A Problem Course in Metric Spaces

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Abstract: The aim is to give a very streamlined development of a course in metric space topology emphasizing the most useful concepts, concrete spaces and geometric ideas. A secondary aim is to treat this as a preparatory ground for a general topology course. It is expected that the teachers will use this set in conjunction with some of my other articles on topics such as subspace topology, compact spaces, connected spaces and Baire category theorem and of course, other books on metric spaces and topology. The discerning experts will perceive that I have preferred definitions that use only the primitive concept of an open ball rather than secondary concepts. (See, for instance, the definitions of open sets, bounded sets, dense sets etc.) Teachers who use this outline should feel free to use their discretion to change the order of the topics and smoothen out the rough ends of the notes. I usually prefer to introduce continuity as early as possible, immediately after open sets and convergent sequences are introduced.

Definition and Examples

Exercise 1. Define a metric space.

Exercise 2. Prove that $|a+b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$. When does the equality holds? Conclude that $(\mathbb{R}, d)$, where $d(x,y) = |x-y|$, is a metric space.

Exercise 3. Define an inner product $\langle\ ,\ \rangle$ on a real (complex) vector space. State and prove Cauchy-Schwarz inequality:

$$|\langle x, y\rangle| \leq \|x\| \, \|y\| \text{ for all } x, y.$$

When does the equality hold?

Exercise 4. Define the norm associated with an inner product. Deduce the triangle inequality: $\|x+y\| \leq \|x\| + \|y\|$, for all x, y in the inner product space.

Exercise 5. Define a norm on a (real/complex) vector space and a normed linear space (nls). Define the associated metric.

Exercise 6. Show that the following are norms on $\mathbb{R}^n$:
(i) $\|x\|_1 := \sum_{k=1}^n |x_k|$.
(ii) $\|x\|_\infty := \max\{|x_k| : 1 \leq k \leq n\}$.
(iii) $\|x\|_2 := (\sum_{k=1}^n |x_k|^2)^{1/2}$. This is called the Euclidean norm. In the sequel unless otherwise specified, we shall assume that $\mathbb{R}^n$ is equipped with this norm.

Exercise 7. We generalize the above to suitable spaces of functions as follows:

(i) Let X be a nonempty set. Let $B(X)$ be the set of all bounded real (or complex) valued functions. Then $\|f\|_\infty := \sup\{|f(x)| : x \in X\}$ defines a norm on $B(X)$.

(ii) Let $X := [0,1]$, the closed unit interval. Then

$$\|f\|_1 := \int_0^1 |f(t)|\, dt$$

defines a norm on the set of all continuous real/complex valued functions on $[0,1]$.

(iii) Let $X := [0,1]$, the closed unit interval. Then

$$\|f\|_s := \left(\int_0^1 (|f(t)|)^2\, dt\right)^{1/2}$$

defines a norm on the set of all continuous real/complex valued functions on $[0,1]$.

Exercise 8. Let the notation be as in the last exercise. Find the distance between the functions $f(t) := t$ and $g(t) = t^2$ for $t \in [0,1]$. What are their distances in the metrics induced by the norms $\| \ \|_\infty$ and $\| \ \|_1$?

Exercise 9 (Young's inequality)**.** Let x, y be nonnegative real numbers. Let $p > 1$ and q be defined in such a way that $\frac{1}{p} + \frac{1}{q} = 1$ holds. (If $p = 2$, what is q?) Prove Young's Inequality:

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

When does equality hold?

Exercise 10 (Hölder's inequality)**.** Let X be $\mathbb{K}^n$ and, for $1 \leq p < \infty$, let $\|x\|_p := (\sum_i |x_i|^p)^{1/p}$ and for $p = \infty$, let $\|x\|_\infty := \max\{|x_i| : 1 \leq i \leq n\}$. For $p > 1$, let q be such that $(1/p) + (1/q) = 1$. For $p = 1$ take $q = \infty$. Prove Hölder's inequality:

$$\sum_i |a_i|\,|b_i| \leq \|a\|_p \|b\|_q\,, \text{ for all } a, b \in \mathbb{K}^n.$$

Hint: Take $x = \frac{|a_i|}{\|a\|_p}$ and $y = \frac{|b_i|}{\|b\|_q}$ and sum over i.

When does equality occur?

Exercise 11 (Minkowski inequality)**.** Let $1 \leq p < \infty$. For $x \in \mathbb{R}^n$, let $\|x\|_p := (\sum_{k=1}^n |x_k|^p)^{1/p}$. Show that this defines a norm on $\mathbb{R}^n$. Prove that $(\mathbb{R}^n, \| \ \|_p)$ is a normed linear space for $1 \leq p \leq \infty$. $\| \ \|_p$ is called the L^p-norm on $\mathbb{R}^n$. *Hint:* For $1 < p < \infty$, observe

$$\begin{aligned}\sum_i |a_i + b_i|^p &= \sum_i |a_i + b_i|\,|a_i + b_i|^{p-1} \\ &\leq \sum_i |a_i|\,|a_i + b_i|^{p-1} + \sum_i |b_i|\,|a_i + b_i|^{p-1}.\end{aligned}$$

Apply Holder's inequality to each of the summands. The triangle inequality for $\| \ \|_p$ is called Minkowski inequality.

Exercise 12 (p-adic valuation on $\mathbb{Q}$). Let p be a prime. If x is a rational number then x can be uniquely written as $x = p^n(a/b)$ where n, a, b are integers, $b > 0$, a and b are relatively prime to each other and to the prime p. Define $|x|_p := p^{-n}$ if $x \neq 0$ and $|0|_p = 0$. Then $x \mapsto |x|_p$ is a called p-adic valuation on $\mathbb{Q}$ and $d_p(x, y) = |x - y|_p$ is a metric on $\mathbb{Q}$.

Exercise 13. Let X be a nonempty set. Define $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. Show that d is a metric on X, called the discrete metric.

Exercise 14. Let (X, d) be a metric space. Let $A \subset X$ be nonempty. Define for $x, y \in A$, $\delta(x, y) := d(x, y)$. Then δ is a metric on A, called the induced metric on the subset A.

Exercise 15. Let (X, d) be a metric space. Define $\delta(x, y) := \min\{1, d(x, y)\}$ for all $x, y \in X$. Show that δ is a metric on X.

Exercise 16. Let (X, d) be a metric space. Define

$$\delta(x, y) := \frac{d(x, y)}{1 + d(x, y)}, \text{ for all } x, y \in X.$$

Show that δ is a metric on X.

Exercise 17 (Product Metric). Let (X, d) and (Y, d) be metric spaces. Show that

$$d((x_1, y_1), (x_2, y_2)) := \max\{d(x_1, x_2), d(y_1, y_2)\}$$

defines a metric on the product set $X \times Y$. Can you think of other metrics on $X \times Y$ coming from the original metrics on X and Y?

Exercise 18. Suppose (X, d) and (Y, δ) be metric spaces. Is there a metric on $X \cup Y$ induced by d and δ? (Assume $X \cap Y = \emptyset$).

