 0 \\ 0 & \overline{(t_k/t_i) - 1} \end{pmatrix}
= \begin{pmatrix} t_{k/t_i} - 1 & 0 \\ 0 & (ti/_{tk} - 1) \end{pmatrix}
= \begin{pmatrix} \frac{t_k - t_i}{t_i} & 0 \\ 0 & \overline{\frac{t_k - t_i}{t_k}} \end{pmatrix}.
$$

Remark: Let $\alpha \in \mathbb{C}^*$. The Jacobian of the map $z \mapsto \alpha z$ on $\mathbb{C}$ is $|\alpha|^2$.

Thus $d\psi_{(1T,1)}$ on the space $T_{1T}(G\mid_T) \oplus T_1(T)$ can be written as
$$\begin{pmatrix} \frac{t_k - t_i}{t_i} & 0 & & & \\ 0 & \overline{\frac{t_k - t_i}{t_k}} & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$
and hence

$$|\text{jacobian}| = \Delta\overline{\Delta} = \omega(\mathrm{r}, \mathrm{t}) \text{ where } \Delta = \prod_{\mathrm{i}<\mathrm{k}} (\mathrm{t_i} - \mathrm{t_k}).$$

At other points (xT, t) the above calculation shows that $\omega(x, t) = \omega(1, t)$, so that the |jacobian|-density needed for the change of variable formula is given as $\omega(x, t) = \Delta\overline{\Delta}$.

Thus we have proved the Weyl integral formula:

$$\int_{G_r} f(g)dg = \frac{1}{|S_n|}\int_{G/T}\left(\int_T f(xtx^{-1})\Delta\overline{\Delta}(t)\, dt\right) dx. \tag{1}$$

In particular if f is a class-function i.e., $f(xgx^{-1}) = f(g)$ for all $x, g \in G$, then we have

$$\int_{G_r} f(g)dg = \frac{1}{\mid S_n \mid}\int f(t)\Delta(t)\overline{\Delta}(t)dt. \tag{2}$$

(Here dt is the normalized Haar measure on T, dx then G-invariant measure on G/T).

Weyl Character Formula

We can use the Weyl integral formula for the class-function to write down all the irreducible characters of $U(n)$.

Let χ be an irreducible character of G, i.e. the character of an irreducible representation of G. We make a series of observations from which we deduce the explicit form of the characters:

1) χ is a class-function and $G = \cup gTg^{-1}$ imply that it is enough to know $\chi\mid_T$.

2) S_n acts on T as we have seen earlier, and $t = (t, \cdots, t_n)$ and $(t_{\sigma(1), \cdots} t_{\sigma(n)}) = t_\sigma$ are conjugate and hence $\chi(t) = \chi(t_\sigma)$.

3) Let (π, H) be the irreducible representation such that $\chi_\pi = \chi$. When we restrict π to T, then $(\pi\mid_T, H)$ is a unitary representation of the compact, abelian group T and hence H splits as a direct sum $\oplus H_i$ such that $(\pi\mid_T, H) = \oplus(\pi\mid_T, H_i)$, and $(\pi\mid_T, H_i)$ is an irreducible 1-dimensional representation of $T = S^1 \times \cdots \times S^1$. We know all the irreducible 1-dimensional representations of T. They are of the form $(e^{i\theta_1}, \cdots, e^{i\theta_n}) \mapsto e^{ir_1\theta_1 + \cdots + ir_n\theta_n}$, where $(r_1, \ldots, r_n) \in \mathbb{Z}^n$. We write this, as $\chi_r(t) = e^{ir\theta}$, for $r = (r_1, \cdots, r_n) \in \mathbb{Z}^n$. $\theta = (\theta_1, \cdots, \theta_n) \in ([0, 2\pi))^n, r\theta = \sum_j r_j\theta_j$. Choosing as a basis for H, a non zero vector from each one of the H_i, we can write $\pi(t) = \begin{pmatrix} \chi_r(t) & & & & \\ & \chi_s(t) & & & \\ & & \ddots & & \\ & & & \chi & \\ & & & & \ddots \end{pmatrix}$ for $r, s, \cdots \in \mathbb{Z}^n$ so that

$$\begin{aligned} \chi_\pi(t) &= \chi(t) \\ &= \chi_r(t) + \chi_s(t) + \cdots \\ &= e^{ir.\theta} + e^{is.\theta} + \cdots \end{aligned}$$

