Winding Numbers of Loops

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In this article we define the index or the winding number of a loop in S^1 based at 1. Its geometric interpretation is it quantifies the number of times the loop winds or goes around the circle. We assume the path lifting and homotopy lifting properties of a covering space.

We start with a lemma which is very basic to what follows.

Lemma 1. *Let $p\colon (E, e) \to (B, b)$ be a covering map. Assume that E is simply connected. Then there is a bijection $\varphi\colon \pi_1(B, b) \to p^{-1}(b)$.*

Proof. We define $\varphi\colon \pi_1(B, b) \to p^{-1}(b)$ by setting $\varphi([c]) := \gamma(1)$, where γ is the lift of c starting at e. We need to show that φ is well-defined. That is, if c_1 also represents $[c]$ and γ_1 is left of c_1 starting at e then $\gamma(1) = \gamma_1(1)$.

Let c_0 and c_1 be homotopic loops in B based at b. Let $H := \{c_t\}$ be a homotopy of c_0 to c_1 with end points fixed. Applying the homotopy lifting lemma, we get a lift of H, say, F. The paths $\gamma_t(s) := F(t, s)$ are lifts of the paths c_t. We claim that all the paths start at e. For, observe that the path $t \mapsto F(t, 0)$ is the lift of the constant path at b: $t \mapsto H(t, 0)$. Hence by the uniqueness of the lifts, the lift must coincide with the constant path at e. Similarly, the map $t \mapsto F(t, 1)$ is the unique lift in E, starting at $y := F(0, 1)$ of the constant path at b. Hence $y = F(t, 1)$ for all t. In particular, the lift γ_t of c_t in E starting at e all terminate at y.

We therefore conclude that the terminal point of all loops in the same homotopy class coincide. Thus φ is well-defined.

We now show that φ is onto. Let $y \in p^{-1}(b)$. Let γ be a path in E which goes from e to y. Let $c := p \circ \gamma$. Then c is a loop, γ is a lift of c and $\varphi([c]) = \gamma(1) = y$.

We now show that φ is one-to-one. Let c_0 and c_1 be loops in B based at b with $\varphi([c_0]) = \varphi([c_1])$. Let γ_i be lifts of c_i in E starting at e. Since $\varphi([c_i]) = \gamma_i()$, we see that $\gamma_0(1) = \gamma_1(1)$. Consequently, the path $\gamma_0\gamma_1^{-1}$ is a loop at e. Since E is simply connected, there is a homotopy F from $\gamma_0\gamma_1^{-1}$ to the constant loop at e. But then $H := p \circ F$ is a homotopy from $c_0 \circ c_1^{-1}$ to the constant loop at b. In particular, $[c_0] = [c_1]$ and hence φ is one-to-one. $\square$

Definition 2. Let $p\colon (\mathbb{R}, 0) \to (S^1, 1)$ be the covering map given by $p(t) = e^{2\pi i t}$. Let c be a loop in S^1 based at 1. Then the lift of c is a map $h\colon [0, 1] \to \mathbb{R}$ such that $h(0) = 0$ and $e^{2\pi i h(t)} = c(t)$, for $0 \le t \le 1$. The terminal point $h(1)$ is necessarily an integer, which is called the *index* of c. We denote this integer by $\operatorname{ind}(c)$. Thus $\operatorname{ind}(c) \in p^{-1}1$. Note that the index depends only on the homotopy class of the loop in view of Lemma 1.

Theorem 3. *1) Two loops c and γ in S^1 based at 1 are in the same homotopy class iff they have the same index.*
2) The map $[c] \mapsto \operatorname{ind}(c)$ is an isomorphism of $\pi_1(S^1, 1)$ and $\mathbb{Z}$.

Proof. 1) is already observed and is included here only for reference.