Exercise 19. Let $M(n, \mathbb{R})$ denote the set of all $n \times n$ real matrices. We identity any $A = (a_{ij}) \in M(n, \mathbb{R})$ with the vector

$$(a_{11}, a_{12}, \ldots, a_{1n}, \ldots, a_{n1}, \ldots, a_{nn}) \in \mathbb{R}^{n^2}.$$

This map is a linear isomorphism between $M(n, \mathbb{R})$ and $\mathbb{R}^{n^2}$. Using this linear isomorphism, we define

$$\| A \| := (\sum_{i,j} |a_{ij}|^2)^{1/2} = \| (a_{11}, \ldots, a_{nn}) \| .$$

Thus $M(n, \mathbb{R})$ is an nls.

Exercise 20. Let $1 \leq p < \infty$. Let ℓ_p be defined as follows:

$$\ell_p := \{(a_n)_{n=1}^{\infty} : a_n \in \mathbb{R} \text{ or } \mathbb{C}, \text{ and } \sum_{n=1}^{\infty} |a_n|^p < \infty\}.$$

Show that ℓ_p is a normed linear space with the norm $\| (a_n) \| := (\sum_{n=1}^{\infty} |a_n|^p)^{1/p}$.

Let ℓ_∞ stand for $(B(\mathbb{N}), \| \ \|_\infty)$.

Open Balls and Open Sets

Exercise 21. Define open balls $B(x, r)$, closed balls $B[x, r]$.

Exercise 22. Let (X, d) be a discrete metric space and $x \in X$. Find the following: (a) $B(x, 1/2)$, (b) $B(x, 3/4)$, (c) $B(x, \sqrt{2})$, (d) $B(x, 10^2+1)$, (e) $B(x, 1/e)$ and (f) $B(x, r)$, $0 < r \leq 1$.

Exercise 23. Draw figures of the open unit ball $B(0, 1)$ in the following metric spaces.
(a) $(\mathbb{R}^2, \| \ \|_2)$, (the standard Euclidean norm).
(b) $(\mathbb{R}^2, \| \ \|_1)$, (the L^1-norm).
(c) $(\mathbb{R}^2, \| \ \|_\infty)$, (the max or sup norm).

Exercise 24. Show that in an nls $(X, \| \ \|)$, we have

$$B(x, r) = x + rB(0, 1), \quad x \in X, r > 0.$$

Thus if we know the unit ball in an nls, we know all the open balls!

Exercise 25. Let $X = I = [0, 1]$. Let $V := C[0, 1]$ be the nls of continuous real valued functions on $[0, 1]$ under the sup norm. How will you visualize $B(0, \varepsilon)$?

Exercise 26. A subset $U \subset X$ of a metric space is said to be *open* if for each $x \in U$, there exists $r > 0$ such that $B(x, r) \subset U$. Find all open sets in a discrete metric space.

Exercise 27. Prove that an open interval in $\mathbb{R}$ is open.

Exercise 28. Show that any open ball in a metric space is open.

Exercise 29. Is the empty set $\emptyset \subset X$ open in the metric space (X, d)?

Exercise 30. Let $U := \mathbb{R} \setminus \mathbb{Z}$. Is U open in $\mathbb{R}$?

Exercise 31. Show that $U := \{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$ is open in $\mathbb{R}^2$.

Exercise 32. Let $U := \{(x, y) \in \mathbb{R}^2 : x > 0 \ \& \ y > 0\}$. Draw the figure of this subset. Is this open? Prove your claim.

Exercise 33. Let $U := \{(x, y) \in \mathbb{R}^2 : x \notin \mathbb{Z}, y \notin \mathbb{Z}\}$. Is U open in $\mathbb{R}^2$. Substantiate your claim.

Exercise 34. Let $E := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\}$. Show that E is open in $\mathbb{R}^2$. (Think geometrically and then make your ideas rigorous.)

Exercise 35. Let A be any finite set in a metric space (X, d). Show that $X \setminus A$ is open.

Exercise 36. Is $\mathbb{Q}$ open in $\mathbb{R}$? How about the set of irrationals?

Exercise 37. Let $\{U_i : i \in I\}$ be a family of open sets in a metric space (X, d). Show that the union $\cup_{i \in I} U_i$ is open in X.

Exercise 38. Let U_i, $1 \leq i \leq n$, be open in (X, d). Show that $\cap_{i=1}^n U_i$ is open in X.

Exercise 39. If $(X, \| \ \|)$ is an nls, U is open in X, then $x + U$ is open for any $x \in X$.

Exercise 40. If $(X, \| \ \|)$ is an nls, U is open in X, then $A + U$ is open for any set $S \subset X$.

Exercise 41. Let $(X, \| \ \|)$ be an nls. Show that if any vector subspace Y of X is open, then $Y = X$.

Can you generalize this?

Exercise 42. Let (X, d) be a metric space. Let $\mathcal{T}$ denote the set of all open sets in X. Show that $\mathcal{T}$ has the following properties:

(i) $\emptyset, X \in \mathcal{T}$.

(ii) Arbitrary union of members of $\mathcal{T}$ again lies in $\mathcal{T}$.

(iii) The intersection of any finite number of members of $\mathcal{T}$ lies again in $\mathcal{T}$.

$\mathcal{T}$ is called the topology determined by d.

Let X be a nonempty set. Assume that we are given a collection $\mathcal{T}$ of subsets of X satisfying the above properties (i)–(iii). Then $\mathcal{T}$ is called a topology on X and members of $\mathcal{T}$ are called the open sets in the topology $\mathcal{T}$.

Show that on any nonempty set X there always exist topologies.

Exercise 43. Let $X := C[0, 1]$ with the sup norm metric $\| \ \|_\infty$. Let E be the set of all functions in X that do not vanish (that is, they do not take the value 0) at $t = 0$. Is E open in X?

Exercise 44. Let X, Y be metric spaces. Consider the product set $X \times Y$ with the product metric (Ex. 17.) Let U (resp. V) be an open set in X (resp. Y). Show that $U \times V$ is open in $X \times Y$.

Exercise 45. Keep the notations above. Let $W \subset X \times Y$ be open in the product metric. Let p_X and p_Y denote the projections of $X \times Y$ onto X and Y respectively. Show that $p_X(W)$ (resp. $p_Y(W)$) is open in X (resp. in Y).

Exercise 46. Let (X, d) be a metric space. Define δ on X as in Ex. 15. Show that a subset U is d-open iff it is δ-open.

Exercise 47 (Interior of a set)**.** Let $A \subset X$ be a subset of a metric space. We say that $x \in A$ is an *interior* point of A if there exists $r > 0$ such that $B(x, r) \subset A$. The set of interior points of A is denoted by A^0. Prove the following:

(a) A is open iff each of its points is an interior point, that is iff $A = A^0$.

(b) For any set A, the set A^0 is the largest open set contained in A.

Exercise 48. Let $A = [0, 1) \subset \mathbb{R}$ have the induced metric from $\mathbb{R}$. Find $B_A(0, r)$ for any $r > 0$. Here $B_A(x, r)$ stands for the open ball *in* A centred at x and radius r w.r.t. to the induced metric.

