An Example illustrating the Concepts of Basic Linear Algebra

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Abstract

There are results which are existential and theoretical in nature in a preliminary course on linear algebra. (We have in mind the results such as the existence of a basis, the extension of a linearly independent set in a vector space V to a basis of V, the proof of rank-nullity theorem etc.) While it trains the young minds to think in abstract, it is equally important to show them how to turn those abstract arguments into something concrete in practical situations. Also, often beginners think of linear maps between $\mathbb{R}^n$ and $\mathbb{R}^m$ only as functions defined by "formulas" in terms of the coordinates. This is in spite of the fact that they have already seen a result which emphasizes that a linear map $T: V \to W$ is determined once we know the images of vectors (under T) in a basis of V.

The purpose of this article is to work out an example explicitly to show how the theory can be implemented in concrete cases. It is also hoped that this example will show how to create linear maps with prescribed kernel and image rather than checking a given map is linear and finding its kernel and image!

Can we find a linear map $T: \mathbb{R}^6 \to \mathbb{R}^4$ such that

$$\ker T = U := \{(x_1, \dots, x_6) \in \mathbb{R}^6 : x_1 = x_2 = x_3 \text{ and } x_4 + x_5 + x_6\} = 0,$$
$$\operatorname{Im} T = V := \{(y_1, \dots, y_4) \in \mathbb{R}^4 : y_1 + y_2 + y_3 + y_4 = 0\}?$$

The first thing we need to check is whether the Rank-Nullity theorem vetoes against the existence of such a map. As we have learnt earlier, $\dim \ker T = 3$ while $\dim \operatorname{Im} T = 3$. Hence $\dim \mathbb{R}^6 = 6 = \dim \ker T + \dim \operatorname{Im} T$. So, the rank-nullity theorem does not rule out the existence of such a linear map. Note that if we required $\ker T = \{x \in \mathbb{R}^6 : x_1 = x_2 = x_3$ or if we demanded that $\operatorname{Im} T = \{y \in \mathbb{R}^4 : y_1 = y_2, y_3 = y_4\}$, then no such maps can exist. (Justify this claim.)

Let us denote the standard basis of $\mathbb{R}^6$ by $e_1, \dots, e_6$ and that of $\mathbb{R}^4$ by $f_1, \dots, f_4$. We fix a basis of U: $u_1 := e_1 + e_2 + e_3$, $u_2 = e_4 - e_6$ and $u_3 := e_5 - e_6$. We fix $v_j := f_j - f_4$, $j = 1, 2, 3$ as a basis of V. The proof of Rank-Nullity theorem shows us how to construct a linear map as required. We need to (i) extend the basis of U (by adding three vectors, say, u_4, u_5, u_6) to a basis of $\mathbb{R}^4$, (ii) map $\{u_1, u_2, u_3\}$ to 0 and (iii) map $\{u_4, u_5, u_6\}$ bijectively onto $\{v_1, v_2, v_3\}$. Finally write down the map explicitly using the standard coordinates and verify it is as required.

Note that $e_1 + e_2 + e_3$ lies in the span of $\{e_1, e_2, e_3\}$. So to find a basis that includes $e_1 + e_2 + e_3$, we can take any two out of e_1, e_2, e_3. For definiteness sake, let us take the basis $\{e_1, e_2, e_1 + e_2 + e_3\}$. Similar reasoning applied to the span of $\{e_4, e_5, e_6\}$ allows to take $\{e_4 - e_6, e_5 - e_6, e_6\}$. Note that we could have taken $\{e_4 - e_6, e_5 - e_6, e_j\}$, $j = 4, 5$ also as a basis for the span of e_4, e_5, e_6.

