Topological Groups — An Introduction

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Welcome and Etiquette

- ☺ SMILE!
- Welcome. Good Morning. Hope all of you are safe and well.
- By default, I have muted all of you.
- If you want to ask something, please use ‘raise-hand’. I shall unmute you and you can ask your doubts.
- As in MTTS camps, I shall ask questions, pause a few seconds so that you can say the answers to yourself! (You are muted!)
- If the audio is bad/weak, please let me know.
- You can use ‘private chat’, if necessary, druing the sessions.

Definition

- A *topological group* is a triple $(G, \tau, \cdot)$ such that (G, τ) is a topological space and $(G, \cdot)$ is a group.
- Both these structures are inter-related in the sense that the group operations are continuous.
- The group multiplication $G \times G \to G$ given by $(x, y) \mapsto xy$ is continuous.
- The inversion map $G \to G$ given by $x \mapsto x^{-1}$ is continuous.

Disciplines

The study of topological group is very basic for the following

- Harmonic Analysis
- Functional Analysis
- Differential Geometry
- Algebraic/Differential Topology
- Theoretical Physics.
- Believe me, Algebraic Number Theory!

Examples

- The most important example is $\mathbb{R}^n$ under addition with the standard topology.
- The multiplicative group of positive reals under the usual topology.
- The most significant example is $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$.
- We shall see more examples later.

References

I grew up with a classic Pontryagin's Topological Groups (1946?). But there are modern books, which I have not really read!

Higgins: Topological Groups

Hewitt and Ross: Abstract Harmonic Analysis

There may be other books which contain a section on topological groups. For instance, books on Lie groups (?)

Topological Subgroups of Topological Groups

Question: How do you define a topological subgroup of a topological group?

If H is a subgroup of a topological group, then H is a topological group with the subspace topology.

Question: Can you use this observation to list a few more topological groups?

Topological Subgroups (Continued)

The following subgroups (of the respective groups) are topological groups with the subspace topology.

- Let $SL(n, \mathbb{K})$ denote the subgroup of $GL(n, \mathbb{K})$ with determinant 1. Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.
- Let $O(n, \mathbb{R})$ denote the subgroup of $GL(n, \mathbb{R})$ of all orthogonal matrices.
- Let $U(n)$ denote the subgroup of all unitary matrices in $GL(n, \mathbb{C})$.
- Let $SO(n, \mathbb{R})$ and $SU(n, \mathbb{R})$ denote respectively the subgroups consisting of elements of $O(n, \mathbb{R})$ and $U(n)$ whose determinant is one.
- Let $GL^+(n, \mathbb{R})$ denote the subgroup of all elements with positive determinant.
- Observe that $O(n, \mathbb{R})$, $SO(n, \mathbb{R})$, $U(n)$ and $SU(n)$ are compact groups.

Warm-up Exercises

- The group $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Is it a topological group? What is the topology on it?
- Let G be any group. There are two obvious topologies on G. Which of them make G a topological group?
- Let G_i, $1 \leq i \leq n$, be topological groups. Is their product a topological group? What is the topology on it? Can you think of a generalization?
- Let G be a topological group. Let $H \leq G$ be a subgroup of G Is there a natural way to make H a topological group?

Playtime

- Fix $a \in G$. Consider $L_a \colon x \mapsto ax$. What can you say of the map L_a? How are L_a and $L_{a^{-1}}$ are related?
- Let $U \ni e$ be an (open) neighbourhood of $e \in G$. Let $a \in G$. What can you say of aU?
- Let $V \ni a$ be open. Is V of the form aU for some $u \ni e$?
- Do last two items remind you of some facts from the norm topology on any normed linear space, say, $\mathbb{R}^n$?

$$B(a, r) = a + B(0, r).$$

- If U is open and A is any set, what do you know about $AU := \{ax : a \in A, x \in U\}$?

Playtime (Continued)

- If U is open what can you say of $A^{-1} = \{x^{-1} : x \in U\}$?
- Let $(G, \cdot)$ be a group, τ a topology on G. Assume that the map $(x, y) \mapsto xy^{-1}$ is continuous as a map from $(G \times G, \tau \times \tau)$ to (G, τ). What would you like to conclude?
- Is there an obvious question you would like to ask?

