Axiom of Choice and its Equivalents

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Let $\{X_i : i \in I\}$ be a family of nonempty sets. A *choice function* f is a map $f : I \to \cup_i X_i$ such that $f(i) \in X_i$ for each $i \in I$. The axiom of choice says that there always exists a choice function. Another way of saying is that the Cartesian product $\prod_{i \in I} X_i$ is nonempty.

Why do we need the axiom of choice? When set theoretic concepts were being introduced, there was a need for an assurance that if an infinite collection of nonempty sets are give, we can choose an element from each set of the collection. To understand this, let an infinite collection of pairs of shoes be given. Then we can select the left shoe from each of this pair. On the other hand, if we are given an infinite collection of pairs of socks, there is not definite way of selecting a sock from each of the pairs. We belabour the idea by a few more examples. If we are given a collection of nonempty subsets of $\mathbb{N}$, we can select the least element from each subset. If we are given a collection of nonempty intervals, we can choose the mid-point of each interval. If we are given a collection of nonempty subsets of $\mathbb{R}$, there is no definite choice of elements from each set of the collection!

Definition 1. A relation $\leq$ on a set X is said to be a *partial order* if (i) it is reflexive, that is, $x \leq x$ for all $x \in X$, (ii) is is transitive, that is, if $x \leq y$ and $y \leq z$, then $x \leq z$ and (iii) it is antisymmetric, that is, if $x \leq y$ and $y \leq x$, then $x = y$.

Let $(P, \leq)$ be a partially ordered set. A subset C is said to be a *chain* if $(C, \leq)$ is a totally ordered set.

A chain C is a partially ordered set P is said to be an *initial segment* if $x, y \in P$ with $x \leq y$ and $y \in C$, then $x \in C$, that is, C contains along with any element $y \in C$ all elements which are less than y.

A totally ordered set C is said to be *well-ordered* iff every nonempty subset of C has a least element.

Ex. 2. Give at least a couple of examples for each of the concepts above.

Ex. 3. A subset A of a well-ordered chain C is an initial segment iff either $A = C$ or there exists an element $x \in C$ such that $A = \{y \in C : y < x\}$.

Lemma 4. *If $\mathcal{C}$ is a collection of well-ordered subsets of a partially ordered set P such that for all $A, B \in \mathcal{C}$, either A is an initial segment of B or B is an initial segment of A, then $C := \cup_{X \in \mathcal{C}} X$ is well-ordered. Furthermore, any $A \in \mathcal{C}$ is an initial segment of C.*

Proof. First of all observe that C is totally ordered. For, if $x, y \in C$, then there exist chains A and B such that $x \in A$ and $y \in B$. Using the hypothesis, we may assume without loss of generality that A is an initial segment of B. Since B is a totally-ordered, it follows that x and y can be compared.

We show that C is well-ordered. Let $\emptyset \subset A \subset C$. Then $A \cap B \neq \emptyset$ for some $B \in \mathcal{C}$. Then for any such B, the nonempty subset $A \cap B$ of the well-ordered set B has a least element. Call it m_B. We claim that all these elements m_B coincide. Let B and D be any two elements of $\mathcal{C}$ such that $A \cap B \neq \emptyset$ and $A \cap D \neq \emptyset$. Without loss of generality we may assume that B is an initial segment of D. Hence $A \cap B \subset A \cap D$. Hence $m_B \geq m_D$. Since B is an initial segment of D and $m_D \leq m_B$ it follows that $m_D \in B$. Hence $m_D \in A \cap B$. Since m_B is the minimum of $A \cap B$, we see that $m_D \geq m_B$. We therefore conclude that $m_B = m_D$ thereby establishing our claim. Let m be the common element m_B for any $B \in \mathcal{C}$ with $A \cap B \neq \emptyset$. We claim that m is the least element of A. Let $a \in A$. Let $B \in \mathcal{C}$ be such that $a \in B$. Then $a \in A \cap B$ and hence $m = m_B \leq a$

To complete the proof, we need to show that any $B \in \mathcal{C}$ is an initial segment of C. Let $b \in B$. Let $x \in C$ be such that $x \leq b$. We must show that $x \in B$. Let $D \in \mathcal{C}$ be such that $x \in D$. Now there are two possibilities: (i) either B is an initial segment of D or (ii) D is an initial segment of B. In the first case, since $b \in B$ and $x \leq b$, it follows that $x \in B$. In the second case, $D \subset B$ and hence $x \in B$! □

