Row Rank of a Matrix Equals its Column Rank

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Let F be a field. If you wish, you may take $F = \mathbb{R}$, the field of real numbers in the sequel.

Let $F^{m\times n}$ be the set of matrices of type $m \times n$ with values in F.

We consider $F^{1\times n}$ as the n-dimensional vector space F^n_{row}.

We consider $F^{m\times 1}$ as the m-dimensional vector space F^m_{col} consisting of all column vectors $(y_1,\dots,y_m)^t := \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$ with $y_i \in F$ for $1 \le i \le m$.

Let $A = (a_{ij}) \in F^{m\times n}$ be an $m \times n$ matrix over a field F. We denote by A_i, the i-throw of A: $A_i := (a_{i1},\dots,a_{in})$. The j-th column of A is denoted by A'_j and is given by $(a_{1j},\dots,a_{mj})^t$.

We usually consider the row-vectors A_i as elements of the n-dimensional vector space $F^n_{\rm row}$ consisting of all row vectors $(x_1,\dots,x_n)$ with $x_i \in F$ for $1 \le i \le n$. We may consider $F^n_{\rm row}$ as the n-dimensional vector space $F^{1\times n}$ consisting of matrices of type $1 \times n$ with entries in F.

The row rank of the matrix A is the number of elements in a maximal linearly independent subset of $\{A_i : 1 \le i \le m\}$. Let $W_r \subset F^n_{\rm row}$ be the vector subspace spanned by the vectors A_i, $1 \le i \le m$. The subspace W_r is known as the row-subspace of the matrix A. Then the row rank of A is $\dim W_r$.

Similar considerations apply to the column vectors A'_j. The column rank of the matrix A is the number of elements in a maximal linearly independent subset of $\{A'_j : 1 \le j \le n\}$. The subspace W_c spanned by $\{A'_j : 1 \le j \le n\}$ is known as the column space of the matrix A. The the column rank of A is nothing other than $\dim W_c$.

The result of the tile says that the column and row ranks of a matrix are equal.

We give a simple proof of this result which is also visually appealing. The proof depends on the following two observations which you might have learned earlier in a course on linear algebra.

Observation 1. Let V be a vector space over F. Let $S := \{v_1,\dots,v_k\} \subset V$. Let W be the vector subspace spanned by S. That is, W consists of all finite linear combinations of the from $\sum_i c_i v_i$, $c_i \in F$. Then $\dim V \le qk$.

This follows from the well-known result that there exists a subset $B \subset S$ which is a basis of V. (Recall that nay basis is a minimal spanning set!)

Observation 2. Let the notation be as in Observation 1. Let $Z \le V$ be a vector subspace of V such that $V \subset W$. Then $\dim Z \le \dim W \le k$.

With the preliminaries over, we are ready to state and prove the theorem.

Theorem 3. *The row rank and the column rank of a matrix A are equal.*

Proof. Let r be the row rank of A.

Let $B_1, \ldots, B_r$ be a set of linearly independent rows of A. Let us write $B_i = (b_{i1}, \ldots, b_{ij}, \ldots, b_{in})$. Then any i-th row of A is a linear combination of B's. We write these linear combinations explicitly.

$$
\begin{aligned}
(a_{11}, \ldots, a_{1n}) &= c_{11}(b_{11}, \ldots, b_{1j}, \ldots, b_{1n}) + \cdots + c_{1r}(b_{r1}, \ldots, b_{rj}, \ldots, b_{rn}) \\
&\vdots \\
(a_{i1}, \ldots, a_{in}) &= c_{i1}(b_{11}, \ldots, b_{1j}, \ldots, b_{1n}) + \cdots + c_{ir}(b_{r1}, \ldots, b_{rj}, \ldots, b_{rn}) \\
&\vdots \\
(a_{m1}, \ldots, a_{mn}) &= c_{m1}(b_{11}, \ldots, b_{1j}, \ldots, b_{1n}) + \cdots + c_{mr}(b_{r1}, \ldots, b_{rj}, \ldots, b_{nj})
\end{aligned}
$$

Let us read these vertically and write the j-th column of this array of equations.[1] We get

$$
\begin{aligned}
\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} &= \begin{pmatrix} c_{11} \\ c_{21} \\ \cdots \\ c_{m1} \end{pmatrix} b_{1j} + \cdots + \begin{pmatrix} c_{1r} \\ c_{2r} \\ \cdots \\ c_{mr} \end{pmatrix} b_{rj}, \quad \text{for } 1 \leq j \leq n \\
&= b_{1j} C_1 + \cdots + b_{rj} C_r, \quad \text{say.}
\end{aligned}
$$

That is, the columns are linear combinations of C_k, $1 \leq k \leq r$. Hence the maximum number of linearly independent columns is at most r, the row rank of A. Thus the column rank of A is less than or equal to the row rank of A.

> We have made use of the Observations. Let W be the vector subspace of F^m_{row} spanned by the the column vectors $C_1, \ldots, C_r$. Then $W \leq W_c$ is a vector subspace of W. By Observation 1, $\dim W \leq r$ and by Observation 2, $\dim W_c \leq r$.

Starting with columns, we may prove that the row rank of A is less than or equal ot the column rank of A. Or, we observe that the column rank (respectively, row rank) of A^T is the row rank (respectively, column rank) of A. Thus we get

$$\text{row-rank of } A = \text{column rank of } A^T \leq \text{row rank of } A^T = \text{column rank of } A.$$

Hence both the ranks are equal. □

The proof above is the same as the one in my book "Linear Algebra–A Geometric Approach". Since my knowledge of LaTeX is better now, it is typeset better to bring our the visual appeal.

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[1] If you remove the brackets of the terms on the left side and stack the rows, you get the matrix A.