In particular, $\chi\mid_T$ is a finite Fourier series. Also each representation χ_r in $\pi\mid_T$ can have multiplicity $C \geq 1$. Hence collecting them together, we may write $\chi\mid_T(t) = Ce^{ir.\theta} + C'e^{is\theta} + \cdots$, $C, C^1 \in \mathbb{Z}^+$.

4) Let $P = \{r \in \mathbb{Z}^n$, where χ_r occurs in $\chi\mid_T\}$ we call P the set of weights of π. We have the lexicographic ordering on $P : r = (r_1, \ldots, r_n) \geq s = (s_1, \ldots, s_n)$ if the first non-zero difference $r_j - s_j \geq 0$. Since the lexicographic order is a total order and since P is a finite set, we have a highest element, say, $r \in P : r > s$, for any $s \in P \setminus \{r\}$ (i.e., $r \geq s$, $r \neq s$). The coefficient C of χ_r in $\chi\mid_T$ is a *positive* integer.

We observe that $\chi\mid_T(t) = C\chi_r(t) + C'\chi_s(t) + \cdots$ and $\chi\mid_T(\sigma.t) = \chi\mid_T(t)$ and

$$\chi_s(\sigma.t) = e^{(is_1 t_{\sigma(1)} + \cdots)} = \chi_{\sigma^{-1}(s)}(t),$$

where

$$\sigma^{-1}(s) = \sigma^{-1}(s_1, \cdots, s_n) = (s_{\sigma^{-1}(1)}, \ldots, s_{\sigma^{-1}(n)}),$$

for *any* $s \in \mathbb{Z}^n$. We deduce that whenever $s \in P$, we also have $\sigma^{-1}s \in P$ and that the coefficients of ψ_s and $\psi_{\sigma^{-1}s}$ in $\chi\mid_T$ are the same for any $\sigma \in S_n$.

In other words S_n-acts on P as above and

$$\chi\mid_T = C\left(\sum_{\sigma\in S_n}\chi_{\sigma.r}\right) + C'\left(\sum_{\sigma\in S_n}\psi_{\sigma.s}\right) + \cdots.$$

This is referred to as the S_n- symmetry of $\psi\mid_T$.

5). By Schur's orthogonality relations, we know that χ is an irreducible character iff $\int_G \chi\overline{\chi} = 1$. Since χ is a class-function, we have, by Weyl integral formula.

$$\int_G \chi(g)\overline{\chi}(g)dg = \frac{1}{\mid S_n\mid}\int_T \chi(t)\Delta(t)\overline{\chi}(t)\overline{\Delta}(t)dt.$$

This suggests that we consider the function. $\xi(t) = \chi(t)\Delta(t)$. We first observe that Δ is an S_n-antisymmetric function: $\Delta(\sigma.t) = \mathrm{sgn}\,(\sigma).\ \Delta(t)$, for $\sigma\in S_n$. For, if $\sigma = (i, i+1)$,

$$\Delta(\sigma.t) = (t_1 - t_2)\cdots(t_{i+1} - t_i)\cdots = -\Delta(t).$$

This is true for all transpositions and hence for all $\sigma\in S_n$.