2) We need only show that φ is a homomorphism: $\operatorname{ind}(c_1 \cdot c_2) = \operatorname{ind}(c_1) + \operatorname{ind}(c_2)$ for loops c_i based at 1. Choose lifts $h_j\colon [0,1] \to \mathbb{R}$ such that $h_j(0) = 0$ and $c_j(t) = e^{2\pi i h_j(t)}$, for $0 \le t \le 1$. Define

$$h(t) := \begin{cases} h_1(2s), & 0 \le t \le 1/2, \\ h_1(1) + h_2(2t-1), & 1/2 \le t \le 1. \end{cases}$$

Then h is continuous, $h(0) = 0$ and $(c_1 \cdot c_2)(t) = e^{2\pi i h(t)}$ for $0 \le t \le 1$. Consequently, $\operatorname{ind}(c_1 \cdot c_2) = \operatorname{ind}(c_1) + \operatorname{ind}(c_2)$. □

We now give applications of the index. We regard the unit ball B in $\mathbb{R}^2$ as the unit disk in $\mathbb{C}$. The applications will depend on the following

Lemma 4. *Let $f\colon B \to S^1$ be a map such that $f(1) = 1$. Then the loop c defined by $c(s) = f(e^{2\pi i s})$, $0 \le s \le 1$ has index 0.*

Proof. Define the loop $\gamma(s) := e^{2\pi i s}$ in B. Then $c = f \circ \gamma$. Since B is convex, c is homotopic to the constant loop at 1 in S^1. Hence $\operatorname{ind}(c) = 0$ by Thm. 3. □

Theorem 5 (No Retraction Lemma)**.** *There is no map $f\colon B \to S^1$ such that $f(z) = z$ for all $z \in S^1$.*

Proof. If there were such a map f, then the loop $s \mapsto e^{2\pi i s}$ in S^1 would have index 0 by the last lemma. However the index of this loop is 1. □

We indicate a proof which is more standard in Algebraic Topology. Let $j\colon S^1 \hookrightarrow B$ be the inclusion. Since $f \circ j$ is the identity map of S^1, $(f \circ j)_* = f_* \circ j_*$ is the identity of $\mathbb{Z} \simeq \pi_1(S^1, 1)$. However, since $\pi_1(B, 1) = 0$, both f_* and j_* are both zero homomorphisms.

Theorem 6 (Brouwer Fixed Point Theorem)**.** *Any map $f\colon B \to B$ has a fixed point.*

Proof. Suppose that f has no fixed point. For each $z \in B$, let $g(z)$ be the point of S^1 at which the line starting from $f(z)$ and passing through z meets S^1. One shows that $g\colon B \to S^1$ is a continuous map such that $g(z) = z$ for all $z \in S^1$. This contradicts Thm. 5. □

Theorem 7 (Borsuk-Ulam Theorem)**.** *Let $f\colon S^2 \to \mathbb{R}^2$ be map. Then there exist antipodal points $\pm x$ such that $f(x) = f(-x)$.*

Proof. Define a map $g\colon S^2 \to \mathbb{R}^2$ by $g(x) = f(x) - f(-x)$. We must show that g vanishes at some point of S^2. Note that $g(-x) = -g(x)$.

Consider the map $h\colon B \to \mathbb{R}^2$ by $h(x, y) := (g(x, y, \sqrt{1 - x^2 - y^2})$. We note that $h(-z) = -h(z)$, for $z \in S^1$. We plan to show that any map h from B to $\mathbb{R}^2$ satisfying $h(-z) = -h(z)$ vanishes at some point of B.

Suppose that such an h does not vanish at all on B. Then

$$\varphi(z) := \frac{h(z)|h(1)|}{|h(z)|h(1)}, \qquad z \in B,$$

is map $\varphi\colon B \to S^1$ with the following properties: (i) $\varphi(-z) = -vfi(z)$, (ii) $\varphi(1) = 1$. By Lemma 1, the loop $c(s) := \varphi(e^{2\pi i s})$ has index 0.

We arrive at a contradiction by showing that the index of c is odd.

Let $k\colon [0, 1] \to \mathbb{R}$ be the lift of φ starting at 0. Then $\operatorname{ind}(\varphi) = k(1)$. Property (i) of φ shows that

$$\exp(2\pi i k(s + 1/2)) = -\exp[2\pi i k(s)] = \exp[2\pi i\,(k(s) + 1/2)], \qquad 0 \le s \le 1/2.$$

For each fixed $s \in [0, 1/2]$, the number $k(s + 1/2) - k(s) - 1/2$ is an integer. Since this is a continuous function of s and has discrete range, it is a constant, say, n. Thus

$$k(s + 1/2) - k(s) = n + 1/2, \qquad 0 \le s \le 1/2.$$

Then

$$\begin{aligned} \operatorname{ind}(c) = k(1) &= k(1) - k(1/2) + k(1/2) - k(0) \\ &= n + 1/2 + n + 1/2 = 2n + 1. \end{aligned}$$

This contradiction establishes the theorem. □

A physical interpretation of this result is that at any given instant of time, there are two antipodal points on the surface of the earth at which the temperatures are the same.

