Arzela-Ascoli Theorem – Cantor's Diagonal Trick

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The theorem states conditions under which a sequence (f_n) of continuous functions on a closed and bounded interval $[a, b]$ has a subsequence which is uniformly convergent on $[a, b]$. The proof uses Cantor's diagonal trick. The reader might have seen a proof of uncountability of $[0, 1]$ using the non-terminating decimal expansion and the diagonal trick.

> We shall briefly explain the proof of uncountability of $[0, 1]$. Assume that $[0, 1]$ is countable. Since it is an infinite set, there exists a bijection $f : \mathbb{N} \to [0, 1]$. We let $x_n := f(n)$. Let $x_n = 0.x_{n1}x_{n2} \dots x_{nk} \dots$ be its decimal expansion. Note that the numbers of the form $k/10^n$ admit two decimal expansion which represent the same number. For example, $1/2 = 0.5 = 0.49999 \dots$. We form $y = 0.y_1y_2 \dots$ where y_1 is any number between 0 and 8 and $y_1 \neq x_{11}$. The next decimal digit y_2 is chosen from 0 and 8 and $y_2 \neq x_{22}$. Then $y \in [0, 1]$. Therefore, $y = x_n$ for some n. But then the n-th decimal digits of y and x_n differ! (The subtle point of the proof is that the real number y can admit only one decimal expansion as 9's are excluded as its decimal digits.)

Let us first try to see consequences of a uniformly convergent sequences of continuous functions. Let $f_n \rightrightarrows f$ on $[a, b]$. Let $M_n > 0$ be such that for all $x \in [a, b]$, we have $|f_n(x)| \leq M_n$. Given $\varepsilon > 1$, there exists $N \in \mathbb{N}$ such that for $k \geq N$ and $x \in [a, b]$ we have $|f_k(x) - f_N(x)| < 1$. Hence we see that

$$\forall k \geq N, \quad \forall x \in [a, b], \text{ we have } |f_k(x)| \leq 1 + M_N.$$

If we take $M := \max\{M_1, M_2, \dots, M_{N-1}, 1 + M_N\}$, we then have $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in [a, b]$. That is $\{f_n\}$ is "uniformly bounded".

Keep the assumptions of the last paragraph. Let $\varepsilon > 0$ be given. For each n, we have $\delta_n > 0$ due to the uniform continuity of f_n corresponding to $\varepsilon/3$:

$$|x - y| < \delta_n \implies |f_n(x) - f_n(y)| < \varepsilon/3. \tag{1}$$

For the same $\varepsilon > 0$, we have an $N \in \mathbb{N}$ such that

$$k \geq N, \quad x \in [a, b] \implies |f_k(x) - f_N(x)| < \varepsilon/3. \tag{2}$$

Let $\delta := \min\{\delta_1, \dots, \delta_N\}$. Then $\delta > 0$. We claim that

$$\forall n \in \mathbb{N}, \quad x, y \in [a, b] \text{ with } |x - y| < \delta \text{ we have } |f_n(x) - f_n(y)| < \varepsilon.$$

For $1 \leq n \leq N$, the claim is obviously true, since $\delta \leq \delta_n$. If $k > N$, then we observe

$$\begin{aligned} |f_k(x) - f_k(y)| &\leq |f_k(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3, \end{aligned}$$

where we used (2) to estimate the first and the thrid terms while (1) was used to estimate the middle term.

These observations suggest the following definitions.

Definition 1. We say that a family of $\mathcal{F}$ of functions from a set $X \subset \mathbb{R}$ to $\mathbb{R}$ is *uniformly bounded* if there exists $M > 0$ such that for all $f \in \mathcal{F}$ and $x \in X$ we have $|f(x)| \leq M$.

We say that $\mathcal{F}$ is (uniformly) *equicontinuous* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f \in \mathcal{F}$ and for all $x, y \in X$ we have

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Observe that any member f of the family is uniformly continuous on X.