We thus end up with a basis $\{e_1 + e_2 + e_3, e_1, e_2, e_4 - e_6, e_5 - e_6, e_6\}$. The new elements, e_1, e_2, e_6 that augment the basis of $\ker T$ are to be mapped bijectively to the chosen basis of $\operatorname{Im} T$. We shall define the linear map T by specifying its action on the above basis of $\mathbb{R}^6$:

$$Te_1 = v_1, Te_2 = v_2, Te_6 = v_3, T(u_1) = T(e_1 + e_2 + e_3) = 0, T(e_4 - e_6) = 0, T(e_5 - e_6) = 0.$$

To write the linear map in terms of the standard basis, we need to see what T does to the elements of the standard basis. Already, we know the images of e_1, e_2, e_6. Hence we need only find out $T(e_3), T(e_4), T(e_5)$. Since $T(e_1 + e_2 + e_3) = 0$,

$$Te_3 = -Te_1 - Te_2 = -v_1 - v_2 = -(f_1 - f_4) - (f_2 - f_4) = -f_1 - f_2 + 2f_4.$$

Since $T(e_4 - e_6) = 0 = T(e_5 - e_6)$, we conclude that

$$Te_4 = Te_5 = Te_6 = v_3 = f_3 - f_4.$$

Thus

$$\begin{aligned}
T(x_1, \dots, x_6) &= x_1 Te_1 + \cdots + x_6 Te_6 \\
&= x_1(f_1 - f_4) + x_2(f_2 - f_4) + x_3(-f_1 - f_2 - 2f_4) + x_4(f_3 - f_4) \\
&\qquad + x_5(f_3 - f_4) + x_6(f_3 - f_4) \\
&= (x_1 - x_3) + (x_2 - x_3)f_2 + (x_4 + x_5 + x_6)f_3 \\
&\qquad + (-x_1 - x_2 + 2x_3 - x_4 - x_5 - x_6)f_4 \\
&= (x_1 - x_3, x_2 - x_3, x_4 + x_5 + x_6, -x_1 - x_2 + 2x_3 - x_4 - x_5 - x_6).
\end{aligned}$$

It is trivial to see that $\ker T = U$ and $\operatorname{Im} T = V$.

It is to be noted that such maps are far from being unique. Can you think of another linear map different from the one given above?

Ex. 1. Let $U := \{x \in \mathbb{R}^{2k} : x_1 + 3x_3 + (2k-1)x_{2k-1} = 0, x_2 = x_4 = \cdots = x_{2k}\}$. Construct a linear map from $\mathbb{R}^{2k}$ onto some $\mathbb{R}^l$ (l to be specified by you!) with U as its kernel.

Remark 2. This remark is due to Bhaba Kumar Sarma, IITG. In the above discussion, we extended a linearly independent set to a basis of V using an already known basis of V. This is very practical and useful.

Let us explain the strategy. Let $\{u_i : 1 \le i \le k\}$ be a linearly independent set in V. Let $\{v_1, \dots, v_n\}$ be a basis of V. Note that $\{u_1, \dots, u_k, v_1, \dots, v_n\}$ is a spanning set. The idea is to keep adding v_j to the set $\{u_i : 1 \le i \le k\}$ provided the resulting set is linearly independent. Let us show this in action. If v_1 is not in the linear span of u_i's, then we have a linearly independent set $B_1 := \{u_1, \ldots, u_k, v_1\}$. If not, we discard v_1 and let $B_1 := \{u_1, \ldots, u_k\}$. So at the end of the first step, either have $B_1 := \{u_i : 1 \leq i \leq k\}$ or $B_1 := \{u_1, \ldots, u_k, v_1\}$ as a linearly independent set in such a way that $\operatorname{span} B_1 = \operatorname{span}\{u_1, \ldots, u_k, v_1\}$. Now we add v_2 to B_1 and proceed as earlier. If $v_2 \in \operatorname{span}\{u_1, \ldots, u_k, v_1\}$ then $B_2 := \{u_1, \ldots, u_k, v_1\}$, otherwise $B_2 := \{u_1, \ldots, u_k, v_1, v_2\}$. Observe that $\operatorname{span} B_2 = \operatorname{span}\{u_1, \ldots, u_k, v_1, v_2\}$. After a finite number of steps, we arrive at a basis of V which extends u_i's. Do you see why?