Exercise 49. Let $X = \{(x, y) : x \geq 0, y \geq 0\}$ be endowed with the induced metric as a subset of $\mathbb{R}^2$ with the Euclidean metric. Draw $B_{(X,d)}(\mathbf{0}, 1)$.

Exercise 50. Let (X, d) be a metric space. Let $A \subset X$ and $a \in A$. Let $B_A(a, r)$ denote the open ball in A with the induced metric. Give a description of $B_A(a, r)$.

Exercise 51 (Subspace Topology)**.** Let Y be subset of a metric space. A subset $A \subset Y$ is said to be *open in* Y if it is an open subset of the metric space (Y, d) where d is the induced metric on Y. Show that $A \subset Y$ is open in Y iff there exists an open set U in X such that $A = Y \cap U$.

Show that the collection $\mathcal{T}_Y$ of all sets open in Y is a topology on Y, called the subspace topology.

Exercise 52. Let $X = \mathbb{R}$ and $Y = \mathbb{Z}$. Which subsets of $\mathbb{Z}$ are open in $\mathbb{Z}$?

Exercise 53. Let $Y \subset X$ be open in X. Then $Z \subset Y$ is open in Y iff Z is open in X.
The result is not true if Y is not open in X

Closed Sets

Exercise 54. Define a closed subset of a metric space as the complement of an open set. Show that $\emptyset$ and X are both open and closed in any metric space.

Exercise 55. Any finite subset of a metric space is closed.

Exercise 56. Let (X, d) be a discrete metric space. Find all closed sets in X.

Exercise 57. Let $X = \mathbb{R}^2$ with the standard metric. Let A be the union of the x and y-axes, that is the set $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$. Show that A is closed. You are required to find the best possible $r_p > 0$ such that $B(x, r_p) \cap A = \emptyset$ for $p \notin A$.

Exercise 58. Let $S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle in $\mathbb{R}^2$. Show that S^1 is closed. You are required to find the best possible $r_p > 0$ such that $B(x, r_p) \cap S^1 = \emptyset$ for $p \notin S^1$.

Exercise 59. Is the set $\mathbb{Q}$ of rationals closed in $\mathbb{R}$? How about the set of irrationals?

Exercise 60. Show that arbitrary intersections of closed sets is closed and a finite union of closed sets is closed. Find "counterexamples" to obvious generalizations.

Exercise 61. Let E be the xy-plane in $\mathbb{R}^3$. Is it closed? More generally, if P is any plane, say, given by $ax + by + cz = d$, is it closed? (Geometric thinking is encouraged.)

Exercise 62. Show that any closed ball $B[x, r] := \{x' \in X : d(x, x') \leq r\}$ is a closed set.

Exercise 63. Let Δ denote the sides along with the "inside" of the triangle whose vertices are at $(-1, 0)$, $(1, 0)$ and $(0, 1)$. Show that Δ is closed.

Exercise 64. If F is a closed subset $\mathbb{R}^n$ and $x \in \mathbb{R}^n$, is $x + F$ still closed? Can you generalize this question?

Exercise 65. Show that in an nls, if X is closed and λ is scalar, then λF is closed.

Exercise 66. Show that there exist closed sets F and C in $\mathbb{R}^n$ such that their sum $F + C$ is not be closed. (Think geometrically in $n = 2$. Think of a hyperbola and the line passing through its vertices.)

Exercise 67. $\star$ Show that $\mathbb{Z} + \sqrt{\mathbb{Z}}$ is not closed in $\mathbb{R}$.

Exercise 68. Show that the only nonempty subset of $\mathbb{R}$ which is both open and closed in $\mathbb{R}$ is $\mathbb{R}$. (Later we shall see this in other disguise.)

Exercise 69 (Closure of a set)**.** Let (X, d) be a metric space and $A \subset X$. Let $\lim(A)$ denote the set of all limit points of A. Show that it is the smallest closed set containing A. It is denoted by $\overline{A}$. It is called the *closure* of A.

Exercise 70. Show that in a metric space $\overline{B(x,r)} \subset B[x,r]$. Give an example to show that $\overline{B(x,r)}$ can be a proper subset of $B[x,r]$.

Exercise 71. In $\mathbb{R}^n$, show that $\overline{B(x,r)} = B[x,r]$.

Exercise 72. Investigate the relation between the closures of the sets $A \cup B$, $A \cap B$, $A \subset B$ and the sets $\overline{A}$, $\overline{B}$.

Convergent Sequences and Limit Points

Exercise 73. Define a convergent sequence in (X,d). Show that any subsequence of a convergent sequence converges to the limit of the given sequence.

Exercise 74. Show that the limit of a sequence in a metric space is unique.

Exercise 75. Show that a (x_n) converges to x in a metric space iff the sequence $d(x_n,x)$ converges to .... Complete the sentence and prove it.

Exercise 76. Let $x_k = (x_{k1},\dots,x_{kn}) \in \mathbb{R}^n$. Then x_k converges to $x = (x_1,\dots,x_n) \in \mathbb{R}^n$ iff $x_{ki} \to x_i$ as $i \to \infty$ for each k.

Exercise 77. Let $x_k \in \mathbb{R}^n$ converge to $x \in \mathbb{R}^n$. Show that $\|x_k\| \to \|x\|$. Is the converse true?

Exercise 78. Let $x_k \to x$ and $y_k \to y$ in $\mathbb{R}^n$. Prove that $x_k + y_k \to x + y$ and that $\langle x_k, y_k\rangle \to \langle x, y\rangle$.

Exercise 79. What does it mean to say that a sequence of inetegers converge to 0 in $\mathbb{Q}$ with the p-adic metric?

Exercise 80. Let X, Y be metric spaces. Let $X \times Y$ be endowed with the product metric. Show that a sequence $(x_n, y_n) \in X \times Y$ converges to $(x,y) \in X \times Y$ iff $x_n \to x$ in X and $y_n \to y$ in Y.

Exercise 81. Let $X = C[0,1]$ with sup norm $\| \ \|_\infty$. Show that a sequence f_n converges to $f \in X$ iff f_n converges to f uniformly on $[0,1]$. For this reason, the sup norm is also called the uniform norm.

Exercise 82. Let (x_n) be a sequence in a discrete metric space. When does it converge? (Classify all convergent sequences in a discrete metric space.)

Exercise 83. Consider $M(n,\mathbb{R})$ as an nls as in Ex. 19. Then a sequence (A_k) in $M(n,\mathbb{R})$ converges to $A \in M(n,\mathbb{R})$ iff the matrix entries $a_{ij}^k \to a_{ij}$ as $k \to \infty$ for all i,j.

Exercise 84. Let the notation be as above. Let $A_k \to A$. Then $A_k^2 \to A^2$.

Exercise 85. Define the limit point of a set in a metric space as follows: $x \in X$ is a limit point of a set E iff for every $r > 0$, we have $B(x,r) \cap E \neq \emptyset$.

Show that any $x \in E$ is a limit point of E.