Example 2. If the family consists of only one function f, then it is equicontinuous iff f is uniformly continuous on X.

Example 3. Recall that one easy way of finding a uniformly continuous function on $[a, b]$ is to find a function f which is differentiable on $[a, b]$ and such that f' is bounded on $[a, b]$. This suggests the next example. Let $\mathcal{F}$ be a family of differentiable functions on $[a, b]$ such that all their derivatives are bounded by M: for all $f \in \mathcal{F}$ and $x \in [a, b]$, we have $|f'(x)| \leq M$. Then the family is equicontinuous. Given $\varepsilon > 0$, choose $\delta > 0$ such that $\delta < \varepsilon/M$. Let $f \in \mathcal{F}$ and $x, y \in [a, b]$ with $|x - y| < \delta$. Then we observe

$$|f(x) - f(y)| = \left|f'(z)(x - y)\right| \leq M\,|x - y| < M\delta < \varepsilon.$$

Example 4. Let $g\colon [a, b] \times [c, d] \to \mathbb{R}$ be continuous. For each $y \in [c, d]$, we define $f_y(x) := g(x, y)$. We claim that the family $\{f_y : y \in [c, d]\}$ is equicontinuous on $[a, b]$. Let $\varepsilon > 0$ be given. Since g is uniformly continuous on the closed and bounded rectangle $[a, b] \times [c, d]$, for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$d((x_1, y_1) < (x_2, y_2)) < \delta \implies |g(x_1, y_1) - g(x_2, y_2)| < \varepsilon.$$

In particular, if $|x_1 - x_2| < \delta$, then $d((x_1, y), (x_2, y)) < \delta$. Consequently,

$$|f_y(x_1) - f_y(x_2)| = |g(x_1, y) - g(x_2, y)| < \varepsilon.$$

Example 5. Let $f_n, f\colon [a, b] \to \mathbb{R}$ be continuous functions. Assume that $f_n \rightrightarrows f$ on $[a, b]$. The paragraph before the definition shows that the family $\{f_n\}$ is equicontinuous.

Theorem 6 (Arzela-Ascoli). *Let $f_n\colon [a, b] \to \mathbb{R}$ be a sequence of (continuous) functions. Assume that the family $\{f_n : n \in \mathbb{N}\}$ is uniformly bounded on $[a, b]$ and (uniformly) equicontinuous on $[a, b]$. Then there exists a subsequence of (f_n) which is uniformly convergent on $[a, b]$.*

Remark 7. Arzela-Ascoli is a Bolzano-Weierstrass type theorem for a sequence of functions. No wonder that Bolzano-Weierstrass theorem is decisively used in the proof below!

Proof. Let $M > 0$ be such that for all n and $x \in [a,b]$ we have $|f_n(x)| \leq M$. Since the set $\mathbb{Q} \cap [a,b]$ of rationals in $[a,b]$ is countable, using a bijection of this set with $\mathbb{N}$, we can list the set $\mathbb{Q} \cap [a,b]$ as a sequence (r_n) of rationals.

> We are now going to find a subsequence of (f_n) which will converge pointwise on $\mathbb{Q} \cap [a,b]$. The idea of the proof is similar to that of the proof of Bolzano-Weierstrass theorem for $\mathbb{R}^2$ (which says that any bounded sequence $((x_n, y_n))$ in $\mathbb{R}^2$ has a convergent subsequence.). Since (x_n) is a bounded sequence of real numbers, there exists a convergent subsequence (x_{n_k}). Recall that this is same as saying that there exists an infinite subset $S_1 \subset \mathbb{N}$ such that the subsequence $(x_n)_{n \in S_1}$ is convergent. We consider the bounded sequence (y_{n_k}) and by Bolzano Weierstrass there exists an infinite subset $S_2 \subset S_1 \subset \mathbb{N}$ such that $(y_n)_{n \in S_2}$ is convergent. It follows that the subsequence $((x_n, y_n))_{n \in S_2}$ is convergent.

