Dimension of a Vector Space

S. Kumaresan
kumaresa@gmail.com

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Abstract

We give three proofs of the fact that any two bases of a finite dimensional vector space have the same number of elements. The first proof is the standard one which you will find in many textbooks. The second proof is the simplest one which you will find in our book. The third is a self-contained version of the second proof and is due to Jayanthan@Goa University.

Let V be a vector space over $\mathbb{R}$ (or over a field $\mathbb{F}$). Let $S \subset V$ be any nonempty subset. Recall that $\mathrm{LS}(S)$ stands for the set of finite linear combinations of elements from S. Any typical element in $\mathrm{LS}(S)$ is of the form $c_1v_1 + \cdots + c_kv_k$ where $k \in \mathbb{N}$ and $c_j \in \mathbb{F}$ and $v_j \in S$ for $1 \leq j \leq k$. We also know that $\mathrm{LS}(S)$ is the smallest vector subspace of V which contains S. We read $\mathrm{LS}(S)$ as the linear span of S.

Ex. 1. If $S = \emptyset \subset V$, how would you like to define $\mathrm{LS}(\emptyset)$?

Ex. 2. If $S \subset T \subset V$, what is the relation between $\mathrm{LS}(S)$ and $\mathrm{LS}(T)$?

Definition 3. A vector space V is *finite dimensional* if there exists a finite set $S \subset V$ such that $V = \mathrm{LS}(S)$. If V is not finite dimensional, we say that it is infinite dimensional.

Ex. 4. (i) Prove that $\mathbb{R}^n$ is finite dimensional. (ii) Prove that the vector space P_n of polynomials of degree less than or equal to $n \in \mathbb{N}$ is finite dimensional.

Ex. 5. Prove that the vector space of polynomials of all degrees is infinite dimensional.

Remark 6. If $V = (0)$ is the zero vector space, it is finite dimensional. Ex. 1 may be of use. See Remark 21.

We make a couple of observations which will make us confident of the entity $\mathrm{LS}(S)$ and these will also be useful later.

Observation 7. Let $S := \{v_1, \dots, v_k\}$. Assume that S is linearly independent. If $v \notin \mathrm{LS}(S)$, what can you say about the set $A := \{v, v_1, \dots, v_k\}$?

The obvious answer is: A is is linearly independent. This is obvious if you have understood the way we have approached the notion of linear dependence and independence. if you are new to our approach, let us do it your way.

Let $av + a_1v_1 + \cdots + a_kv_k = 0$. We claim that $a = a_1 = \cdots = a_k = 0$. If $a \neq 0$, then $v = \frac{1}{a}(\sum_{j=1}^k a_jv_j) \in \mathrm{LS}(S)$, a contradiction to our hypothesis that $v \notin \mathrm{LS}(S)$. Hence we conclude that $a = 0$. The above expression then yields $a_1v_1 + \cdots + a_kv_k = 0$. Since S is linearly independent, we obtain $a_i = 0$ for each i. Hence the claim.

Observation 8. Let the notation be as in the last observation. Let $v \in \mathrm{LS}(S)$. We can write $v = \sum_{i=1}^k c_iv_i$. We claim that the coefficients are unique. For, if $v = \sum_{i=1}^k d_iv_i$, then $0 = v - v = \sum_{i=1}^k (c_i - d_i)v_i = 0$. Since S is linearly independent, we deduce that for each i, we have $c_i - d_i = 0$ or $c_i = d_i$.

Observation 9. Keep the hypothesis of Observation 7. Let $w \in \mathrm{LS}(S)$. Assume that $w \neq 0$. Then we claim that we can replace one $v_j \in S$ by w and the set $S_j := \{w\} \cup \{v_i : 1 \leq i \leq k, i \neq j\}$ is linearly independent. This is again easy if you have a good understanding of linear dependence/independence. In any case, we shall go through the standard arguments.