Since χ was already observed to be S_n-symmetric their product ξ is S_n- anti-symmetric. Also note that, in ξ, 'the highest term' is of the form $C(e^{i(r_1+n-1)\theta_1}e^{r_2+(n-2)\theta_2)}\cdots)$ and the coefficient of this highest term is the same positive integer C which occurs in $\chi = C\chi_r + \cdots$. Also note that if any s appears in ξ with coefficient C', that is, $\xi = \cdots + C'e^{is\theta} + \cdots$, then $\sigma\cdot s$ also appears in ξ with coefficient $\mathrm{sgn}\,(\sigma).C$.

6. What are the simplest S_n-anti-symmetric functions on T one can think of ?

Given $\ell = (\ell_1,\cdots\ell_n)\in\mathbb{Z}^n$. We form the elementary S_n-anti-symmetric sum

$$\xi_\ell(\theta) = \sum_{\sigma\in S_n}\mathrm{sgn}\,(\sigma)e^{\sigma.\ell).\theta},\quad \sigma.\ell = (\ell_{\sigma(1),\cdots},\ell_{\sigma(n)}).$$

Note that unless ℓ_i are mutually distinct, $\xi_\ell = 0$ and that we can write

$$\xi_\ell(\theta) = \det(t_i^{\ell_j}) = (t_i^{\ell_j}),\quad t = (e^{i\theta_1},\cdots,e^{i\theta_n}) = (t_1,\cdots,t_n).$$

In particular, $\Delta = \xi_{n-1,\cdots,1,0}$ an elementary anti-symmetric sum.

7). Now in $\xi = \chi\Delta$, if we denote the highest weight $((r_j + n - j)) = (\ell_j) = \ell$, then

$$\xi = C\xi_\ell + \cdots,\ \text{where } C,\cdots\in\mathbb{Z},\ \text{and } C > 0.$$

That is, $e^{i(\sigma.\ell)\theta}$ occurs in ξ with coefficient $\mathrm{sgn}\,(\sigma)C$ for any $\sigma\in S_n$. Also $\sigma.\ell < \ell$, for any $\sigma\neq 1, \sigma\in S_n$. So, in particular, for $\sigma = (i, i+1), \sigma.\ell < \ell \Rightarrow \ell_i > \ell_{i+1} \Rightarrow \ell_1 > \ell_2\cdots > \ell_n$. We can now repeat the above argument of anti-symmetry to $\xi - \xi_\ell$ and so that we can write

$$\xi = C\psi_\ell + C^1\xi_{\ell^1} + \cdots,\ \text{with } C, C^1, \in\mathbb{Z}^+, C > 0$$

and $\ell > \ell^1 > \cdots$.

8). For any $m, \ell \in \mathbb{Z}^n, \ell \neq m$, χ_m and χ_ℓ are distinct irreducible characters of T and so $\int_T \chi_m \chi_\ell dt = 0$. Hence $\int_T \xi_\ell \overline{\xi}_m = \delta_{l,m} n!$. Since χ is an irreducible character,

$$\begin{aligned} 1 &= \int_{G_r} \chi(g)\overline{\chi}(g)dg \\ &= \frac{1}{n!}\int_T \chi(t)\delta(t)\overline{\chi}(t)\overline{\delta}(t)dt \\ &= \frac{1}{n!}\int_T \left(C\xi_\ell + C^1\chi_{\ell^1} + \cdots\right)\left(C\overline{\xi}_\ell + C^1\overline{\xi}_\ell + \cdots\right)dt \\ &= (C^2 + C'^2 + \cdots) = 1. \end{aligned}$$

Therefore, $C = 1, C^1 = \cdots = 0$. Hence $\xi = \xi_\ell$ or

$$\chi(t) = \frac{\sum \operatorname{sgn}(\sigma) e^{(\sigma.\ell)(t)}}{\Delta}, \quad \text{for } t \in T_r.$$

Thus χ is explicitly written for $t \in T_r$. Since $\chi\mid_T$ is, anyhow, a trigonometric polynomial (see 3) and $\chi(t)$, for $t \in T^i$ is also a trigonometric polynomial, χ extends uniquely to all of T. The leading term of the finite Fourier series $\chi\mid_T$ is $t_1^{r_1}\cdots t_n^{r_n}$, $r_j = \ell_j - (n-j)$, and so $r_1 \geq r_2 \geq \cdots$. Recall that r is called the highest weight of the representation. Since ξ_ℓ, when ℓ and r are related as above, uniquely determines ξ, it follows that r uniquely determines the representation.