Neighbourhood Base at $e \in G$

Let $\mathcal{U}$ denote the set of all neighborhoods of $e \in G$. We list the properties of $\mathcal{U}$. U is called the fundamental neighbourhood system at $e \in G$.

1. $e \in U$ for all $U \in \mathcal{U}$.
2. If $U_1, U_2 \in \mathcal{U}$, then there exists $U \in \mathcal{U}$ such that $U \subset U_1 \cap U_2$.
   Hint: Use the continuity of the group multiplication at (e, e).
3. If $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $VV^{-1} \subset U$.
   Hint: Use the continuity of $(x, y) \mapsto xy^{-1}$ at (e, e).

Neighbourhood Base at $e \in G$ (Continued)

1. If $U \in \mathcal{U}$ and $a \in U$, then there exists $V \in \mathcal{U}$ such that $aV \subset U$.
2. If $U \in \mathcal{U}$ and $a \in G$, there exists $V \in \mathcal{U}$ such that $aVa^{-1} \subset U$.
   Hint: Use the continuity of $L_a \circ R_{a^{-1}}$ at e.

Let G be *any* group. Let $\mathcal{U}$ be a collection of subset of G satisfying the 5 conditions above. Then there is a natural topology on G which makes it into a topological group.

Question: What is the system of neighbourhoods at $a \in G$.

Neighbourhood Base at $e \in G$ (Continued)

1. If $U \in \mathcal{U}$ and $a \in U$, then there exists $V \in \mathcal{U}$ such that $aV \subset U$.
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   Hint: Use the continuity of $L_a \circ R_{a^{-1}}$ at e.

Let G be *any* group. Let $\mathcal{U}$ be a collection of subset of G satisfying the 5 conditions above. Then there is a natural topology on G which makes it into a topological group.

- Question: What is the system of neighbourhoods at $a \in G$ for the above mentioned topology?
  Answer: $\{aU : U \in \mathcal{U}\}$.

Topology and Subgroups

Theorem

If H is a subgroup of G, then the closure $\overline{H}$ is also a subgroup.

Proof.

- Easy, if G is metrizable.
- Let $x, y \in \overline{H}$, $U \in \mathcal{U}$. To prove $xyU \cap H \neq \emptyset$ and $x^{-1}U \cap H \neq \emptyset$.
- $\exists V \in \mathcal{U}$ such that $xVyV \subset xyU$. Why? Continuity of $(x, y) \mapsto xy$.
- Let $z_1 \in xV \cap H$ and $z_2 \in yV \cap H$. (Why do they exist?) Then $z_1 z_2 = xv_1 yv_2 \in H$ as well as $z_1 z_2 \in xVyV \subset xyU$.
- Let $z \in U^{-1}x \cap H$. (Why does z exist?) Then $z = u^{-1}x \in H$ and hence $z^{-1} = x^{-1}u \in x^{-1}U \cap H$.

□

Topology and Subgroups (Continued)

- If H is a normal subgroup of G, then the closure $\overline{H}$ is also a normal subgroup.
- If H is an open subgroup of G, then H is closed.
  Hint: Consider the coset decomposition of G with respect to H. That is, observe that $H = G \setminus \cup_{x \notin H} xH$.

Exercises

1. Let H be a closed subgroup of finite index in a topological group. Prove that H is open.
2. Let G be a topological group and $E \subset G$. Show that
$$\overline{E} = \cap_{U \in \mathcal{U}} UE = \cap_{U \in \mathcal{U}} EU.$$
Question: Have you seen an analogous result earlier?
3. Let G be a topological group. Let $\mathcal{U}$ be the neighbourhood base at e. Show that G is Hausdorff iff $\cap_{U \in \mathcal{U}} = \{e\}$. Choose $V \in \mathcal{U}$ such that $aV \cap bV = \emptyset$. If $z \in aV \cap bV$, then $a^{-1}b = v_1 v_2^{-1} \in VV^1$.
4. How will you define the uniform continuity of $f \colon G \to \mathbb{C}$ on any topological group? (G need not be metrizable.)
5. Let G and H be topological groups. Prove that a group homomorphism $f \colon G \to H$ is continuous iff it is continuous at e. (Have you seen an analogous result elsewhere?)