Before we do it, let us look at a simple example. Let $V = \mathbb{R}^n$, $n \geq 3$ and let $v_i = e_i$ be the standard basic vectors. Let $S := \{e_1, e_2, e_3\}$. Let $w = e_1 + e_2$. Do you see that we can replace e_1 (respectively, e_2) to get $S_1 = \{e_1 + e_2, e_2, e_3\}$ (respectively, $S_2 = \{e_1, e_1 + e_2, e_3\}$? Can we replace e_3 by $e_1 + e_2$? What goes wrong?

Since $w \in \mathrm{LS}(S)$, we can write $w = \sum_{i=1}^n c_iv_i$. Since $w \neq 0$, there exists at least one j such that $c_j \neq 0$. (In the above example $c_1 = 1 = c_2$.) We claim that we can exchange v_j for w. That is, we work with S_j as described above.

Assume that there exist scalars a, a_i, $i \neq j$ such that

$$aw + a_1v_1 + \cdots + a_{j-1}v_{j-1} + a_{j+1}v_{j+1} + \cdots + a_kv_k = 0 = aw + \sum_{i \neq j} a_iv_i = 0.$$

We shall prove that $a = 0$ and that $a_i = 0$ for $i \neq j$.

Can $a \neq 0$? If $a \neq 0$, then

$$w = -\frac{1}{a}\left(\sum_{i \neq j} a_iv_i\right) = -\frac{1}{a}\left(\sum_{i \neq j} a_iv_i\right) + 0 \cdot v_j.$$

But we know that $w = \sum_{i=1}^k c_iv_i$. Since S is linearly independent, the linear expression for w in terms of v_i's is unique by Observation 8. That is, the coefficients of each v_i, for *all* $1 \leq i \leq k$, must be equal. So, in particular, the coefficients of v_j in these expressions must be the same. But then we arrive at $c_j = 0$, a contradiction.

So we conclude that $a = 0$. Since $a = 0$, we end up with $\sum_{i \neq j} a_iv_i = 0$. Since S is linearly independent, we conclude that $a_i = 0$ for $i \neq j$. That is we have shown that S_j is linearly independent.

Observation 10. Keep the notation of the last observation 9. We claim that $\mathrm{LS}(S_j) = \mathrm{LS}(S)$.

We need to show that $\mathrm{LS}(S_j) \subset \mathrm{LS}(S)$ and the reverse inclusion.

Let $v \in \mathrm{LS}(S_j)$. Then v is a linear combination of w and v_i's (where $i \neq j$). But w itself is a linear combination of all v_i's. Hence v s a linear combination of v_i's and hence lies in $\mathrm{LS}(S)$. Can you write down a proof with all the symbols?

Let $v = bw + \sum_{i \neq j} b_i v_i \in \mathrm{LS}(S_j)$. Then

$$
\begin{aligned}
v &= b(\sum_{i=1}^{k} c_i vi) + \sum_{i \neq j} b_i v_i \\
&= (bc_1 + b_1)v_1 + \cdots + (bc_{j-1} + b_{j-1})v_{j-1} + bc_j v_j \\
&\quad + (bc_{j+1} + b_{j+1})v_{j-1} + \cdots + (bc_k + b_k)v_k.
\end{aligned}
$$

Hence it follows that $v \in \mathrm{LS}(S)$.

To prove the reverse inclusion, let us note that v_j can be written in terms of w and v_i, $i \neq j$. For, from $w = \sum_{i \neq j} c_i v_i + c_j v_j$, we obtain $v_j = w - \frac{1}{c_j} \sum_{i \neq j} c_i v_i$. Hence any linear combination of v_i's can be expressed in terms of w and v_i $(i \neq j)$.

Can you write down a proof with symbols as above? If $v = \sum_i d_i v_i \in \mathrm{LS}(S)$, then

$$
\begin{aligned}
v &= d_j v_j + \sum_{i \neq j} d_i v_i \\
&= d_j(w - \frac{1}{c_j} \sum_{i \neq j} c_i v_i) + \sum_{i \neq j} d_i v_i.
\end{aligned}
$$

Remark 11. Learn the arguments in Observations 9–10 well. In literature, they are collectively known as Exchange lemma or Replacement lemma. You will come across such analogous arguments in the study of finitely generated abelian groups, finite generated modules over a PID, cyclic vectors towards Jordan canonical forms etc where you exchange one of the elements in the generating set by another suitable element. (B.Sc. students may ignore this remark. It is meant for students who wish to pursue Algebra at a higher level.)