Corollary 8. *No subset of $\mathbb{R}^2$ is homeomorphic to S^2.*

Proof. Thm. 7 tells us that there can be no one-to-one continuous map of S^2 into $\mathbb{R}^2$. □

Corollary 9 (Invariance of Domain)**.** *The open disks in $\mathbb{R}^2$ cannot be homeomorphic to open balls in $\mathbb{R}^n$ for $n \ge n$.*

Proof. If $f\colon D^n \to D^2$ is a homeomorphism, then f maps $S^{n-1} = \partial D^n$ homeomorphically into D^2. Whence it follows that there exists a homeomorphism of S^2 onto a subset of $\mathbb{R}^2$. This contradicts Corollary 8. □

The next result is concerned with the division of volumes of objects in $\mathbb{R}^3$ by planes. It derives its picturesque name from its interpretation: It is possible, with a single knife stroke, to cut two pieces of bread and a piece of ham each into equal halves, no matter how irregular their shapes are or how askew their relative locations.

Theorem 10 (Ham Sandwich Theorem)**.** *Let E_i, $1 \leq i \leq 3$, be Lebesgue measurable nonempty subsets of $\mathbb{R}^3$. Then there exists a plane in $\mathbb{R}^3$ which divides each of them into sets of equal measure.*

Proof. Let $u \in S^2$. Let L be the line $\mathbb{R}u$. Then there is a unique point lying on L such that the plane through the point and perpendicular to u divides E_1 into two parts of equal volume. We indicate a plausible reason for this. Intuitively, one moves a point tu forward along L and observes that the part of E_1 behind the plane is a continuous function of t which increases from 0 to $m(E_1)$. The point required is the point tu at which $h(p)$ is exactly half the volume $m(E_1)/2$. We denote by $g_1(u)$ this value of t so that g_1 is a map from S^2 to $\mathbb{R}$. One knows that g_1 is continuous and by very construction $g_1(-u) = -g_1(u)$ for $u \in S^2$. Similarly, g_i, $i = 2, 3$ are defined. We want to show that there is a $u \in S^2$ at which all g_i's take the same value.

Define a map $f(u) := (g_1(u) - g_2(u), g_1(u) - g_3(u))$ from S^2 to $\mathbb{R}^2$. It suffices to show that f vanishes at some point of S^2. Using the fact that $g_i(-u) = -g_i(u)$ for $1 \leq i \leq 3$, we infer that $f(-x) = -f(x)$ for all $x \in S^2$.

By Borsuk-Ulam theorem, there is a point $u \in S^2$ such that $f(-u) = f(u)$. But since $f(-u) = -f(u)$, we see that $f(u) = 0$. $\square$

Ex. 11. Let A, B and C be open balls in $\mathbb{R}^3$. Describe the plane in $\mathbb{R}^3$ which divides each of the balls in half by volume. When is the plane unique?

Ex. 12. If A and B are bounded connected open sets in $\mathbb{R}^3$ and $p \in \mathbb{R}^3$, then there exists a plane passing through p and dividing the two sets in half by volume.

Ex. 13. A space X is said to have the *fixed point property* iff any continuous map from X to itself has a fixed point. Show that if X has fixed point property and Y is homeomorphic to X then Y has fixed point property.

Ex. 14. A subspace Y of X is a retract of X if there is a continuous map $f\colon X \to Y$ with $f(y) = y$ for all $y \in Y$. Show that the unit ball $B^n \subset \mathbb{R}^n$ has the fixed point property iff its boundary S^{n-1} is a retract of B^n.

Ex. 15. Let A be a 3×3 matrix with positive real entries. Then A has a positive real eigenvalue.