Topology and Topological Groups (Continued)

Theorem

Let G be a connected (topological) group. Let $U = U^{-1}$ be a symmetric open set with $e \in U$. Then $G = \cup_{n \in \mathbb{N}.} U^n$.

Proof.

Why does such U exist?
What does U^n denote?
What is the relation between U^n and $(U^n)^{-1}$?
Let $H := \cup_n U^n$. Is it clear that each U^n is open? How about H?
Is H a subgroup?
Why is H closed?
Why is $H = G$? □

Quotient w.r.t. a subgroup

- Let $H \leq G$ be a subgroup of a topological group.
- Let $X := G/H$ denote the set of left cosets of the form aH. Let $\pi \colon G \to X$ be the quotient map $a \mapsto aH$.
- $\Omega \subset X$ is open if $\pi^{-1}(\Omega)$ is open in G. This defines the quotient topology on X.
- $\pi \colon G \to X$ is , of course, continuous and also open.
- Let $U \subset G$ be open . Then $\pi(U) = \{aH; a \in U\}$. Is it open in X? Observe $\pi^{-1}(\pi(U)) = UH$ is open in G.

When is G/H Hausdorff?

- If G/H is Hausdorff, then $\{eH\} \subset X$ is closed and hence $H := \pi^{-1}(eH)$ is closed.
- Assume H is closed. Let $aH \neq bH$. Then $a^{-1}b \notin H$.
- Let $U \in \mathcal{U}$ be such that $a^{-1}bU \cap H = \emptyset$. (Why does such a U exist?)
- There exists $V \in \mathcal{U}$ such that $(aV)^{-1}bV \subset a^{-1}bU$. (Why?) Use the continuity of $(x, y) \mapsto x^{-1}y$.
- Conclude $(aV)^{-1}bV \cap H \subset a^{-1}bU \cap H = \emptyset$.
- $\pi(aV) \cap \pi(bV) = \emptyset$. (Verify.)
- $\pi(aV)$ and $\pi(bV)$ are open sets containing aH and bH respectively.
- Theorem: The quotient space G/H is Hausdorff iff H is a closed subgroup.

H and G/H connected implies G connected

Assume that $H \leq G$is connected and G/H is connected.

- Let U and $V := G \setminus U$ be proper open subsets of G.
- $\pi(U)$ and $\pi(V)$ provide a 'disconnection' of G/H.
  Each of them is open. (Why?)
  $\pi(U) \cap \pi(V) = \emptyset$. (Why?)
  If $aH \in \pi(U) \cap \pi(V)$, then $aH = xH$ with $x \in U$ and $aH = yH$ with $y \in V$. xH is connected. (Why?)
  $xH \subset U \cup V$. Hence $xH \subset U$. (Why?)
  Similarly, $yH \subset V$. Hence $xH = yH \subset U \cap V$, a contradiction.
- $G/H = \pi(U) \cup \pi(V)$ and G/H is connected. Hence one of them is empty.

Homogeneous Spaces

- Note that G has a natural action on $X := G/H$. In fact, the map $\alpha\colon G \times G/H \to G/H$: $(a, xH) \mapsto axH$ is continuous. (Verify.) Note that this action is transitive.
- Let X be a locally compact 2nd countable Hausdorff space. Let $\alpha\colon G \times X \to X$ be a continuous group action, which is transitive. Then we say that X is a *homogeneous space* or G-space.
- Given two G-spaces X and Y, how do you define a G map $\varphi\colon X \to Y$?
- When are two G-spaces isomorphic?

Classification of Homogeneous Spaces

- Theorem. Let X be a G-space. Then G is isomorphic to G/H for some closed subgroup $H \leq G$.
- Fix $p \in X$. Let $H := G_p$ be the stabilizer/isotropy subgroup at p. What is the natural map from G/H to X?
- The above natural maps turns out to be an isomorphism. It requires a non-trivial proof and needs a version of Baire category theorem. See my notes on Baire Category available from MTTS site.