We now move onto the main result of this article.

Let $B := \{v_1, \ldots, v_m\}$ and $B' := \{w_1, \ldots, w_n\}$ be two bases of a nonzero finite dimensional vector space V. Note that $\mathrm{LS}(B_1) = V = \mathrm{LS}(B_2)$.

Our aim is to show that $m = n$. Note that $m \geq 1$ and $n \geq 1$, since V is nonzero.

First observe that none of the w_i's can be zero. For, if $0 \in B'$, then B' must have one more element. For other wise, $\mathrm{LS}(B') = (0) = V$, a contradiction. Let $w_1 = 0$, then $0 = 1 \cdot 0 + 0 \cdot w_2 + \cdots + 0 \cdot w_n = \sum_j 0 \cdot w_j$. Thus the way of expressing 0 is not unique and hence B' cannot be a basis.

Since $w_1 \in V = \mathrm{LS}(B_1)$, by Observations 8–9, there exists an i_1 such that v_{i_1} can be replaced by w_1. The resulting set $B_1 = \{w_1\} \cup \{v_i : i \neq i_1\}$ is such that $\mathrm{LS}(B_1) = \mathrm{LS}(B) = V$, that is, B_1 is a spanning set for V. Also, B_1 is a linearly independent set. Hence B_1 is a basis of V.

Now repeat the argument above with the pair (B, w_1) replaced by (B_1, w_2) to conclude that there exists i_2 such that $B_2 := \{w_1, w_2\} \cup \{v_i : i \neq i_1, i_2\}$ is a basis of V.

There is a subtlety here. Since B_1 is a basis of V, we can write $w_2 = aw_1 + \sum_{i \neq i_1} a_i v_i$. To mimic the argument above, we look for a term with nonzero coefficient and we want the term to be some v_i. We claim that there exists i such that $a_i \neq 0$. If all $a_i = 0$, then we find that $w_2 = aw_1$ and hence the set $\{w_1, w_2\}$ is a linearly dependent subset of the basis B'. This contradiction shows that there exists i such that $a_i \neq 0$. We call this i as i_2. Understand this well and you need to mimic this argument in the next step too.

Ex. 12. Can you write down the argument for arrive at B_3?

We continue this process.

If $m > n$, then after n steps, we would arrive at a basis

$$B_n = \{w_1, \dots, w_n\} \cup \{v_i : i \neq i_1, \dots i_n\} = B' \cup \{v_i : i \neq i_1, \dots i_n\} = B' \cup C, \ \textit{say}.$$

Since $C \neq \emptyset$, if $v_r \in C$, then $v_r \in V = \mathrm{LS}(B')$. Hence the set $B' \cup \{v_r\}$ is linearly dependent. Hence its super set B_n is linearly dependent. This contradicts the fact that B_n is a basis. Hence we conclude that $m \leq n$.

Interchanging B and B' in the argument above, we can show that $n \leq m$ and hence $m = n$.

Thus we have arrived at the following theorem.

Theorem 13. *Let V be a finite dimensional vector space. Then any two bases of V have the same number of elements.* □

Note that we still do not know whether any finite dimensional vector space has a basis!

Definition 14. Let V be a finite dimensional vector space. The *dimension* of V is the number of elements in any basis. Note that this is well-defined in view of the last theorem. We denote the dimension of a vector space by $\dim V$.

Our second and favourite proof needs the following result. For a simple proof using Gaussian elimination and induction, see [1]. (Most often this is proved using echelon forms.)