Exercise 86. x is a limit point of E iff there exists a sequence (x_n) in E such that $x_n \to x$.

Exercise 87. E is closed in (X,d) iff E contains all its limit points.

Exercise 88. Find the set of all limit points of $\mathbb{Q}$ in $\mathbb{R}$.

Exercise 89. If A is a nonempty bounded subset of $\mathbb{R}$, then its supremum and infimum are limit points of A.

Exercise 90. We say that x is a cluster or an accumulation point of a set E if for each $r > 0$, the set $B(x, r) \cap E$ contains a point *other than* x.

Show that every point $\mathbb{Z} \subset \mathbb{R}$ is limit point of $\mathbb{Z}$ while $\mathbb{Z}$ has no cluster point.

Exercise 91 (Bolzano-Weierstrass-I)**.** Every bounded infinite subset of $\mathbb{R}$ has a cluster point in $\mathbb{R}$.

Exercise 92 (Bolzano-Weierstrass-II)**.** Show that any bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Exercise 93. Extend the last two exercises to $\mathbb{R}^n$. Can one extend these to an arbitrary nls?

Exercise 94. Let F be a finite subset of a metric space. What are its cluster points?

Cauchy Sequences and Completeness

Exercise 95. Define a Cauchy sequence in a metric space.

Exercise 96. Show that any convergent sequence in a metric space is Cauchy. The converse is not true.

Exercise 97. What are the Cauchy sequences in a discrete metric space?

Exercise 98. Show that in a metric space a Cauchy sequence is convergent iff it has a convergent subsequence.

Exercise 99. Define a complete metric space. Show that any discrete metric space is complete.

Exercise 100. Let $D := \{(x, y) : x^2 + y^2 < 1\}$. Is D complete?

Exercise 101. Show that $\mathbb{R}$ is complete. Hence conclude that $\mathbb{R}^n$ is complete.

Bounded Sets

Exercise 102. We define A to be *bounded* in (X, d) if there exists $x_0 \in X$ and $R > 0$ such that $A \subset B(x, r)$. Show that a subset A of a metric space (X, d) is bounded iff for every $x \in X$ there exists $r > 0$ such that $A \subset B(x, r)$.

Exercise 103. Let A be a nonempty subset of a metric space (X, d). Define the diameter $\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}$, in the extended real number system. Show that a subset A is bounded iff either it is empty or its diameter is finite.

Exercise 104. Let $x_n \to x$ in a metric space. Then the set $\{x_n\}$ is bounded.

Exercise 105. Let $(X, \| \ \|)$ be an nls. Show that $A \subset X$ is bounded iff there exists $M > 0$ such that $\|x\| \leq M$ for all $x \in A$.

Exercise 106. Which vector subspaces of an nls are bounded subsets?

Exercise 107. Show that the set $O(n)$ of all orthogonal matrices (that is, the set of matrices satisfying $AA^t = I = A^tA$) is a bounded subset of $M(n, \mathbb{R})$. Here $M(n, R)$ is considered as an nls as in Ex. 19.

Exercise 108. Show that the set $SL(n, \mathbb{R})$ of all $n \times n$ real matrices with determinant 1 is not bounded.

Exercise 109. Let G be a subgroup of the multiplicative group $\mathbb{C}^*$ of the non-zero complex numbers. Assume that as a subset of $\mathbb{C}$ it is bounded. Show that $|g| = 1$ for all $g \in G$.

Exercise 110. Boundedness is metric specific. $\mathbb{R}$ with the standard metric is unbounded while with respect to $\delta := \min\{1, d\}$, it is bounded. However the topologies induced by d and δ are the same. (Ex. 46.)

Continuity

Exercise 111. Let (X, d) and (Y, d) be metric spaces. A function $f : X \to Y$ is said to be continuous *at* $x \in X$ iff for every sequence (x_n) in X converging to x, we have $f(x_n) \to f(x)$.

Show that any constant map from a metric space to another is continuous.

Exercise 112. Show that the maps $\mathbb{R}^2 \to \mathbb{R}$ given by $\alpha : (x, y) \mapsto x + y$ and $\mu : (x, y) \mapsto xy$ are continuous.

Exercise 113. Show that the projection maps $p_i : \mathbb{R}^n \to \mathbb{R}$ given by $p_i(x) = x_i$ where $x = (x_1, \ldots, x_n)$ is continuous.

Exercise 114. Prove the following facts about the set of continuous functions: If $f, g : X \to \mathbb{R}$ are continuous at x, the so are $f+g, fg, af$, for any $a \in \mathbb{R}$. Consequently, the set of functions continuous at x form a real vector space.

Do you appreciate the ease with which you proved these results thanks to our definition of continuity? The core of the argument falls upon established facts on convergent sequences!

Exercise 115. Use the last two exercises to conclude that any polynomial function $p(x_1, \ldots, x_n)$ in the variables $x_1, \ldots, x_n$ will be continuous. (Give examples of such functions!)

Exercise 116. Fix $x \in X$. Then the function f_x defined by $f_x(y) := d(x, y)$ is continuous.

Exercise 117. Consider $M(2, \mathbb{R})$. Let $f(A) = \det(A)$. Show that f is a continuous function. Can you think of a generalization?

Exercise 118. Composite of continuous functions is continuous.

In the next three exercises, the product sets are given the product metrics.

Exercise 119. Show that the vector addition map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by$(x, y) \mapsto x + y$ is continuous.

Exercise 120. Show that the inner product map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $(x, y) \mapsto \langle x, y \rangle$ is continuous.

Exercise 121. Show that the scalar multiplication map $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by $(a, x) \mapsto ax$ is continuous.

Exercise 122. Show that the squaring map $A \mapsto A^2$ on $M(n, \mathbb{R})$ is continuous. Generalizations?

Exercise 123. Let X, Y be metric spaces. Let $f\colon X \to Y$ be a function. Show that the following are equivalent:
(a) f is continuous at $x \in X$.
(b) Given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $d(x, x') < \delta$, then $d(f(x), f(x')) < \varepsilon$.
(c) Given an open set V containing $f(x)$ in Y, we can find an open set U containing x such that $f(U) \subset V$.

Exercise 124. $f\colon X \to Y$ is continuous iff for every open set $V \subset Y$, its inverse image $f^{-1}(V)$ is open in X.

Exercise 125. $f\colon X \to Y$ is continuous iff for every closed set $V \subset Y$, its inverse image $f^{-1}(V)$ is closed in X.

Exercise 126. One can use Ex. 124 and Ex. 125 to show that certain sets are open or closed. For example, redo Ex. 31, Ex. 32, Ex. 34, Ex. 43 and Ex. 61.

For instance the set of points $(x, y) \in \mathbb{R}^2$ such that $\cos(x^2) + x^3 - 47y > e^x - y^2$ is an open subset of $\mathbb{R}^2$.

Exercise 127. Show that the set of all invertible matrices in $M(2, \mathbb{R})$ is open. (Hint: Ex. 117.)