9). Given $r = (r_1 \cdots r_n) \in \mathbb{Z}^n$ with $r_1 \geq r_2 \geq \cdots \geq r_n$. We can form $\chi_r = \frac{\chi_\ell}{\Delta}$. Is χ_r an irreducible character? Yes, it is always, since χ_r is a continuous class-function on G and $\int \chi_r \overline{\psi}_s = 0$ for any $s \in \mathbb{Z}^n$, with $s_1 \geq \cdots \geq s_n$ and $r \neq s$.

10). The dimension $d(r)$ of the representation whose highest weight is r and whose character is given by $\chi_r = \xi\mid_\Delta$, is given by $\chi_r(e)$. But the expression we got for χ_r is valid only for regular elements and $1 \notin G_r$. But observe that $\xi_\ell(1) = 0$ and so we have an indeterminate form of the type $0/0$. So we can apply L'Hopital type argument.

Weyl's dimension formula

Recall that if χ is the character of a finite dimensional representation then $\chi(e)$ is the dimension of that representation. Now the Weyl's character formula for the irreducible representation (with $r = (r_1, \cdots, r_n)$ as highest weight) of $U(n)$ is given by

$$\chi(t) = \frac{\sum \operatorname{sgn}(\sigma)\cdots e^{i(\sigma.\ell).t}}{\sum \operatorname{sgn}(\sigma) e^{i(\sigma.\rho).t}},$$

where $\gamma \in T_r$ is a regular element (that is, with distinct eigenvalues), $\ell_1 = r_1 + (n-1), \ell_2 = r_2 + (n-2)\cdots \ell_{n-1} = r_{n-1} + 1, \ell_n = r_n, \sigma$ runs through S_n. Since $e \in T_r$, we cannot evaluate $\chi(e)$ by substituting $t = e$ in the above formula. Notice that both the numerator and the denominator become zero at e. We can hence use the L'Hopital's rule.

Observe that, in our earlier notation, $\xi(\ell_1, \cdots, \ell_n) = \sum \operatorname{sgn}(\sigma) e^{i(\ell_{\sigma(1)}\theta_1 + \cdots + \ell_{\sigma(n)}\theta_n)}$, can also be written as

$$\begin{pmatrix} t_1^{\ell_1} & \cdots & t_n^{\ell_n} \\ t_2^{\ell_1} & \cdots & t_n^{\ell_n} \\ \vdots & & \\ t_n^{\ell_1} & \cdots & t_1^{\ell_n} \end{pmatrix} = (t_i^{\ell_j}).$$

If we set $t_p = (e^{i(n-1)\theta}, \cdots, e^{i\theta}, 1)$ then we have

$$\chi(e) = \lim_{\theta\to 0}\chi(t_\rho) = \lim_{\theta\to 0}\frac{\xi_\ell(t_\rho)}{\xi_\rho(t_\rho)}.$$

Now

$$\begin{aligned}\xi_\ell(t_\rho) &= \sum \operatorname{sgn}(\sigma) e^{i(\ell_{\sigma(1)}(n-1)\theta_1 + \cdots \ell_{\sigma(n)}\theta_n)} \\ \xi_\rho(t_\ell) &= \sum \operatorname{sgn}(\sigma) e^{i(\rho_{\sigma(1)}\ell_1\theta_1 + \cdots + \rho_{\sigma(n)}\ell_n\theta_n)}\end{aligned}$$