Theorem 15. *Let $m, n \in \mathbb{N}$ with $m < n$. Let $\sum_{j=1}^n a_{ij}x_j = 0$, $1 \leq i \leq m$ be a linear homogeneous system. Then there exists $c := (c_1, \dots, c_n) \in \mathbb{F}^n \setminus \{0\}$ such that c is a (non-zero) solution of the homogeneous system: $\sum_{j=1}^n a_{ij}c_j = 0$ for $1 \leq i \leq m$.* □

Proposition 16. *Let $B := \{v_i : 1 \leq i \leq n\}$ be a set in a vector space V. Let $S := \{w_j : 1 \leq j \leq n+1\}$ be a subset of $LS(B)$. Then S is linearly dependent.*

Proof. To prove the result, we need to find $c := (c_1, \dots, c_{n+1}) \in \mathbb{F}^{n+1} \setminus \{0\}$ such that $\sum_{j=1}^{n+1} c_j w_j = 0$.

Since $w_j \in \mathrm{LS}(B)$ there exist a set of scalars a_{ij}, $1 \leq i \leq n+1$, $1 \leq j \leq n$ such that

$$w_i = a_{i1}v_1 + \cdots + a_{in}v_n, \qquad (1 \leq i \leq n+1).$$

Insert the expressions for w_j in the equation $\sum_{j=1}^{n+1} c_j w_j = 0$. We obtain

$$0 = \sum_{j=1}^{n+1} c_j \sum_{i=1}^{n} a_{ij} v_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n+1} a_{ij} c_j \right) v_i. \tag{1}$$

Can we choose c_j so that for each i, the sum in the bracket on the right side of (1) is zero? That is, can we choose c_j such that

$$a_{i1}c_1 + \cdots + a_{i,n+1}c_{n+1} = 0, \qquad (1 \le i \le n)? \tag{2}$$

We note that this is a homogeneous system of n linear equations in $(n+1)$-unknowns, namely, $c_1, \ldots, c_{n+1}$. By Theorem 15, there exists a nonzero solution $(c_1, \ldots, c_{n+1})$ of (2). Thus we have found scalars c_j (not all zero!) such that (1) is true. That is, $\sum_j c_j w_j = 0$.

Thus we find that the set A is linearly dependent. □

Remark 17. Let V be a finite dimensional vector space. Let $S \subset V$ be a finite set with $|S| = N$. Let B be a basis of V. We claim that B is finite, in fact, $|B| \le N$.

If $|B| > N$, then choose a subset $A := \{v_1, \ldots, V_{N+1}\} \subset B$. Since B is linearly independent, so is A. Since $v_j \in LS(S) = V$, by the last proposition A is linearly dependent. This is a contradiction. So we infer that $|B| \le N$.

Remark 18. As an immediate corollary, we obtain another proof of Theorem 13. Let $B_1 := \{v_i : 1 \le i \le m\}$ and $B_2 := \{w_j : 1 \le j \le n\}$ be bases of a vector space V. Since B_1 is a basis, it is a spanning set and $B_2 \subset \mathrm{LS}(B_1) = V$. We claim $n \le m$. If $n > m$, then $n \ge m+1$ and the set $\{w_j : 1 \le j \le m+1\}$ is linearly dependent by the last proposition. Since $\{w_j : 1 \le j \le m+1\} \subset B_2$, we see that B_2 is linearly dependent. This contradicts the fact that B_2, being a basis of V, is linearly independent. We therefore conclude that $n \le m$. Interchanging roles of B_1 and B_2 in the argument above leads us to conclude that $m \le n$. Hence $m = n$.

We now give a direct proof proof of Proposition 16. It is due to Jayanthan A.J. of Goa University. The main strategy is to adapt the proof of Theorem 15 which uses Gaussian elimination and induction. See my book [1].

Proof. (of Proposition 16. We shall prove it by induction on n.

When $n = 1$, let $B = \{v\}$ and $S := \{w_1, w_2\}$. Then $w_j = t_j v$ and hence $t_2 w_1 - t_1 w_2 = t_2 t_1 v - t_1 t_2 v = 0$. Either $t_1 t_2 = 0$ in which case either $w_1 = 0$ or $w_2 = 0$ and hence S is linearly dependent. Otherwise, $t_1 t_2 \ne 0$ and hence w_1, w_2 are linearly dependent.

Assume the result for $n-1$. What does this mean? If S is a set of n-vectors lying in the linear span of any set if $n-1$ vectors then S is linearly dependent.