Exercise 128. Show that the set $SL(n, \mathbb{R})$ of matrices in $M(n, \mathbb{R})$ with determinant one is a closed subset of $M(n, \mathbb{R})$.

Exercise 129. Define the map $f\colon \mathbb{R} \to \mathbb{R}$ by setting $f(x) = |x - 1|$. Show that f is continuous.

Exercise 130. Consider $\varphi(f) := f(0)$ as a map $\varphi\colon C[0, 1] \to \mathbb{R}$. Show that φ is continuous.

Exercise 131. Consider $C[0, 1]$ with the norm $\| \ \|_\infty$ as in Ex. 7. Show that the map $f \mapsto \int_0^1 f(t)\,dt$ is continuous. Is the map still continuous if we take $\| \ \|_1$ as the norm on $C[0, 1]$?

Exercise 132. Let A be a nonempty subset of a metric space (X, d). Define

$$d_A(x) := \inf\{d(x, a) : a \in A\}, \qquad x \in X.$$

Then d_A is continuous.

Exercise 133. Let $A = (0, 1) \subset \mathbb{R}$. Draw the graph of the function d_A.

Exercise 134. Show that x is a limit point E iff $d_E(x) = 0$.

Exercise 135 (Urysohn's Lemma)**.** Let A, B be two disjoint closed subsets of a metric space. Show that there exists a continuous function $f\colon X \to \mathbb{R}$ such that $0 \le f \le 1$ and $f = 0$ on A and $f = 1$ on B.

Exercise 136. Let $A \in M(n, R)$. Consider vectors of $\mathbb{R}^n$ as column vectors, that is, as matrices of size $n \times 1$ so that the matrix multiplication Ax makes sense. Then the map $x \mapsto Ax$ is continuous.

In fact, we can say more. The joint map from $M(n, \mathbb{R}) \times \mathbb{R}^n$ to $\mathbb{R}^n$ given by $(A, x) \mapsto Ax$ is continuous.

Exercise 137. Let X, Y be nls. Let $T\colon X \to Y$ be a linear map. Prove that T is continuous iff it is continuous at $0 \in X$. Use this to show that T is continuous iff there exists a constant $C > 0$ such that $\|Tx\| \leq C\,\|x\|$.

Exercise 138. Let X be any nls. Show that any linear map $T\colon \mathbb{R}^n \to X$ is continuous.

Exercise 139. Let $(X, \|\ \|)$ be an nls. Show that $\|\ \| : X \to \mathbb{R}$ is continuous.

Exercise 140. Find a continuous function $f\colon (a,b) \to \mathbb{R}$ which is bijective and such that f^{-1} is also continuous.

Exercise 141. Let $f\colon \mathbb{R} \to \mathbb{R}$ be such that $f^{-1}(a,\infty)$ and $f^{-1}(-\infty,b)$ are open for any $a, b \in \mathbb{R}$. Show that f is continuous.

Exercise 142. Let A be a subset of a metric space (X,d). Let (Y,d) be another metric space. A function $f\colon X \to Y$ is said to be continuous on A iff f is continuous at each $a \in A$. Show that f is continuous on A iff $f\colon (A,d) \to (Y,d)$ is continuous.

Exercise 143 (Pasting Lemma-I)**.** Let $\{U_i : i \in I\}$ be a family of open sets such that $X = \cup_i U_i$. Let $f_i\colon U_i \to Y$ be continuous. Assume that $f_i(x) = f_j(x)$ whenever $x \in U_i \cap U_j$. Show that there exists continuous function $f\colon X \to Y$ such that $f(x) = f_i(x)$ if $x \in U_i$.

Exercise 144 (Pasting Lemma-II)**.** Formulate an analogue of Pasting Lemma-I where open sets are replaced by closed sets. (Tread carefully. Go through the earlier proof more closely.)

Exercise 145. Find a continuous function $f\colon \mathbb{C}^* \to S^1 := \{z \in \mathbb{C} : |z| = 1\}$ such that $f(z) = z$ for $z \in S^1$.

Exercise 146. Show that the conjugation map $z \mapsto \overline{z}$ is continuous on $\mathbb{C}$.

Dense Sets

Exercise 147. We say a subset $D \subset X$ of a metric space is *dense in* X if for any given $x \in X$ and $r > 0$, we have $B(x,r) \cap D \neq \emptyset$. In other words, any non-empty open set in X must contain point of D.

Show that $\mathbb{Q}$ is dense in $\mathbb{R}$. is $\mathbb{R} \setminus \mathbb{Q}$ dense in $\mathbb{R}$?

Exercise 148. Show that $D \subset X$ is dense in the metric space (X,d) iff every point of X is a limit point of D.

Exercise 149. Show that $D \subset X$ is dense in the metric space X iff its closure $\overline{D} = X$. (This is the standard definition.)

Exercise 150. * Show that the set $\mathbb{Z} + \sqrt{2}\mathbb{Z}$ is dense in $\mathbb{R}$. Conclude that it cannot be closed in $\mathbb{R}$.

Exercise 151. Does there exist a finite set which is dense in $\mathbb{R}$?

Exercise 152. What are the dense subsets of a discrete metric space?

Exercise 153. Let X, Y be metric spaces and $D \subset X$ be dense. Let $f, g\colon X \to Y$ be continuous functions such that $f(x) = g(x)$ for all $x \in D$. Show that $f = g$ on X.

Exercise 154. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous additive group homomorphism. Show that $f(x) = \lambda x$ for $x \in \mathbb{R}$ where $\lambda = f(1)$.

Exercise 155 (Weierstrass Approximation Theorem)**.** The theorem states: Given any continuous function $f\colon [0,1] \to \mathbb{R}$ and given $\varepsilon > 0$, there exists a polynomial $p(x)$ such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [01]$. Interpret this result using the concept of dense sets.

Compactness

Exercise 156. Define an open cover of a subset in a metric space. Give some examples.

Exercise 157. Show that given any open cover of $A = \{1/n : n \in \mathbb{N}\} \cup \{0\}$ (considered as a subset of $\mathbb{R}$) we can find a finite number of elements in the cover such that their union contains A.

Can we assert such an analogous result for the set $B = \{1/n : n \in \mathbb{N}\}$?

Exercise 158 (Heine-Borel Theorem)**.** Any open cover of $[a, b]$ admits a finite subcover.

Exercise 159. Let $X = \mathbb{Z}$. Can you think of an open cover of X which does not admit a finite subcover?

Exercise 160. Let X be an infinite set. Let X be endowed with discrete metric. Give an open cover of X which does not admit a finite subcover.

Exercise 161. Give an open cover of $(0, 1)$ which does not admit a finite subcover?

Exercise 162. Define a compact subset of a metric space.

Show that any finite subset of a metric space is compact.

Exercise 163. Find all compact subsets of a discrete metric space.

Exercise 164. Show that $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ is compact in $\mathbb{R}$. Can you generalize this? (in a metric space?)

Exercise 165. Show that any compact subset of a metric space is closed ad bounded.

Exercise 166. Show that any closed subset of a compact set is compact.