Theorem 5. *The following are equivalent:*

(i) (Axiom of Choice) *If $\{X_i : i \in I\}$ is a nonempty family of nonempty sets, then $\prod_{i \in I} X_i$ is nonempty.*

(ii) (Zorn's Lemma) *If P is a nonempty partially ordered set with the property that very chain in P has an upper bound then P has maximal element.*

(iii) (Well-Ordering Principle) *Given any set X, there exists an order $\leq$ on X such that $(X, \leq)$ is a well-ordered set.*

Proof. We shall prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). Let P be as in the statement (ii). We shall prove the result by contradiction. Let us therefore assume that P has no maximal element. If S is any chain in P, let $B(S)$ denote the set of upper bounds for S. We observe that $B(S)$ contains at least one upper bound for S that does not lie in S. Assume on the contrary that all the upper bounds for S lie in S. Let b be an upper bound for S. Then we claim that b is a maximal element of P. For, otherwise, there is a $\beta \in P$ such that $\beta > b$. But then β is an upper bound for S which cannot lie in S since $\beta > b$ and b is an upper bound for S. Let $B'(S)$ denote the set of upper bounds for S that do not belong to S. By the axiom of choice, there exists a function φ from the set $\mathcal{S}$ of all chains in P to the set $\{B'(S) : S \in \mathcal{S}\}$ such that for each $S \in \mathcal{S}$, $\varphi(S)$ is an upper bound for S that does not lie in S.

Now fix an element $a \in P$. Let $\mathcal{C}$ denote the set of all subsets S of P with the following properties:

(a) S is a well-ordered chain of P.

(b) a is the least element of S.

(c) For any proper nonempty initial segment $I_x \subset S$, the least element of $S \setminus I_x$ is $\varphi(I_x)$.

Observe that $\{a\} \in \mathcal{C}$ so that $\mathcal{C} \neq \emptyset$. It is easily seen that if S and T are in $\mathcal{C}$, then one of them is an initial segment of the other. For, if A is the union of all common initial segments of S and T, then $A = I_x = I_y$ for some $x \in S$ and $y \in T$. If A is a proper subset of both S and T, then $\varphi(A)$ is the least element of $S \setminus A$ and $T \setminus A$. Note that by (c), the element $\varphi(A)$ lies in S as well as T. But then $A \cup \{\varphi(A)\}$ will be an initial segment common to both S and T. This contradicts the 'maximality' of A. Hence A must be equal to one of S and T, say S. Then S is an initial segment of T.

We now let $C := \cup_{S \in \mathcal{C}} S$. By the lemma, C is a well-ordered chain satisfying the conditions (a) and (b). It also satisfies the condition (c) (and hence is a member of $\mathcal{C}$). For, if I_x is a proper initial segment of C, then there exists $y \in C \setminus I_x$. By the very definition of C, we can find $B \in \mathcal{C}$ such that $y \in B$. By what we observed above, I_x is a proper initial segment of B. Hence, by (c), $\varphi(I_x)$ is the least element of $B \setminus I_x$. As B itself is an initial segment of C, $\varphi(I_x)$ is also the least element of $C \setminus I_x$. Thus C satisfies (a)-(c) and hence is an element of $\mathcal{C}$.

If $C' := C \cup \{\varphi(C)\}$, then C' also lies in $\mathcal{C}$. Hence $C' \subset C$ and in particular, $\varphi(C) \in C$, a contradiction to our definition of φ. This completes the proof of (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii). Let X be given. Let $\mathcal{A}$ be the set of all well-ordered subsets $(A, \leq_A)$ of X. Then $\mathcal{C}$ is nonempty. We define a partial order on $\mathcal{C}$ by setting $(A, \leq_A) \preceq (B, \leq_B)$ iff $A \subset B$, $\leq_B$ is an 'extension' of $\leq_A$ and $(A, \leq_A)$ is an initial segment of $(B, \leq_B)$. It is easy to see that one can apply Zorn's lemma to get a maximal element $(Y, \leq_Y)$ in $\mathcal{A}$. If $Y \neq X$, then choose $x_0 \in X \setminus Y$. Then $Y_0 := Y \cup \{x_0\}$ admits an obvious extension $\leq_0$ of $\leq_Y$ such that $(Y, \leq_Y) \preceq (Y_0, \leq_0)$, contradicting the maximality of Y.

(iii) $\Rightarrow$ (i). Let a family $\{X_i : i \in I\}$ of nonempty sets be given. By (ii) there exists a well-order on each of the X_i's. Let $f(i)$ be the least element of X_i with respect to this order. Then f is a choice function. $\square$

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