The sequence $(f_n(r_1))$ is a bounded sequence of real numbers. By Bolzano-Weierstrass theorem, there exists a convergent subsequence. Thus, there exists an infinite subset $S_1 \subset \mathbb{N}$ such that the sequence $(f_n(r_1))_{n \in S_1}$ is convergent. Using the well-ordering principle, we may exhibit S_1 as $\{k_{11} < k_{12} < \cdots < k_{1n} \cdots\}$. It is most appropriate to denote this subsequence by $(f_{k_{1n}}(r_1)) \equiv (f_{k_{11}}(r_1), f_{k_{12}}(r_1), \ldots, f_{k_{1n}}(r_1), \ldots)$. But following the tradition, we denote this subsequence as $(f_{1n}(r_1))$. Now consider the bounded sequence $(f_{1n}(r_2)) = (f_n(r_2))_{n \in S_1}$. We have a convergent subsequence. There exists an infinite subset $S_2 \subset S_1 \subset \mathbb{N}$ such that the sequence $(f_n(r_2))_{n \in S_2}$ is convergent. We denote this subsequence by $(f_{2n}(r_2))$. Note that the sequence $f_{2n}(r_1)$ is also convergent. Proceeding this way, we arrive a subsequence $(f_{mn}(r_m))$ which is convergent subsequence of $(f_{(m-1)n}(r_{m-1}))$. Note again that the sequences $(f_{mn}(r_j))$ are convergent for $1 \leq j \leq m$. Consider the sequence $(f_{mn}(r_{m+1}))$. This bounded sequence will have a convergent subsequence $(f_{(m+1)n}(r_{m+1}))$. This goes on for all $m \in \mathbb{N}$. We thus end up with a sequence of sequences $(f_{1n}(x)), \ldots, (f_{mn}(x))$ such that (i) for each m, the sequence $(f_{mn}(x))$ is a subsequence of $(f_{(m-1)n}(x))$ and (ii) the sequence $(f_{mn}(r_j))$ is convergent for $1 \leq j \leq m$.

Consider the subsequence $(f_{nn}) := (f_{11}, f_{22}, \ldots)$. This sequence has the property that for each m, the sequence $(f_{nn}(r_m))$ is convergent. That is, the sequence (f_{nn}) is point-wise convergent on the set of rationals in $[a,b]$. Note that so far, we have only used the uniform boundedness of the sequence (f_n).

We now exploit the uniformly equicontinuity of the sequence (f_n) to show that (f_{nn}) is uniformly convergent on $[a,b]$. Of course, we should expect the density of rationals to play a role! Given $\varepsilon > 0$, let $\delta > 0$ be such that $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \varepsilon/3$ for all $n \in \mathbb{N}$. We can find a partition $a < s_1 < \cdots < s_{k-1} < s_k < b$ such that each of the subintervals is of length less than δ and the nodes s_j, $1 \leq j \leq k$ are rationals.

> We subdivide $[a,b]$ into some k subintervals each of length $(b-a)/k < \delta/2$. By the density of rationals, we can find a rational s_j in the j-th subinterval $\left(a + \frac{j-1}{k}(b-a), a + \frac{j}{k}(b-a)\right)$ for $0 \leq j < k$. Let $t_j := a + \frac{j}{k}(b-a)$. We now concentrate on the partition $a < s_1 < s_2 < \cdots < s_k < b$ of $[a,b]$. We have $|a - s_1| < \delta/2$, $|b - s_k| < \delta/2$ and
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> $$|s_j - s_{j-1}| \leq |s_j - t_{j-1}| + |t_{j-1} - s_{j-1}| \leq (t_j - t_{j-1}) + (t_{j-1} - t_{j-2}) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$
>
> Thus each of the partitioning subinterval is of length at most δ.