Exercise 167. Show that any closed and bounded subset of $\mathbb{R}$ is compact.

Exercise 168. A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

In a general metric space, this need not be true. See the next few exercises.

Exercise 169. Let X be infinite and d be the discrete metric on X. Show that X is bounded and closed but not compact.

Exercise 170. The set $(-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ is a closed and bounded subset in $\mathbb{Q}$, but not compact.

Exercise 171. Consider $B[0, 1]$, the closed unit ball in $C[0, 1]$ under the sup norm. Show that it is not compact. (As of now, the proof I can think of uses sequential compactness!)

Exercise 172. Consider the closed unit ball in ℓ^2. It is not compact.

Exercise 173. Show that the square $[-R, R] \times [-R, R]$ is a compact subset of $\mathbb{R}^2$.
Note that the proof generalizes to $\mathbb{R}^n$.

Exercise 174. Show that a subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.

Exercise 175. Which of the following are compact? Justify your answers.
(a) The unit sphere $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$.
(b) The hyperbola $x^2 - y^2 = 1$ in $\mathbb{R}^2$.
(c) The parabola $y^2 = x$ in $\mathbb{R}^2$.
(d) The ellipse $(x^2/a^2) + (y^2/b^2) = 1$.
(e) A 'conic section' in $\mathbb{R}^2$ given by a second degree polynomial in x and y.
(f) The set of points $x \in \mathbb{R}^n$ such that $x_1^2 + 2x_2^2 + \cdots + nx_n^2 \leq (n+1)^2$.
(g) A nonzero vector subspace of an nls.

Exercise 176. Show that the set $O(n)$ of all orthogonal matrices is a compact subset of $M(n, \mathbb{R})$. (Recall that if $A = (C_1, \ldots, C_n)$ is orthogonal with C_k as the k-th column, then $\langle C_k, C_j \rangle = \delta_{jk}$.)

Exercise 177. Any continuous function from a compact metric space to $\mathbb{C}$ is bounded.

Exercise 178. Any continuous function from a compact metric space to $\mathbb{R}$ attains its bounds.

Exercise 179. Show that the continuous image of a compact metric space is compact.

Exercise 180 (A very useful fact)**.** Let X, Y be metric spaces. Assume that X is compact. Let $f \colon X \to Y$ be a bijective continuous map. Show that the inverse map $f^{-1} \colon Y \to X$ is continuous.

Exercise 181. Define a totally bounded subset of a metric space. (Back door entry of compactness notion!) Show that any compact space is totally bounded.

Exercise 182. Show that any bounded subset of $\mathbb{R}$ is totally bounded.
Can you generalize this to $\mathbb{R}^n$? to any metric space?

Exercise 183. Show that if B is totally bounded and $A \subset B$, then A is totally bounded.

Exercise 184 (Lebesgue Covering Lemma)**.** Let (X, d) be a compact metric space. Let $\{U_i\}$ be an open cover of X. Then there is a $\delta > 0$ such that if $A \subset X$ with diameter $\operatorname{diam}(A) < \delta$, then there is an i such that $A \subset U_i$.
If δ is as in the theorem and $0 < \delta' \leq \delta$, then δ' also has the required property. Any δ of the theorem is called a Lebesgue number of the covering $\{U_i\}$.

Exercise 185. Let $X = (0, 1)$ and $U_n = (1/n, 1)$. Does a Lebesgue number exist for this cover?

Exercise 186. If A is totally bounded so is its closure.

Exercise 187 (Characterization of Compact Metric Spaces)**.** For a metric space (X, d), the following are equivalent:

(1) X is compact: every open cover has a finite subcover.

(2) X is complete and totally bounded.

(3) Every infinite set has a cluster point.

(4) Every sequence has a convergent subsequence.

Exercise 188. Given two nonempty subsets A, B of a metric space (X, d) we define the distance $d(A, B)$ between them by $d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$.

Find $d(\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q})$. Find $d(A, B)$ where A is the rectangular hyperbola $xy = 1$ and B is the union of axes given by $xy = 0$.

Exercise 189. Given A, B two compact subsets of a metric space such that $A \cap B = \emptyset$. Show that $d(A, B) > 0$. In fact, show that there exist $a \in A$ and $b \in B$ such that $d(A, B) = d(a, b)$.

Exercise 190. The product metric space of two compact metric spaces is compact.

Exercise 191. Let C be closed and K be compact in $\mathbb{R}^n$. Assume that $C \cap K = \emptyset$. Show that $d(K, C) > 0$.

Exercise 192. Let C be closed and K be compact in $\mathbb{R}^n$. Show that $K + C$ is closed.

Exercise 193. Let A, B be compact subsets of $\mathbb{R}^n$. Show that their sum $A + B$ is compact.

Exercise 194 (Arzela-Ascoli Theorem)**.** The theorem gives a characterization of compact subsets in $(C(X), \| \ \|_\infty)$ where X is a compact (metric) space.

Exercise 195 (Finite Intersection Property and Compactness)**.** Let X be a set. We say that a collection $\mathcal{A}$ of (nonempty) subsets of X has *finite intersection property* (f.i.p., in short) if every finite family $A_1, \ldots, A_n$ of elements in $\mathcal{A}$ has a nonempty intersection.

A topological space is compact iff every family of closed sets with f.i.p. has nonempty intersection. *Hint:* Start with an open cover $\mathcal{U}$ which does not admit a finite subcover. Look at $\{X \setminus U : U \in \mathcal{U}\}$.

Uniform Continuity

Exercise 196. Define uniformly continuous functions from one metric space to another. Prove that the following functions are uniformly continuous:

(a) $f\colon [1, \infty) \to \mathbb{R}$ given by $f(x) = 1/x$.

(b) Any linear map $T\colon \mathbb{R}^m \to \mathbb{R}^n$. (Qn. How about a linear map from $\mathbb{R}^m$ to an nls?)

(c) $f\colon \mathbb{R} \to \mathbb{R}$ such that $f'(x)$ exists and is bounded. (If you have learnt calculus of several variables, extend this result suitably.)

Exercise 197. Show that the following maps are not uniformly continuous:

(a) $f\colon (0, \infty) \to \mathbb{R}$ given by $f(x) = 1/x$.

(b) $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$.

Exercise 198. Go through the solution of Ex. 132. Would you like to improve upon the conclusion?

Exercise 199. Show that any continuous function from a compact metric space to any other metric space is uniformly continuous. (Take sometime to explain why the 'obvious proof' does not work and how we surmount the difficulty.)

Exercise 200. Show that any uniformly continuous function carries bounded sets to bounded sets.

Exercise 201. Show that any uniformly continuous function carries Cauchy sequences to Cauchy sequences.

The converse is not true.

Exercise 202. Let (X, d) be a metric space. Assume that D is a dense subset of X. Let Y be a complete metric space. Let $f\colon (D, d) \to (Y, d)$ be a uniformly continuous function. Show that there exists a uniformly continuous function $g\colon X \to Y$ such that $g(x) = f(x)$ for all $x \in D$. (The function g is called an extension of f from D to X.)