Or,

$$\begin{aligned}\xi_\ell(t_\rho) &= \begin{pmatrix} e^{i(n-1)\theta\ell_n} & \cdots & e^{i(n-1)\theta\ell_n} & \\ & e^{i\theta\ell_n} & \cdots & e^{i\theta} \\ & 1 & \cdots & 1 \end{pmatrix} \\ \xi_\rho(t_\rho) &= \begin{pmatrix} e^{i\ell_n.\theta n-1} & \cdots & e^{i\ell_n\theta.\rho_n} & \\ & e^{i\ell_2\theta n-1} & \cdots & 1 \\ & e^{\ell_1\theta n-1} & \cdots & 1 \end{pmatrix}\end{aligned}$$

and so $\xi_\rho(t_\ell) = \xi_\ell(t_\rho)$.

Also, $\xi_\rho(t) = \prod_{i<k}(t_1 - t_k) = \prod_{j<k}(t_j - t_k)$ and hence

$$\xi_\rho(t_\ell) = \prod_{j<k}(e^{i\ell_j\theta} - e^{i\ell_k\theta}) = \prod_{j<k}((\ell_j - \ell_k)\theta + \text{ higher order terms}).$$

Hence $\frac{\xi_\ell(t_\rho)}{\xi_\rho(t_\rho)} = \frac{\prod_{j<k}((\ell_j-\ell_k)\theta+\cdots)}{\prod_{j<k}((n-j-(n-k))\theta+\cdots)}$. Therefore $\lim_{\theta\to 0}\frac{\xi_\ell(t_\rho)}{\xi_\rho(t_\rho)} = \prod_{j<k}\frac{(\ell_j-\ell_k)}{(k-j)} := d(r)$.

The branching law for $U(n)$

Let $G = U(n)$. If we consider $H = \{g \in U(n) : g.e_n = e_n$ where e_n is the n-th elt of the usual o.n. basis of $\mathbf{C}^n\}$, then $H \simeq U(n-1)$
$H = \int \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}; a \in U(n-1)\}$. Thus we consider $U(n-1)$ as a subgroup of $U(n)$. By Weyl's character formula we know all the irreducible characters of $U(n)$.

Now, if (π, V) is an irreducible representation of G, then $(\pi\mid_H, V)$ is a (f.d) representation of the compact group $U(n-1)$, and hence is a direct sum of irreducible representations of H. We ask: How does it decompose?

Note that if χ_π is the character of π of G then $\chi_\pi \mid_H = m_i\chi_{\sigma_i}$, where $\pi\mid_H = \oplus m_i\sigma_i\sigma_i$ irreducible representation of H. By Schur's theory we know that σ_i is determined completely by χ_{σ_i}. By Weyl's character formula we know all the irreducible characters of G and those of H. Also we know χ_i is determined completely by χ_{σ_i}. By Weyl's character formula we know all the irreducible characters of G and those of H. Also we know $\chi\mid_T$ and $\chi_{\sigma_i}\mid_{T\cap H}$ determine χ and chi_{σ_i} (and hence π and σ_i completely). (Note that $T\cap H$ is the maximal torus of H). So it is enough to write down the decomposition $\chi = \sum m_i\chi_{\sigma_i}$ on $T\cap H$. Note that $T\cap H = \{t = t(\theta) = t(\theta_1\cdots\theta_n) : \theta_n = 0\}$ or $t = (t_1, \cdots t_{n-1}, 1. \mid t_i \mid = 1 \forall t \in T\cap H$.

Now the denominator is given by

$$D(t_1, \cdots, t_{n-1}, (t_1 - 1)(t_2 - 1).(t_{n-1} - 1) : \qquad 1$$

Recall that $D(\underline{t})$ can be written as a determinant $\begin{pmatrix} t_1 & \cdots & t_{n-1} & 1 & \\ t_2 & \cdots & 1 & t_1 & \\ t, & t_{n-1} & \cdots & t_2 & t_1 \end{pmatrix}$. So subtract the last column of $D(\underline{t})$ from each of the previous ones and factor the resulting $(n-1)$ order determinant. (Recall $\underline{D(t)}$ = the difference product $= \pi_{i<g}(t_i - t_j)$. Then 1 follows trivially. In fact, the proof is given above).