Hence if $x \in [a, b]$ lies in the j-th subinterval, then $|x - s_j| < \delta$. Hence if $x \in [a, b]$ and s_j is chosen so that $|x - s_j| < \delta$, then we have

$$|f_{nn}(x) - f_{nn}(s_j)| < \varepsilon/3. \tag{3}$$

Since $(f_{nn}(r))$ is convergent for each rational $r \in [a, b]$, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\forall m, n \geq N, 1 \leq j \leq k \text{ we have } |f_{mm}(s_j) - f_{nn}(s_j)| < \varepsilon/3. \tag{4}$$

With these observations, we now show that (f_{nn}) is uniformly Cauchy on $[a, b]$. Let $x \in [a, b]$. Choose j such that $|x - s_j| < \delta$. We obtain, for $m, n \geq N$,

$$\begin{aligned} |f_{nn}(x) - f_{mm}(x)| &\leq |f_{nn}(x) - f_{nn}(s_j)| + |f_{nn}(s_j) - f_{mm}(s_j)| + |f_{mm}(s_j) - f_{mm}(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned} \tag{5}$$

We have used (3) to estimate the first and the last terms while we used (4) to estimate the middle term. □

Remark 8. Did you observe that we employed the curry-leaf trick in (5)? The s_j chosen to 'suit' the x under consideration was the curry leaf!

Remark 9. The first part of the proof above generalizes to the following situation. If X is a metric space with a countable dense subset, then any uniformly bounded sequence of functions will have a subsequence (f_{nn}) which converges point-wise on the countable dense subset.

The second part can be carried over to any "totally bounded" metric space. These are the spaces in which for any $\delta > 0$, we can find a finite set $A \subset X$ such that for any $x \in X$, there exists $a \in A$ such that $d(x, a) < \delta$.

To state a general version of the theorem and also to cast the theorem in proper perspective we need some notation. Let X be a compact metric space. The set $C(X, \mathbb{R})$ of continuous functions from X to $\mathbb{R}$ is a real vector space. We endow $C(X, \mathbb{R})$ with the norm $\|f\|_\infty \equiv \|f\| := \text{lub } \{|f(x)| : x \in X\}$. This induces a metric defined by $d(f, g) := \|f - g\|$. Under this metric, the space is complete in the sense that any Cauchy sequence is convergent. Ascoli theorem characterizes the compact subsets of this metric space as the subsets of functions which are (uniformly) equicontinuous, closed (in the metric topology) and bounded in the metric (which is same as uniformly bounded, as defined earlier).

One has various characterizations of compact subsets of a metric space. One of them says that a subset $K \subset X$ (X any metric space) is compact iff it is complete and totally bounded. One may use this characterization to give an alternate proof of Arzela-Ascoli. We refer the reader to our book "Topology of Metric Spaces" for a more detailed discussion and examples.

Remark 10. An often overlooked point of the proof is: Why (f_{nn}) is a subsequence? Note that if we agree to write $S_k := \{n_{k,1} < n_{k,2} < \cdots < n_{k,r} < n_{k,r+1} < \cdots\}$, then by induction we can show that $n_{k,k} < n_{k+1,k+1}$ for $k \in \mathbb{N}$. (Note that then $f_{kk} := f_{n_{kk}}$, see how ugly the notation turns out to be!) Let us start proving that $n_{1,1} < n_{2,2}$. Note that $n_{21} \in S_1$ and since n_{11} is the least element of S_1, we know that $n_{21} \geq n_{11}$. Hence $n_{22} > n_{21} \geq n_{11}$.

Assume, by induction, that we have shown that $n_{r,r} > n_{r-1,r-1}$ for $1 \leq r \leq k$. and that $n_{k,r} \geq n_{k-1,r}$, $1 \leq r \leq k-1$. Then we have $n_{k+1,k+1} > n_{k+1,k} \geq n_{k,k}$. The result now follows by induction.