Exercise 203. A function $f\colon (X, d) \to (Y, d)$ is said to be *Lipschitz* if the exists a constant $L > 0$ (called a Lipschitz constant of f) such that for all $x_1, x_2 \in X$, we have

$$d(f(x_1), f(x_2)) \leq Ld(x_1, x_2).$$

Show that nay Lipschitz continuous function is uniformly continuous.

Exercise 204. Show that the functions of Ex. 196-(c) are Lipschitz.

Exercise 205. Let $T\colon X \to Y$ be a linear map between two nls. Show that T is continuous iff it is Lipschitz.

Exercise 206. Let $f\colon (X, d) \to (Y, d)$ be a map of metric spaces. Assume that f is locally Lipschitz, that is, for each $x \in X$, there exists an open ball B_x containing x and a constant $L_x > 0$ such that

$$d(f(x_1), f(x_2)) \leq L_x d(x_1, x_2) \text{ for all } x_1, x_2 \in B_x.$$

If X is compact, then f is Lipschitz.

Connectedness

Exercise 207. A (metric) space X is said to be *connected* if the only sets which are both open and closed in X are $\emptyset$ and the full space X.

$\mathbb{R}$ is connected. (Ex. 68.)

Exercise 208. A space X is not connected iff there exist two disjoint proper non-empty subsets A and B such that A and B are both open and closed in X and $X = A \cup B$. In such a case, we say that the pair (A, B) is a *disconnection* of X.

Exercise 209. Let X be a set such that $|X| \geq 2$ with discrete metric. Then X is not connected.

Exercise 210 (Modify this exercise suitably!)**.** A subset A of a topological space X is said to be connected if A is a connected space when considered as a topological space with the induced (or subspace) topology. In the case of metric space (X, d), this amounts to saying that (A, δ) is connected, where δ is the restriction of the metric d on X to A.

Exercise 211 (A Most Important Characterization of Connected Spaces)**.** A topological space X is connected iff every continuous function $f\colon X \to \{\pm 1\}$ is a constant function.

A subset A of X is connected iff every continuous function $f\colon A \to \{\pm 1\}$ is a constant function.

Exercise 212. A set $J \subseteq \mathbb{R}$ is connected iff J is an interval.

Exercise 213. The $O(n, \mathbb{R})$ of orthogonal matrices of order n is not connected.

Exercise 214. Let X be a topological space. Let A and B be two connected subsets of X such that $A \cap B \neq \emptyset$. Then $A \cup B$ is connected.

Exercise 215. Let A be a connected subset of a space X. Let $A \subset B \subset \overline{A}$. Then B is connected.

Exercise 216. Let $\{A_i : i \in I\}$ be a collection of connected subsets of a space X with the property that for all $i, j \in I$ we have $A_i \cap A_j \neq \emptyset$. Then $A := \cup_i A_i$ is connected.

Exercise 217. Let X be a connected topological space and $g \colon X \to Y$ be a continuous map. Then $g(X)$ is connected.

Exercise 218. Show that the set $GL(2, \mathbb{R})$ is not connected.

Exercise 219. Show that the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected.

Exercise 220. Show that the set $SO(2, \mathbb{R}) := \{A \in O(2, \mathbb{R}) : \det A = 1\}$ is connected. *Hint:* Write down all elements of $SO(2, \mathbb{R})$ explicitly.

Exercise 221. Let X and Y be connected spaces. Then the product space $X \times Y$ is connected.

Exercise 222. We say that $f \colon X \to Y$ is *a locally constant* function if for each $x \in X$, there exists an open set U_x containing x with the property that f is a constant on U_x.

If X is connected, then any locally constant function is constant on X.

Exercise 223. Let U be an open connected subset of $\mathbb{R}^n$ and $f \colon U \to \mathbb{R}$ be a differentiable function such that $Df(p) = 0$ for all $p \in U$. Then f is a constant function.

Exercise 224. Let $f \colon X \to \mathbb{R}$ be a nonconstant continuous function on a connected space. Show that $f(X)$ is uncountable.

Exercise 225. Let (X, d) be a connected metric space. Assume that X has at least two elements. Then $|X| \geq |\mathbb{R}|$.

Exercise 226. Which of the following sets are connected subsets of $\mathbb{R}^2$?
(a) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
(b) $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$.
(c) $\{(x, y) \in \mathbb{R}^2 : xy = 1\}$.
(d) $\{(x, y) \in \mathbb{R}^2 : xy = c$ for some fixed $c \in \mathbb{R}\}$.
(e) $\{(x, y) \in \mathbb{R}^2 : (x^2/a^2) + (y^2/b^2) = 1\}$.

Exercise 227. Let $A \subset X$. What does it mean to say that the characteristic function χ_A continuous? (Assume some condition on A if necessary.)

Path Connected spaces

Exercise 228. Let X be a topological space. A continuous map $\gamma \colon [0, 1] \to X$ is called a *path* in X. If $\gamma(0) = x$ and $\gamma(1) = y$, then γ is said to be a path joining the points x and y or simply a path from x to y.

Give an example of a path in $\mathbb{R}^2$ that connects $(-1, 0)$ and $(1, 0)$ and passes through (0,1).

Exercise 229. A topological space X is said to be *path connected* if for any pair of points x and y in X, there exists a path $\gamma \colon [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Any interval in $\mathbb{R}$ is path connected.

Exercise 230. The space $\mathbb{R}^n$ is path connected. Any two points can be joined by a line segment: $\gamma(t) := x + t(y - x)$, for $0 \leq t \leq 1$. We call this path γ a linear path.

Exercise 231. For every $r > 0$, the circle $C_r := \{(x, y) \in \mathbb{R}^2 : x^2+y^2 = r^2\}$ is path connected.

Exercise 232. The set $\{(x, y) \in \mathbb{R}^2 : x \geq 0 \quad \& \quad x^2 - y^2 = 1\}$ is path connected.
The hyperbola is not path connected.

Exercise 233. The parabola $\{(x, y) \in \mathbb{R}^2 : y^2 = x\}$ is path connected.

Exercise 234. The union of the two parabolas $\{(x, y) \in \mathbb{R}^2 : y^2 = x\}$ and $\{(x, y) \in \mathbb{R}^2 : y = x^2\}$ is path connected.

Exercise 235. The union of the parabolas $\{(x, y) \in \mathbb{R}^2 : y^2 = x\}$ and $\{(x, y) \in \mathbb{R}^2 : y^2 = -x\}$ is path connected.

Exercise 236. The unit sphere $S^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2+x_2^2+x_3^2 = 1\}$ is path connected.

Exercise 237. Let $\gamma_1 \colon [0, 1] \to X$ and $\gamma_2 \colon [0, 1] \to X$ be two paths such that $\gamma_1(1) = \gamma_2(0)$. Then there exists a path $\gamma_3 \colon [0, 1] \to X$ such that $\gamma_3(0) = \gamma_1(0)$ and $\gamma_3(1) = \gamma_2(1)$.

Exercise 238. Let X be path connected. Then X is connected.

Exercise 239. * The converse of the last exercise is not true.