To divide the numerator by $(t_1 - 1).(t_{n-1} - 1)$. We subtract the 2nd column from the Ist, 3rd from the 2nd, ... and n-th from $(n-1)$ -th. Then the last row is $(0, \cdots 0, 1)$. The definition is then reduced to one of order $n-1$. Now divide each elt. in the j-th row by $(t_j - 1)$, using

$$t^{\ell_1} - t^{\ell_2}/_{t-1} = t^{\ell_1} - 1 + \cdots + t\ell_2.$$

Of course we assume $\chi = \chi_\ell$, where $\ell = (\ell_1 \cdots \ell_n)$

$$\ell_1 > \ell_2 > \cdots > \ell_n, \ell_i \in \mathbb{Z}.$$

Therefore the resulting expression is, for $t \in H \cap T$

$$\chi_{\underline{\ell}}(t) =\mid t^{\ell_1 - 1} + \cdots + t^{\ell_2}, t^{\ell_2 - 1} + \cdots + t^{\ell_3} \cdots \mid .$$

But this is the sum of all $(n-1) \times (n-1)$ determinants of the form $\mid t^{r_1} \cdots t_n^{r_n - 1} \mid$.

$$\ell_1 \supset r_1 \geq \ell_2 > r_2 \geq \ell_3 > \cdots > r_{n-1} \geq \ell_n.$$

Since $\underline{\rho} = \underline{\ell} - \rho$ is the highest weight of the representation with irreducible ch. $\chi_{\underline{\ell}}, f_i = \ell_i - (n-i)$.
So if we subtract $(n-j)$ from r_j, then we get f' as the highest weight of $\chi_{\underline{r}}$, an irreducible character of $V(n-1)$. Thus we have proved

Theorem: If $\underline{f} = (f_1, \cdots f_n)$ is the highest weight of an irreducible representation of G, then that representation when restricted to H becomes a direct sum of irreducible representations of H with highest weight $\underline{f'}$, where f and f' are related as follows: $f_1 \geq f_1^1 \geq f_2 \geq \cdots \geq f'_{n-1} \geq f_{n'}$.

Home Work

1). G a connected loc. cpt. gp. Γ a discrete subgroup of $G\Gamma \subseteq$ center (G). If $[G, G]$ is dense in $G.d$ if $G \mid \Gamma$ is compact then G is compact.

2). G a compact s.s. connected Lie group with Lie algebra . If every representation of is the differential of a representation of G, then G is simply connected.

3). Show that the two sheeted (or any non-trivial) covering group $\tilde{G}$ of $G = SL(2, \mathbf{R})$ does not have a faithful representation.

(Hints: 1) There does exist such non-trivial covering groups of $SL(2,\mathbf{R})$. 2) $G_{\mathbb{C}} = SL(2,\mathbb{C})$ is simply connected. 3). Remember Weyl's theorem on the con. between representation of groups and those of the Lie algebras).

4). Let χ be the character of an irreducible representation of a compact group G. Then

$$\chi(a)\chi(b) = dim_{\chi(e)} \int_G \chi(xax^{-1}b)dx(\int_G dx = 1).$$

Conversely any of its $fn.\varphi$ satisfying $\varphi(a)\varphi(b) = \int \varphi(xax^{-1}bdx$ is, but for a scalar,an irreducible character.

5). Any irreducible representation of a compact group on a Banach (or any 'decent' top vector space) is finite dimensional.

Note: 1) Make an honest attempt to solve those on your own.
2) In case of inability to solve any, ask me to give you hints.