Let us deal with the case n. Keep the notation of the statement. If one of w_j's is 0, the set S is linearly dependent. So we assume $w_j \ne 0$ for any $1 \le j \le n+1$.

We write $w_i = a_{i1}v_1 + \ldots + a_{in}v_n$. (Why is it possible?) Since $w_1 \ne 0$, one of $a_{1j} \ne 0$. Without loss of generality, let us assume that $a_{11} \ne 0$. We use this assumption to "eliminate" v_1. Let E_i stand for the i-th 'equation': $w_i = a_{i1}v_1 + \ldots + a_{in}v_n$. For $i \ge 2$, we consider $a_{i1}E_1 - a_{11}E_i$. In concrete terms, look at

$$a_{11}w_i - a_{i1}w_1 = \sum_{r=1}^{n} a_{11}a_{ir}v_r - \sum_{r=1}^{n} a_{i1}a_{1r}v_r.$$

Can you expand this avoiding the summation notation? What do you get? We get

$$u_i := a_{11}w_i - a_{i1}w_1 = \sum_{r=2}^{n} a_{11}a_{ir}v_r - \sum_{r=2}^{n} a_{i1}a_{1r}v_r = \sum_{r=2}^{n} b_{ir}v_r, \quad (2 \le i \le n+1),$$

where $b_{ir} := a_{11}a_{ir} - a_{i1}a_{1r}$.

What have we achieved? We have shown that a set $\{u_i : 2 \le i \le n+1\}$ lies in the linear span of the set $\{v_r : 2 \le r \le n\}$ of $n-1$ vectors. Hence by induction hypothesis, the set $\{u_i : 2 \le i \le n+1\}$ is linearly dependent.

Therefore, there exists $(c_2, \dots, c_{n+1}) \in \mathbb{F}^n \setminus \{0\}$ such that $\sum_{i=2}^{n+1} c_i u_i = 0$.

What are u_i's? We use the fact that $u_i := a_{11}w_i - a_{i1}w_1$ in the equation above to arrive at

$$\begin{aligned}\sum_{i=2}^{n+1} c_i u_i &= \sum_{i=2}^{n+1} c_i(a_{11}w_i - a_{i1}w_1) \\ &= (c_2a_{21} + \cdots + c_{n+1}a_{n+1,1})w_1 - a_{11}(c_2w_2 + \cdots + c_{n+1}w_{n+1}) \\ &= 0.\end{aligned}$$

Since by assumption $(c_2, \dots, c_{n+1}) \neq (0, \dots, 0)$, let us assume that $c_j \neq 0$. Since $a_{11} \neq 0$, in the above linear combination we see that the coefficient $-a_{11}c_j \neq 0$ of w_j for some $j \ge 2$. Thus we have proved that set S is linearly dependent. □

Remark 19. Theorem 13 is again an immediate corollary of the last proposition. The argument of Remark 18 goes through verbatim.

Ex. 20. Go through the second proof of Proposition 16. Adapt it to give a proof of Theorem 15.

Remark 21. Let $V = (0)$ be the zero vector space. Can we assign the notion of dimension to it? Either we can decree that $\dim((0)) = 0$ or we can 'prove' it. The proof depends on a proper interpretation of $\mathrm{LS}(S)$. We already know that $\mathrm{LS}(S)$ is the smallest vector subspace containing S. So, what is $\mathrm{LS}(\emptyset)$? It is clearly (0). Next, if B is a basis of a nonzero vector space, then B is a minimal spanning set and $\dim V = |B|$, the number of elements in B. Now revisit the equality $\mathrm{LS}(\emptyset) = (0)$. Is it clear that $\emptyset$ is the minimal spanning set for the zero vector space (0)? If you say 'yes', that is it. We rest the case.

Ex. 22. Let V be a vector space of dimension $\dim V = n$. Prove that any set $S \subset V$ which contains any $n+1$ elements is linearly dependent. (This is a most often used fact.)

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References

[1] S Kumaresan, *Linear Algebra – A Geometric Approach*, PTI, 22nd reprint (2020).