Exercise 240. Let U be an open connected subset of $\mathbb{R}^n$. Then U is path connected.

Exercise 241. Let (X, d) be an unbounded connected metric space. Let $x \in X$ and $r > 0$ be arbitrary. Show that there exists $y \in X$ such that $d(x, y) = r$.

Exercise 242. Assume that a path $\gamma \colon [0, 1] \to \mathbb{R}^n$ connects a point in $x \in B(0, 1) \subset \mathbb{R}^n$ to a point y with $\|y\| > 1$. Show that there exists $t \in [0, 1]$ such that $\|\gamma(t)\| = 1$.

Complete Metric Spaces

Exercise 243. Let (X, d) be a complete metric space and $E \subset X$. Then E is closed in X iff (E, d) is a complete metric space.

Exercise 244 (Nested Interval Theorem)**.** State and prove the nested interval theorem in $\mathbb{R}$. Did you observe that the condition $\ell(J_n) \to 0$ is needed only to prove the uniqueness of the common point?

Exercise 245 (Cantor's Intersection Theorem)**.** Let (X, d) be a complete metric space. Let a nonempty closed subset F_n be given for any $n \in \mathbb{N}$ such that $F_{n+1} \subset F_n$. Assume further that $\operatorname{diam}(F_n) \to 0$. Show that $\cap_{n=1}^{\infty} F_n$ consists exactly of one point.

Could you have concluded that $\cap_n F_n \neq \emptyset$ if you did not assume that $\operatorname{diam}(F_n) \to 0$? Compare this with the nested interval theorem.

Exercise 246. Let (X, d) and (Y, d) be complete metric spaces. Show that the product metric space is also complete.

Exercise 247. Show that the space $(C[0, 1], \| \ \|_\infty)$ is complete.
Does the proof still go through if we replace $[0, 1]$ by any compact metric space?
How about any metric space but we restrict ourselves to the set of bounded continuous functions?

Exercise 248. Show that the space $(B[0,1], \| \ \|_\infty)$ is complete.

Does the proof go through if we replace $[0,1]$ by any set X?

Exercise 249. Show that $(C[-1,1], \| \ \|_1)$ is not complete.

Exercise 250. Let $\mathbf{c}_0$ denote the real vector space of all real sequences that converge to 0. Show that $\|x\| := \sup\{|x_n|\}$ defines a norm. Is $\mathbf{c}_0$ complete with respect to this norm?

Exercise 251. Let $\mathbf{c}_{00}$ denote the real vector space of all real sequences such that $x_n = 0$ for all n greater than some N (which may depend on x). Show that $\|x\| := \sup\{|x_n|\}$ defines a norm. Is $\mathbf{c}_{00}$ complete with respect to this norm?

Exercise 252. Define an isometry of a metric space. Show that if A is an $n \times n$ real orthogonal matrix, then the map $x \mapsto Ax$ is an isometry of $\mathbb{R}^n$.

Exercise 253. Fix a unit vector $u \in \mathbb{R}^n$. Define

$$R_u(v) = v - 2\langle v, u\rangle u.$$

Show that R_u is an isometry. (Geometrically, it is the reflection with respect to the 'plane' determined by the equation $\langle x, u\rangle = 0$.)

Exercise 254. Define a completion of a metric space. Show that $\mathbb{R}$ is a completion of $(\mathbb{Q}, | \ |)$.

What is the completion of the space of irrationals with respect to the absolute value metric?

Exercise 255. Let V be the vector space of all polynomials with real coefficients endowed with the norm $\|p\| := \sup\{|p(x)| : 0 \le x \le 1\}$. Show that $(C[0,1], \| \ \|_\infty)$ is a completion of V.

Exercise 256 (Completion of a Metric Space)**.** Let (X,d) be any metric space. Fix a point $o \in X$. For each $x \in X$, consider the function $f_x(y) := d(y,x) - d(y,o)$. Show that $f_x \in B(X)$ and that the map $x \mapsto f_x$ is an isometry of X into $(B(X), \| \ \|_\infty)$.

Conclude that every metric space has a completion.

Do you think we can use this method to complete $\mathbb{Q}$?

Baire Category Theorem

Exercise 257. A subset $A \subset X$ of a topological space is said to be *nowhere dense* in X, if given any nonempty open set U, we can find a nonempty open subset $V \subset U$ such that $A \cap V = \emptyset$.

Show that the set $\mathbb{Z}$ is nowhere sense in $\mathbb{R}$.

Exercise 258. Let V be any proper vector subspace of $\mathbb{R}^n$. Then V is nowhere dense in $\mathbb{R}^n$.

More generally, let X be a normed linear space. Let V be any proper vector subspace of X. Then V is nowhere dense in X.

Exercise 259. Show that $A \subset X$ is nowhere dense in X iff the interior of the closure of A is empty, that is, $(\overline{A})^0 = \emptyset$. (This is the standard definition.)

Exercise 260 (Baire Category Theorem). Let (X, d) be a complete metric space.

(1) Let U_n be open dense subsets of X, for $n \in \mathbb{N}$. Then $\cap_n U_n$ is dense in X.

(2) Let F_n be nonempty closed subsets of X such that $X = \cup_n F_n$. Then at least one of F_n's has nonempty interior. In other words, a complete metric space cannot be a countable union of nowhere dense closed subsets.

Exercise 261. $\mathbb{R}^n$ cannot be the union of a countable collection of lower dimensional subspaces.

Exercise 262. Let X be an infinite dimensional complete normed linear space. Then X cannot be countable dimensional.

Exercise 263 (Uniform Boundedness Principle). Let X be a complete metric space and $\mathcal{F}$ a family of continuous real valued functions on X. Assume that for all $x \in X$, there exists $C_x > 0$ such that $|f(x)| \leq C_x$ for all $f \in \mathcal{F}$. Then there exists a non-empty open set $U \subseteq X$ and a constant C such that $|f(x)| \leq C$ for all $f \in \mathcal{F}$ and $x \in U$.

Banach's Contraction Principle

Exercise 264. Let X and Y be metric spaces. A map $T: X \to Y$ is said to be a *contraction* if there exists a constant c, $0 < c < 1$ such that

$$d(T(x), T(x')) \leq c d(x, x'), \qquad \text{for all } x, x' \in X.$$

Observe any contraction is Lipschitz continuous and hence it is uniformly continuous.

Exercise 265. Let $f: [a, b] \to [a, b]$ be differentiable and $|f'(x)| \leq c$ with $0 < c < 1$. Then f is a contraction of $[a, b]$.

Exercise 266 (Banach Contraction Principle). Let (X, d) be a complete metric space. Assume that $T: X \to X$ is a contraction. Then f has a unique fixed point — a point $x \in X$ such that $f(x) = x$.

In fact, if we take any $x_0 \in X$ and let $x_n := T(x_{n-1})$ be defined recursively for $n \geq 1$, then (x_n) converges to an $x \in X$. Furthermore we have

$$d(T^n x_0, x) \leq \frac{c^n}{1-c} d(x_0, T x_0).$$