$\mathbb{Q}$ does not have the LUB Property

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Abstract

We shall give a simple proof of the fact that the field of rational numbers $\mathbb{Q}$ does not enjoy the LUB property, that is, $\mathbb{Q}$ is not order-complete. The proof also gives the existence of $a \in \mathbb{R}$ such that $a^2 = 2$, that is, $\sqrt{2}$ exists in $\mathbb{R}$. In particular, $\mathbb{Q}$ is a proper subset of $\mathbb{R}$.

What is the LUB property of $\mathbb{Q}$? It goes as follows: Given any nonempty subset $A \subset \mathbb{Q}$ which is bounded above in $\mathbb{Q}$, (that is, there exists $\alpha \in \mathbb{Q}$ which is an upper bound of A), there exists $a \in \mathbb{Q}$ such that $a = \text{LUB } A$.

We know that if $a, b \in \mathbb{R}$ with $a < b$, then the LUB of $[a, b)$ is b. This suggests that we consider $E := [0, \sqrt{2}) \cap \mathbb{Q}$. But we still may not know that $\sqrt{2} \in \mathbb{R}$! How do we surmount this problem?

Let us consider $E := \{t \in \mathbb{Q} : t \geq 0 \ \& \ t^2 \leq 2\}$. Since, $0, 1 \in E$, E is not empty. Can we find an upper bound of E in $\mathbb{Q}$? Is 2 an upper bound of E? Yes, for, otherwise, there exists a $t \in E$ such that $t > 2$. But then $t^2 > 2^2 = 4$. This leads to a contradiction since as an element of E, $t^2 \leq 2$. So, we conclude that 2 is an upper bound of E.

If $\mathbb{Q}$ enjoys the LUB property, then there exists $a \in \mathbb{Q}$ such that $a = \text{LUB } E$. Note that $a \geq 1$. We claim that $a^2 = 2$. If the claim is true, then there is a solution of the equation $X^2 = 2$ in $\mathbb{Q}$, that is, in high-school language, $a = \sqrt{2}$ is rational. This absurdity shows that our assumption that $\mathbb{Q}$ enjoys the LUB property is false.

So, we wish to prove that $a^2 = 2$. If $a^2 \neq 2$, then either $a^2 < 2$ or $a^2 > 2$. We shall prove that each of these possibilities lead to a contradiction.

Let, if possible, $a^2 < 2$. We shall show that there exists $k \in \mathbb{N}$ such that $(a + \frac{1}{k})^2 < 2$. What does this lead to? First of all, note that $a + \frac{1}{k} \in \mathbb{Q}$. So, if $(a + \frac{1}{k})^2 < 2$, then $a + \frac{1}{k} \in E$. Since $a = \text{LUB } E$, we must have $a + \frac{1}{k} \leq a$ or $1/k \leq 0$, an absurdity. So, $a^2 < 2$ is not tenable/possible.

How do we find a k such that $(a + \frac{1}{k})^2 < 2$? That is, we must find $k \in \mathbb{N}$ such that $a^2 + \frac{2a}{k} + \frac{1}{k^2} < 2$. Since $\frac{1}{k^2} \leq \frac{1}{k}$, we find that $a^2 + \frac{2a}{k} + \frac{1}{k^2} \leq a^2 + \frac{2a+1}{k}$. Therefore, it suffices to find a $k \in \mathbb{N}$ such that $a^2 + \frac{2a+1}{k} < 2$ or what is the same, to find k such that $\frac{1}{k} < \frac{2-a^2}{1+2a}$.

Note that $1 + 2a \neq 0$. (Why?) Thus, we need to find $k > \frac{1+2a}{2-a^2}$. Since $\frac{1+2a}{2-a^2} \in \mathbb{Q}$ and $\mathbb{N}$ is not bounded above in $\mathbb{Q}$[1], there exists $k \in \mathbb{N}$ such that $k > \frac{1+2a}{2-a^2}$.

We now verify any k chosen as above works. Let $a = \text{LUB } E$. Consider $\frac{1+2a}{2-a^2}$. It lies in $\mathbb{Q}$. Since $\mathbb{N}$ is not bounded above in $\mathbb{Q}$, there exists $k \in \mathbb{N}$ such that $k > \frac{1+2a}{2-a^2}$. We claim that $(a + \frac{1}{k})^2 < 2$. For,

$$
\begin{aligned}
(a + \frac{1}{k})^2 &= a^2 + \frac{2a}{k} + \frac{1}{k^2} \\
&\leq a^2 + \frac{2a}{k} + \frac{1}{k} \\
&\leq a^2 + \frac{1+2a}{k} \\
&< a^2 + (1+2a)\frac{2-a^2}{1+2a} \\
&= 2.
\end{aligned}
$$

Since $a \in \mathbb{Q}$, $a + \frac{1}{k} \in \mathbb{Q}$ and $(a + \frac{1}{k})^2 < 2$. Hence, $a + \frac{1}{k} \in E$. Since $a = \text{LUB } E$, it is an upper bound of E and we must have $a + \frac{1}{k} \leq a$, that is, $1/k \leq 0$. This absurdity leads us to conclude that $a^2 < 2$ is not possible. (Note that we did not use the fact that a is the LUB of E.)

Is it possible that $a^2 > 2$? Assume that $a^2 > 2$. We shall find a $k \in \mathbb{N}$ such that $(a - \frac{1}{k})^2 > 2$. This will lead us to a contradiction, as we shall see later.

We proceed as earlier and try find such a k. We wish to have $(a - \frac{1}{k})^2 = a^2 - \frac{2a}{k} + \frac{1}{k^2} > 2$. This certainly happens, if $a^2 - \frac{2a}{k} > 2$, that is, if $a^2 - 2 > \frac{2a}{k}$ is true. This means that we need to choose $k \in \mathbb{N}$ such that $k > \frac{2a}{a^2-2}$. Since $\frac{2a}{a^2-2} \in \mathbb{Q}$ is not an upper bound of $\mathbb{N}$, there exists k such that $k > \frac{2a}{a^2-2}$. Fix such a k and we have $(a - \frac{1}{k})^2 > 2$. (We urge the reader to write a formal proof as we did above!)

Now where does this lead us to? We now use the fact that $a = \text{LUB } E$. Since $a - \frac{1}{k} < a$, we deduce that $a - \frac{1}{k}$ is not an upper bound of E. Hence there exists $t \in E$ such that $t > a - \frac{1}{k}$. It follows that $t^2 > (a - \frac{1}{k})^2 > 2$, that is, $t^2 > 2$. This is a contradiction, since t is an element of E, we have $t^2 \leq 2$. Hence we conclude that $a^2 > 2$ is not admissible.

By law of trichotomy in $\mathbb{Q}$, we conclude that $a^2 = 2$. Since $a \in \mathbb{Q}$, this means that "$\sqrt{2} \in \mathbb{Q}$". We arrived at this contradiction due to our assumption that $\mathbb{Q}$ enjoys the LUB property. So, we conclude that $\mathbb{Q}$ does not have the LUB property. □

Remark 1. We offer two proofs for the fact that $\mathbb{N}$ is not bounded above in $\mathbb{Q}$. (This is a sort of Archimedean property of $\mathbb{Q}$.) We prove this by contradiction.

First proof: Let $\frac{p}{q} \in \mathbb{Q}$ be an upper bound of $\mathbb{N}$. Note that $2 \leq p/q$. Hence $p \geq q$. We can then write $p = mq + p'$ where $p' < p$. Hence $p/q = m + \frac{p'}{q}$, where $m \in \mathbb{N}$. Note that $\frac{p'}{q} < 1$. Therefore $\frac{p}{q} < m + 1$. But $m + 1 \in \mathbb{N}$ and hence $\frac{p}{q}$ is not an upper bound of $\mathbb{N}$.

Second proof: Let $p/q \in \mathbb{Q}$ be an upper bound of $\mathbb{N}$. Again, $p/q \geq 2$ and hence we may as well assume that $p, q \in \mathbb{N}$. For any $k \in \mathbb{N}$, we have $k \leq \frac{p}{q}$ or what is the same, we have

[1] See Remark1 below for a proof.

$qk \leq p$ for any $k \in \mathbb{N}$. Since $q \geq 1$, we see that $k \leq qk$. Thus we arrive at $k \leq qk \leq p$ for any $k \in \mathbb{N}$. If we take $k = p+1$, then we conclude that $p+1 \leq p$, a contradiction.

Remark 2. The proof of Q not enjoying the LUB property yields more than we aimed for. A slight modification of the proof establishes the existence of a real number $a \in \mathbb{R}$ such that $a^2 = 2$.

We may still work with $E := \{t \in \mathbb{Q} : t \geq 0 \ \& \ t^2 \leq 2\}$. But when we deal with the case $a^2 < 2$, $a + \frac{1}{k}$ is a real number and may not be in E. A way around would be to invoke the density of $\mathbb{Q}$ in $\mathbb{R}$. There exists $s \in \mathbb{Q}$ such that $a < s < a + \frac{1}{k}$ and we have $s^2 < 2$. Therefore, $s \in E$ but $s > a$, an upper bound of E, a contradiction.

Thus we arrive at the fact that there exists a real number a whose square is 2. Hence $\mathbb{Q}$ is a proper subset of $\mathbb{R}$.

We may also start $E := \{t \in \mathbb{R} : t \geq 0 \ \& \ t^2 \leq 2\}$. Then E is a nonempty subset of $\mathbb{R}$ which is bounded above by 2. Hence by the LUB property of $\mathbb{R}$, there exists $a \in \mathbb{R}$ such that $a = \text{LUB } E$. The proof above shows that $a^2 = 2$. The only point to note is that in the proof for $\mathbb{R}$, we needed to choose $k > \frac{1+2a}{2-a^2}$ or $k > \frac{2a}{a^2-2}$. The numbers $\frac{1+2a}{2-a^2}$ and $\frac{2a}{a^2-2}$ lie in $\mathbb{R}$. We need to invoke the Archimedean property of $\mathbb{R}$ to find such k.

The reasoning above can be extend to prove the existence of n-th root of any positive real number. We refer the reader to "A Basic Course in Real Analysis", by Ajit Kumar and Kumaresan.

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